

# ON THE POSITIVITY SET OF A LINEAR RECURRENCE SEQUENCE

BY

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ABSTRACT

We consider real sequences  $(f_n)$  that satisfy a linear recurrence with constant coefficients. We show that the density of the positivity set of such a sequence always exists. In the special case where the sequence has no positive dominating characteristic root, we establish that the density is positive. Furthermore, we determine the values that can occur as density of such a positivity set, both for the special case just mentioned and in general.

## 1. Introduction and main results

A sequence  $(f_n)_{n \geq 0}$  of real numbers is called a (*linear*) *recurrence sequence* if it satisfies a linear recurrence

$$f_{n+h} = c_1 f_{n+h-1} + \cdots + c_{h-1} f_{n+1} + c_h f_n, \quad n \geq 0,$$

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\* Supported in part by the Christian Doppler Research Association (CDG). S. Gerhold gratefully acknowledges a fruitful collaboration and continued support by Bank Austria and ÖBFA through CDG. Supported in part by the FWF grant F1305.

Received June 21, 2005

with constant coefficients  $c_k \in \mathbb{R}$ . (Since we are concerned with questions of positivity, we restrict attention to real sequences.) One of the most charming and celebrated results in the theory of recurrence sequences is the Skolem–Mahler–Lech theorem. It asserts that the zero set

$$\{n \in \mathbb{N}: f_n = 0\}$$

of a recurrence sequence is the union of a finite set and finitely many arithmetic progressions. The recent comprehensive monograph by Everest et al. [3] contains references to the substantial literature devoted to this result and to related questions. However, not much seems to be known about the positivity set

$$(1) \quad \{n \in \mathbb{N}: f_n > 0\}.$$

In the following section we establish that the density of the set (1) always exists, where the density of a set  $A \subseteq \mathbb{N}$  is defined as

$$\delta(A) := \lim_{x \rightarrow \infty} x^{-1} \#\{n \leq x: n \in A\},$$

provided that the limit exists.

**THEOREM 1:** *Let  $(f_n)$  be a recurrence sequence. Then the density of the set  $\{n \in \mathbb{N}: f_n > 0\}$  exists.*

Recall that a recurrence sequence can be written as a generalized power sum

$$(2) \quad f_n = \sum_{k=0}^m P_k(n) \gamma_k^n, \quad n \geq 0,$$

with non-zero polynomials  $P_k \in \mathbb{C}[n]$  and roots  $\gamma_k \in \mathbb{C}$  that are roots of the characteristic polynomial

$$z^h - c_1 z^{h-1} - \dots - c_{h-1} z - c_h$$

of the recurrence. The roots of largest modulus are called *dominating roots* of  $(f_n)$ .

It does not come as a surprise that recurrence sequences with no positive dominating root have oscillating behaviour. Indeed, in Section 3 we prove that for such a sequence  $(f_n)$  the densities of (1) and the negativity set

$$(3) \quad \{n \in \mathbb{N}: f_n < 0\}$$

are always positive. This generalizes the following known result [1, 4]: If a recurrence sequence has at most four dominating roots, and none of them is real positive, then the sets (1) and (3) both have infinitely many elements.

**THEOREM 2:** *Let  $(f_n)$  be a nonzero recurrence sequence with no positive dominating characteristic root. Then the sets  $\{n \in \mathbb{N} : f_n > 0\}$  and  $\{n \in \mathbb{N} : f_n < 0\}$  have positive density.*

In Section 4 we investigate which numbers actually occur as density of the positivity set of some recurrence sequence. It turns out that all possible values occur, both for sequences with no dominating positive root and in general.

Finally (Section 5), we return to the Skolem–Mahler–Lech theorem. Our approach yields the following weak version: The density of the zero set exists and is a rational number.

**2. The density of the positivity set**

*Notation:* We write  $(f_n) \equiv 0$  if  $f_n = 0$  for all  $n \geq 0$ . The Lebesgue measure of a set  $B \subset \mathbb{R}^m$  is denoted by  $\lambda(B)$ .

The goal of this section is to prove Theorem 1. Dividing  $f_n$  by  $n^D|\gamma_1|^n$ , where  $\gamma_1$  is a dominating root of  $(f_n)$  and  $D$  is the maximal degree of the  $P_k$  with  $|\gamma_k| = |\gamma_1|$ , we obtain from (2)

$$n^{-D}|\gamma_1|^{-n} f_n = \sum_{i=1}^d a_i \cos(2\pi\theta_i n + \beta_i) + v - r_n,$$

where  $(r_n)$  is a recurrence sequence with  $r_n = O(1/n)$ ,  $\theta_1, \dots, \theta_d$  are in  $]0, 1[$ , and  $a_i, \beta_i, v \in \mathbb{R}$ . From now on we will assume w.l.o.g.  $D = 0$  and  $|\gamma_1| = 1$ . As a first step we get rid of any integer relations that the  $\theta_i$  might satisfy.

**LEMMA 3:** *Let  $\theta_1, \dots, \theta_d$  be real numbers. Then there is a basis  $\{\tau_1, \dots, \tau_{m+1}\}$  of the  $\mathbb{Z}$ -module*

$$M = \mathbb{Z} + \mathbb{Z}\theta_1 + \dots + \mathbb{Z}\theta_d$$

such that  $1/\tau_{m+1}$  is a positive integer and  $1, \tau_1, \dots, \tau_m$  are linearly independent over  $\mathbb{Q}$ .

*Proof:*  $M$  is finitely generated and torsion free, hence it is free [7, Theorem III.7.3]. Let  $\{\alpha_1, \dots, \alpha_{m+1}\}$  be a basis. Since  $1 \in M$ , there are integers  $e_1, \dots, e_{m+1}$  such that

$$e_1\alpha_1 + \dots + e_{m+1}\alpha_{m+1} = 1.$$

We complete  $(e_1/g, \dots, e_{m+1}/g)$ , where  $g := \gcd(e_1, \dots, e_{m+1})$ , to a unimodular integer matrix  $\mathbf{C}$  with last row  $(e_1/g, \dots, e_{m+1}/g)$  [7, §XXI.3]. Then

$$(\tau_1, \dots, \tau_{m+1})^T := \mathbf{C}(\alpha_1, \dots, \alpha_{m+1})^T$$

yields a basis of  $M$  with  $\tau_{m+1} = 1/g \in \mathbb{Q}$ . Now suppose

$$u_1\tau_1 + \dots + u_m\tau_m = u$$

for integers  $u_1, \dots, u_m, u$ . Since  $u$  has also the representation

$$ug\tau_{m+1} = u,$$

it follows that  $u_1 = \dots = u_m = u = 0$ . ■

Take  $\tau_1, \dots, \tau_{m+1}$  as in Lemma 3, with  $\tau_{m+1} = 1/g$ . Roughly speaking, we have put all integer relations among the  $\theta_i$  into the rational basis element  $\tau_{m+1}$ . There are integers  $b_{ij}$  with

$$\theta_i = \sum_{j=1}^{m+1} b_{ij}\tau_j.$$

Now we split the sequence  $(f_n)$  into the subsequences  $(f_{gn+k})_{n \geq 0}$  for  $0 \leq k < g$ . We have

$$f_{gn+k} = G_n - s_n,$$

where  $s_n := r_{gn+k}$  and  $G_n$  is the dominant part. Defining the integer matrix

$$\mathbf{B} := (gb_{ij})_{\substack{1 \leq i \leq d \\ 1 \leq j \leq m}} \in \mathbb{Z}^{d \times m}$$

and the real vector  $\mathbf{c} = (c_1, \dots, c_d)$  with

$$c_i := 2\pi k \sum_{j=1}^{m+1} b_{ij}\tau_j + \beta_i, \quad 1 \leq i \leq d,$$

it can be written as (cos is applied component-wise)

$$G_n = \mathbf{a}^T \cos(2\pi n\mathbf{B}\tau + \mathbf{c}) + v.$$

We show that the density of  $\{n \in \mathbb{N}: f_{gn+k} > 0\}$  exists for each  $k$ . Since  $(s_n)$  is a recurrence sequence with fewer characteristic roots than  $(f_n)$ , we may assume inductively that  $\delta(\{n \in \mathbb{N}: s_n < 0\})$  exists. Thus, if  $(G_n) \equiv 0$ , we are done. Now let  $k$  be such that  $(G_n)$  is not the zero sequence. It is plain that  $G_n = H(n\tau)$ , where

$$H(\mathbf{t}) := \mathbf{a}^T \cos(2\pi\mathbf{B}\mathbf{t} + \mathbf{c}) + v, \quad \mathbf{t} \in [0, 1]^m.$$

The following theorem shows that the function  $H$  can be used to evaluate the density of the positivity set of  $(G_n)$ , which equals, as we will see below, that of the set  $\{n: f_{gn+k} > 0\}$ .

**THEOREM 4 (Kronecker–Weyl):** *Let  $\tau_1, \dots, \tau_m$  be real numbers such that  $1, \tau_1, \dots, \tau_m$  are linearly independent over  $\mathbb{Q}$ . Then the sequence  $(n\tau)$  is uniformly distributed modulo one, i.e., for every Jordan measurable set  $A \subseteq [0, 1]^m$  we have*

$$\delta(\{n \in \mathbb{N} : n\tau \bmod 1 \in A\}) = \lambda(A).$$

*Proof:* We refer to Cassels [2, Theorems IV.I and IV.II]. ■

We define

$$(4) \quad L_\varepsilon := \{n \in \mathbb{N} : G_n \geq \varepsilon\} \quad \text{and} \quad S_\varepsilon := \{n \in \mathbb{N} : |G_n| < \varepsilon\}.$$

The corresponding sets for the function  $H$  are defined as

$$\tilde{L}_\varepsilon := \{\mathbf{t} \in [0, 1]^m : H(\mathbf{t}) \geq \varepsilon\} \quad \text{and} \quad \tilde{S}_\varepsilon := \{\mathbf{t} \in [0, 1]^m : |H(\mathbf{t})| < \varepsilon\}.$$

Since for all  $\varepsilon \geq 0$

$$L_\varepsilon = \{n \in \mathbb{N} : n\tau \bmod 1 \in \tilde{L}_\varepsilon\},$$

we have  $\delta(L_\varepsilon) = \lambda(\tilde{L}_\varepsilon)$  for all  $\varepsilon \geq 0$  by Theorem 4. Similarly,

$$(5) \quad \delta(S_\varepsilon) = \lambda(\tilde{S}_\varepsilon), \quad \varepsilon > 0.$$

Note that the boundary of the bounded set  $\tilde{S}_\varepsilon$  (respectively  $\tilde{L}_\varepsilon$ ) is a Lebesgue null set (as seen by applying the following lemma with  $F(\mathbf{t}) = H(\mathbf{t}) - \varepsilon$ ), hence  $\tilde{S}_\varepsilon$  and  $\tilde{L}_\varepsilon$  are Jordan measurable, and Theorem 4 is indeed applicable. Lemma 5 seems to be known [6], but we could not find a complete proof in the literature.

**LEMMA 5:** *Let  $F: \mathbb{R}^m \rightarrow \mathbb{R}$  be a real analytic function. Then the zero set of  $F$  has Lebesgue measure zero, unless  $F$  vanishes identically.*

The proof of Lemma 5 is postponed to the end of this section. Since  $(G_n)$  is not the zero sequence, the function  $H$  does not vanish identically on  $[0, 1]^m$ . By the Lebesgue dominated convergence theorem and Lemma 5 we thus find

$$\lim_{\varepsilon \rightarrow 0} \lambda(\tilde{S}_\varepsilon) = 0 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \lambda(\tilde{L}_\varepsilon) = \lambda(\tilde{L}_0).$$

This yields  $\delta(\{n \in \mathbb{N} : G_n > s_n\}) = \lambda(\tilde{L}_0)$  by the following lemma, which completes the proof of Theorem 1.

**LEMMA 6:** *Let  $(G_n)$  and  $(s_n)$  be real sequences with  $s_n = o(1)$ , and let  $L_\varepsilon, S_\varepsilon$  be as in (4). Suppose that  $\delta(L_\varepsilon)$  and  $\delta(S_\varepsilon)$  exist for all  $\varepsilon \geq 0$ , and that*

$$\lim_{\varepsilon \rightarrow 0} \delta(L_\varepsilon) = \delta(L_0) \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \delta(S_\varepsilon) = 0.$$

Then

$$\delta(\{n \in \mathbb{N}: G_n > s_n\}) = \delta(L_0).$$

*Proof:* For any set  $A \subseteq \mathbb{N}$  we write  $A(x) := \{n \leq x: n \in A\}$ . Define

$$P := \{n \in \mathbb{N}: G_n > s_n\}.$$

Let  $\varepsilon > 0$  be arbitrary. Take  $n_0$  such that  $|s_n| < \varepsilon$  for  $n > n_0$ . It follows that

$$\begin{aligned} \#P(x) &= \#\{n \leq n_0: G_n > s_n\} + \#\{n_0 < n \leq x: G_n \geq \varepsilon\} \\ &\quad + \#\{n_0 < n \leq x: s_n < G_n < \varepsilon\}, \end{aligned}$$

hence

$$|\#P(x) - \#L_\varepsilon(x)| \leq \#S_\varepsilon(x) + o(x)$$

as  $x \rightarrow \infty$ . Thus we have

$$\begin{aligned} |x^{-1}\#P(x) - \delta(L_0)| &\leq |x^{-1}\#P(x) - x^{-1}\#L_\varepsilon(x)| + |x^{-1}\#L_\varepsilon(x) - \delta(L_0)| \\ &\leq x^{-1}\#S_\varepsilon(x) + |x^{-1}\#L_\varepsilon(x) - \delta(L_0)| + o(1). \end{aligned}$$

The right hand side tends to

$$\delta(S_\varepsilon) + |\delta(L_\varepsilon) - \delta(L_0)|$$

as  $x \rightarrow \infty$ . By assumption, this can be made arbitrarily small, which implies  $\delta(P) = \delta(L_0)$ . ■

*Proof of Lemma 5:* For  $m = 1$  this is clear, since then the zero set is countable. Now assume that we have established the result for  $1, \dots, m - 1$ . Put

$$V := \{(t_2, \dots, t_m) \in \mathbb{R}^{m-1}: F(\cdot, t_2, \dots, t_m) \text{ vanishes identically}\}.$$

Take a real number  $s$  such that  $F(s, \cdot, \dots, \cdot)$  is not identically zero. Clearly,  $F(s, t_2, \dots, t_m) = 0$  for all  $(t_2, \dots, t_m) \in V$ . By the induction hypothesis, this implies  $\lambda(V) = 0$ . Note that  $V$  is closed, hence measurable. Since  $F$  is real analytic in the first argument, we have

$$\int_{\mathbb{R}} \chi_Z(t_1, \dots, t_m) d\lambda(t_1) = 0$$

for all  $(t_2, \dots, t_m) \notin V$ , where  $\chi_Z$  is the characteristic function of the zero set

$$Z := \{(t_1, \dots, t_m) \in \mathbb{R}^m: F(t_1, \dots, t_m) = 0\}.$$

Since  $V$  has measure zero, this implies

$$\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \chi_Z(t_1, \dots, t_m) d\lambda(t_1) \cdots d\lambda(t_m) = 0.$$

This argument works for any order of integration, hence we obtain  $\int_{\mathbb{R}^m} \chi_Z = 0$  by Tonelli's theorem. ■

### 3. Sequences with no positive dominating root

In this section we prove Theorem 2. We begin by settling the two special cases where the  $\theta_i$  are all irrational or all rational, and then put them together.

LEMMA 7: *Let  $\theta_1, \dots, \theta_d$  be irrational numbers, and let  $a_i, \beta_i$  be real numbers such that the sequence*

$$u_n = \sum_{i=1}^d a_i \cos(2\pi\theta_i n + \beta_i)$$

*is not identically zero. Let further  $(r_n)$  be a recurrence sequence with  $r_n = o(1)$ . Then the set  $\{n \in \mathbb{N} : u_n > r_n\}$  has positive density.*

*Proof:* Proceeding as in the proof of Theorem 1, we can write

$$G_n := u_{gn+k} = \mathbf{a}^T \cos(2\pi n \mathbf{B}\tau + \mathbf{c}),$$

where  $\mathbf{B}$  is an integer matrix no row of which is zero,  $\mathbf{c}$  is a real vector, and  $1, \tau_1, \dots, \tau_m$  are linearly independent over  $\mathbb{Q}$ . If  $k$  is such that  $(G_n) = (u_{gn+k}) \equiv 0$ , then the density of  $\{n \in \mathbb{N} : G_n > s_n\}$ , where  $s_n = r_{gn+k}$ , exists by Theorem 1, but may be zero. Now choose a  $k_0$  such that the corresponding sequence  $(G_n) = (u_{gn+k_0})$  is not the zero sequence. We have  $G_n = H(n\tau)$ , where

$$H(\mathbf{t}) := \mathbf{a}^T \cos(2\pi \mathbf{B}\mathbf{t} + \mathbf{c}).$$

Moreover, with the notation of the proof of Theorem 1, we have

$$\delta(\{n \in \mathbb{N} : G_n > s_n\}) = \lambda(\tilde{L}_0).$$

The function  $H$  is not identically zero on  $[0, 1]^m$ . But

$$(6) \quad \int_0^1 \cdots \int_0^1 H(t_1, \dots, t_m) dt_1 \cdots dt_m = 0,$$

because no row of  $\mathbf{B}$  is the zero vector. Hence  $H$  has a positive value on  $[0, 1]^m$ , and since it is continuous, we have  $\lambda(\tilde{L}_0) > 0$ . ■

Observe that the integral in (6) need not vanish if  $\mathbf{B}$  has a zero row, which can only happen if the  $\theta_i$  corresponding to this row is a rational number. This is the reason why we consider rational  $\theta_i$ s separately.

LEMMA 8: *Let  $\theta_1, \dots, \theta_d$  be rational numbers in  $]0, 1[$ , and let  $a_i, \beta_i$  be real numbers such that the purely periodic sequence*

$$u_n = \sum_{i=1}^d a_i \cos(2\pi\theta_i n + \beta_i)$$

*is not identically zero. Then  $u_n$  has a positive and a negative value.*

*Proof:* By the identity

$$\sum_{k=0}^{q-1} \cos \frac{2\pi kp}{q} + i \sum_{k=0}^{q-1} \sin \frac{2\pi kp}{q} = \sum_{k=0}^{q-1} e^{2\pi i kp/q} = 0,$$

valid for integers  $0 < p < q$ , and the addition formula of  $\cos$  we obtain

$$u_0 + \dots + u_{q-1} = 0,$$

where  $q$  is a common denominator of  $\theta_1, \dots, \theta_d$ . ■

*Proof of Theorem 2:* It suffices to consider the positivity set. We may write

$$f_n = u_n + v_n - r_n,$$

where  $(r_n)$  is a recurrence sequence with  $r_n = o(1)$ ,

$$u_n = \sum_{i=1}^d a_i \cos(2\pi\theta_i n + \beta_i),$$

$$v_n = \sum_{i=d+1}^e a_i \cos(2\pi\theta_i n + \beta_i),$$

$\theta_1, \dots, \theta_d$  are irrational,  $\theta_{d+1}, \dots, \theta_e$  are rational numbers in  $]0, 1[$  with common denominator  $q > 0$ , and  $(u_n + v_n) \not\equiv 0$ . If  $(v_n) \equiv 0$ , then the result follows from Lemma 7. Now suppose  $(v_n) \not\equiv 0$ . Then for each  $k$  the density of the set  $\{n \in \mathbb{N}: f_{qn+k} > 0\}$  exists by Theorem 1. By Lemma 8 there is  $k_0$  such that



$v_{qn+k_0} = : v > 0$ . It suffices to show that the set  $\{n \in \mathbb{N}: f_{qn+k_0} > 0\}$  has positive density. This is clear if  $(u_{qn+k_0}) \equiv 0$ . Otherwise, notice that

$$\{n \in \mathbb{N}: f_{qn+k_0} > 0\} \supseteq \{n \in \mathbb{N}: u_{qn+k_0} > r_{qn+k_0}\},$$

and the latter set has positive density by Lemma 7. ■

#### 4. The possible values of the density

In this section we investigate which values from  $[0, 1]$  occur as density of some recurrence sequence. In its basic form, the question is readily answered:

EXAMPLE 9: *Let  $w$  be a real number and define*

$$f_n := \sin(2\pi n\sqrt{2}) - w.$$

*Then, by Theorem 4,*

$$\begin{aligned} \delta(\{n \in \mathbb{N}: f_n > 0\}) &= \lambda(\{t \in [0, 1]: \sin(2\pi t) > w\}) \\ &= \begin{cases} 1 & w \leq -1, \\ \frac{1}{2} - \frac{1}{\pi} \arcsin w & -1 \leq w \leq 1, \\ 0 & w \geq 1. \end{cases} \end{aligned}$$

*Since the range of arcsin is  $[-\pi/2, \pi/2]$ , for every  $\kappa \in [0, 1]$  this yields a recurrence sequence  $(f_n)$  such that*

$$\delta(\{n \in \mathbb{N}: f_n > 0\}) = \kappa.$$

The following proposition generalizes this example. Note that the density of the zero set of a recurrence sequence is always a rational number by the Skolem–Mahler–Lech theorem.

PROPOSITION 10: *Let  $\kappa$  be a real number and  $r$  be a rational number with  $0 \leq \kappa, r \leq 1$  and  $\kappa + r \leq 1$ . Then there is a recurrence sequence  $(f_n)$  such that*

$$\delta(\{n \in \mathbb{N}: f_n > 0\}) = \kappa \quad \text{and} \quad \delta(\{n \in \mathbb{N}: f_n = 0\}) = r.$$

*Proof:* Suppose that  $r = p/q$  for positive integers  $p$  and  $q$ . As seen in Example 9, there is a recurrence sequence  $(g_n)$  such that the density of the zero set of  $(g_n)$  is zero and the density of its positivity set is  $\kappa/(1 - r)$  (The case  $r = 1$  is trivial.) The interlacing sequence

$$f_{bn+k} := \begin{cases} 0, & 0 \leq k < p \\ g_n, & p \leq k < q \end{cases}$$

is a recurrence sequence [3, Section 4.1]. Clearly, the density of its zero set is  $r$ , and the density of its positivity set is

$$\delta(\{n \in \mathbb{N}: f_n > 0\}) = \frac{q-p}{q} \times \frac{\kappa}{1-r} = \kappa,$$

as required. ■

If we restrict attention to sequences without real positive dominating roots, then Theorem 2 tells us that the density of the positivity set can be neither zero nor one. Computer experiments with numerical values for the parameters usually yield approximations of the density that are close to  $\frac{1}{2}$ . Still, all values in between occur.

**THEOREM 11:** *Let  $\kappa \in ]0, 1[$ . Then there is a recurrence sequence  $(f_n)$  with no positive dominating characteristic root and  $\delta(\{n \in \mathbb{N}: f_n > 0\}) = \kappa$ .*

*Proof:* Let  $\varepsilon > 0$  be arbitrary. We define a function  $H$  on  $[0, \frac{1}{2}]$  by

$$H(t) := \begin{cases} \frac{(\varepsilon-1)^2}{\varepsilon} (1 - \frac{2t}{\varepsilon}), & 0 \leq t \leq \frac{\varepsilon}{2} \\ \varepsilon - 2t, & \frac{\varepsilon}{2} \leq t \leq \frac{1}{2} \end{cases}$$

and extend it to an even, 1-periodic function  $H$  on  $\mathbb{R}$  (see Figure 1). It is continuous and satisfies

$$\int_0^1 H(t)dt = 0 \quad \text{and} \quad \lambda(\{t \in [0, 1]: H(t) > 0\}) = \varepsilon.$$

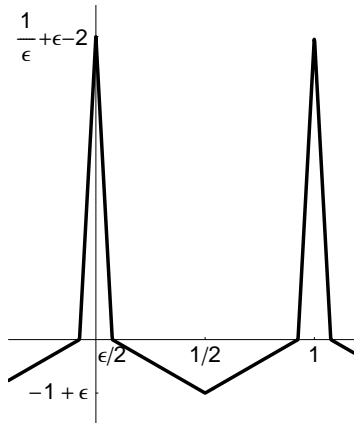


Figure 1. The function  $H$ .

Expanding  $H$  into a Fourier series, we find that there are real  $a_j$  such that  $H$  is the pointwise limit of

$$H_m(t) := \sum_{j=1}^m a_j \cos(2\pi jt)$$

as  $m \rightarrow \infty$ . Since the zero set of  $H$  is a null set, the Lebesgue dominated convergence theorem yields

$$\lim_{m \rightarrow \infty} \lambda(\{t \in [0, 1]: H_m(t) > 0\}) = \varepsilon.$$

We fix an  $m$  such that

$$\lambda(\{t \in [0, 1]: H_m(t) > 0\}) \leq 2\varepsilon.$$

The function

$$\phi(A_1, \dots, A_m) := \lambda\left(\left\{t \in [0, 1]: \sum_{j=1}^m A_j \cos(2\pi jt) > 0\right\}\right)$$

is continuous on  $\mathbb{R}^m \setminus \{0\}$ . To see this, observe that  $\phi$  is continuous at all points  $(A_1, \dots, A_m)$  for which  $\sum_{j=1}^m A_j \cos(2\pi jt)$  is not identically zero and appeal to the uniqueness of the Fourier expansion. Since  $\phi(1, 0, \dots, 0) = \frac{1}{2}$  and  $\phi(a_1, \dots, a_m) \leq 2\varepsilon$ , the function  $\phi$  assumes every value from  $[2\varepsilon, \frac{1}{2}]$  by the intermediate value theorem.

Hence the positivity sets of the sequences

$$f_n := \sum_{j=1}^m A_j \cos(2\pi jn\sqrt{2})$$

assume all densities from  $[2\varepsilon, \frac{1}{2}]$  for appropriate choices of  $(A_1, \dots, A_m)$  by Theorem 4. Repeating the whole argument with  $-H$  instead of  $H$  yields the desired result for  $\kappa \in [\frac{1}{2}, 1 - 2\varepsilon]$ . Since  $\varepsilon$  was arbitrary, the theorem is proved. ■

### 5. A weak version of Skolem–Mahler–Lech

Without using the Skolem–Mahler–Lech theorem, it follows from Theorem 1 that the density of the zero set of a recurrence sequence  $(f_n)$  exists. We can show a bit more with our approach. Recall, however, that we only deal with real sequences, whereas the Skolem–Mahler–Lech theorem holds for any field of characteristic zero.

PROPOSITION 12: *The density of the zero set of a (real) recurrence sequence  $(f_n)$  is a rational number.*

*Proof:* Let  $k$  be a natural number, and let  $g, G_n,$  and  $s_n$  be as in the proof of Theorem 1. If  $k$  is such that  $(G_n) \equiv 0$ , then the density of the zero set of  $(f_{gn+k})$  is rational, since we may assume inductively that the density of  $\{n: s_n = 0\}$  is rational.

Now suppose  $(G_n) \not\equiv 0$ . The zero set of  $(f_{gn+k})$  can be partitioned as

$$\{n \in \mathbb{N}: G_n = s_n\} = \{n: G_n = s_n, |G_n| < \varepsilon\} \cup \{n: G_n = s_n, |G_n| \geq \varepsilon\},$$

where  $\varepsilon \geq 0$  is arbitrary. The latter set is finite, and the first one is contained in  $S_\varepsilon$ , defined in (4). Hence

$$\delta(\{n \in \mathbb{N}: G_n = s_n\}) \leq \delta(S_\varepsilon)$$

for all  $\varepsilon \geq 0$ . But we know that  $\lim_{\varepsilon \rightarrow 0} \delta(S_\varepsilon) = 0$  from the proof of Theorem 1, which yields

$$\delta(\{n \in \mathbb{N}: G_n = s_n\}) = 0.$$

Thus, the zero sets of all subsequences  $(f_{gn+k})_{n \geq 0}, 0 \leq k < g$ , have rational density, which proves the desired result. ■

### 6. Conclusion

There is no algorithm known that decides, given a recurrence sequence  $(f_n)$ , whether  $f_n > 0$  for all  $n$ , nor has the problem been shown to be undecidable. When we are talking about algorithmics, it is natural to assume that the recurrence coefficients and the initial values are rational numbers. In this case Gourdon and Salvy [5] have proposed an efficient method for ordering the characteristic roots w.r.t. their modulus. Thus, the dominating characteristic roots can be identified algorithmically. If none of them is real positive, then we know that the sequence oscillates by Theorem 2. On the other hand, sequences where a positive dominating root is accompanied by complex dominating roots seem to pose difficult Diophantine problems. For instance, we do not know if the sequence

$$(7) \quad f_n := \cos(2\pi\theta n) + 1 + \left(-\frac{1}{2}\right)^n$$

is positive for  $\theta = \sqrt[3]{2}$ , say. It is positive for  $n \leq 10^5$ . It can be shown, however, that the set of  $\theta$ s for which the corresponding sequence  $(f_n)$  (defined by (7)) is eventually positive has measure one [4, Theorem 7.2].

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