Mathematical Finance 2: Continuous Time Models

Stefan Gerhold
FAM (Financial and Actuarial Mathematics)
TU Wien

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Contents

1 Black-Scholes model

1.1 Basics

Samuelson 1965, Black-Scholes 1973, Merton 1973

Finite time interval $t \in [0, T]$. (Sometimes $t \in [0, \infty)$ is considered.)

Riskless asset $B_t = e^{rt}$, where $r \in \mathbb{R}$ is deterministic (same interest rate for savings and loans).

Risky asset: geometric Brownian motion (see below).

No market frictions:

1. Agents can trade in continuous time
2. Any position size is possible (no shortselling constraints, assets are divisible)
3. No bid-ask spread (*one* price process for the risky asset)
4. No market impact

Brownian motion (BM): $(W_t)_{0 \le t \le T}$ defined on a filtered probability space satisfying the usual conditions. BM satisfies

1. $W_0 = 0$
2. W is adapted and has continuous paths
3. For $s < t$, $W_t - W_s$ is independent of $\mathcal{F}_s$
4. $W_t - W_s \sim \mathcal{N}(0, t - s)$

Geometric Brownian motion: $S_0 > 0$,

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad \text{drift } \mu \in \mathbb{R}, \text{ volatility } \sigma > 0.$$

Implicit definition (SDE). Explicitly:

Proposition 1.1. *The unique solution of this SDE is*

$$S_t = S_0 \exp(\sigma W_t + (\mu - \tfrac{1}{2}\sigma^2)t).$$

Proof. Define $Y_t := \sigma W_t + (\mu - \frac{1}{2}\sigma^2)t$. First, assume $S_0 = 1$ and then multiply by S_0. Apply Ito's formula to $S_t = h(Y_t)$ with $h(x) = e^x$.

Uniqueness: Exercise (or from general theory of SDEs). □

Note: For $\mu = 0$, S is a martingale (more on this later). Indeed,

$$\mathbb{E}[e^{\sigma(W_t - W_u)} | \mathcal{F}_u] = \mathbb{E}[e^{\sigma(W_t - W_u)}] = e^{\frac{1}{2}\sigma^2(t-u)}.$$

Definition 1.2. *A trading strategy is a pair of progressively measurable processes, $\phi = (\phi_t^1, \phi_t^2)_{0 \le t \le T}$. Its value process is*

$$V_t(\phi) := \phi_t^1 S_t + \phi_t^2 B_t.$$

The strategy is self-financing, if $\int_0^T (\phi_u^1)^2 du$ and $\int_0^T |\phi_u^2| du$ are a.s. finite and

$$V_t(\phi) = V_0(\phi) + \int_0^t \phi_u^1 dS_u + \int_0^t \phi_u^2 dB_u, \quad 0 \le t \le T.$$

Define the discounted asset price $S^* := S/B$.

$$\begin{aligned} S_t^* &= S_0^* \exp(\sigma W_t + (\mu - r - \tfrac{1}{2}\sigma^2)t), \\ dS_t^* &= (\mu - r) S_t^* dt + \sigma S_t^* dW_t. \end{aligned}$$

S^* is a submartingale for $\mu > r$, and a supermartingale for $\mu < r$.

Definition 1.3. *A measure $\mathbb{Q} \sim \mathbb{P}$ is an equivalent martingale measure (EMM), if S^* is a local $\mathbb{Q}$-martingale.*

Theorem 1.4. *$\mathbb{Q}$ is an EMM $\Longleftrightarrow$ for every self-financing trading strategy ϕ, the discounted value process $V^* := V(\phi)/B$ is a local $\mathbb{Q}$-martingale.*

Proof. $\Longleftarrow$: Consider the buy-and-hold strategy $\phi_t = (1, -S_0)$. Then

$$V_t^*(\phi) = B_t^{-1}(S_t - S_0 B_t) = S_t^* - S_0.$$

$\Longrightarrow$: Calculate $dV_t^* = d(B_t^{-1} V_t)$ by the Ito product rule:

$$\begin{aligned} dV^* = d(B^{-1}V) &= B^{-1} dV + V dB^{-1} + d\underbrace{[B^{-1}, V]}_{0} \\ &= B^{-1}(\phi^1 dS + \phi^2 dB) + (\phi^1 S + \phi^2 B) dB^{-1} \\ &= \phi^1 (S dB^{-1} + B^{-1} dS) + B^{-1}\phi^2 B r dt + \phi^2 B(-r B^{-1}) dt = \phi^1 dS^*. \ \Box \end{aligned}$$

Example 1.5 (Simplified Girsanov)**.** *Let $dW_t^* := dW_t - \lambda(t)dt$, where λ is a deterministic L^2 function, and define*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(\int_0^T \lambda(s) dW_s - \frac{1}{2} \int_0^T \lambda(s)^2 ds \right).$$

An easy calculation shows that $\mathbb{E}_{\mathbb{Q}}[e^{uW_t^*}] = e^{u^2 t/2}$, $u \in \mathbb{R}$, and so W^* has the one-dimensional marginals of a standard BM under $\mathbb{Q}$.

Theorem 1.6. *In the Black-Scholes model, the unique EMM is given by*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(\frac{r-\mu}{\sigma}W_T - \frac{1}{2}\frac{(r-\mu)^2}{\sigma^2}T\right).$$

$W_t^* := W_t - \frac{r-\mu}{\sigma}t$ *is a* $\mathbb{Q}$*-Brownian motion, and* $dS_t^* = \sigma S_t^* dW_t^*$.

Proof. By Girsanov's theorem (see the exercise class). □

We have

$$S_t^* = S_0 e^{-rt}\exp(\sigma W_t + (\mu - \tfrac{1}{2}\sigma^2)t) = S_0 \exp(\sigma W_t^* - \tfrac{1}{2}\sigma^2 t).$$

Thus, under $\mathbb{Q}$, S^* is a geometric BM with drift 0, and thus a *true* $\mathbb{Q}$-martingale (not just local). Dynamics of S under $\mathbb{Q}$:

$$dS = \mu S dt + \sigma S dW = \mu S dt + \sigma S dW^* + (r-\mu)S dt = rS dt + \sigma S dW^*.$$

The following definition serves to rule out riskless gains by so-called "doubling strategies" (see Section 1.7).

Definition 1.7. *A self-financing trading strategy is admissible w.r.t.* $\mathbb{Q}$*, if* $V^*(\phi)$ *is a* $\mathbb{Q}$*-martingale.*

From now on, "strategy" in the BS model always means a self-financing trading strategy which is admissible w.r.t. the unique EMM.

Definition 1.8. *The BS model is the market* (S, B) *with the set of* $\mathbb{Q}$*-admissible trading strategies, for the unique EMM* $\mathbb{Q}$*.*

Definition 1.9. *A market model is free of arbitrage, if there is no strategy with* $V_0(\phi) = 0$*,* $V_T(\phi) \geq 0$ *and* $\mathbb{P}[V_T(\phi) > 0] > 0$*.*

Theorem 1.10. *The BS model is free of arbitrage.*

Proof. If $V_T(\phi) \geq 0$ and this is positive with positive probability, the same holds under $\mathbb{Q}$, and we get $V_0(\phi) = V_0^*(\phi) = \mathbb{E}_{\mathbb{Q}}[V_T^*(\phi)] > 0$. □

Note that this argument does not work if V^* is a strict local martingale. In Section 1.7, we will se an example of such a trading strategy.

Definition 1.11. *A replicable claim is an* $\mathcal{F}_T$*-measurable random variable* X *such that there is a strategy* ϕ *with* $V_T(\phi) = X$ *a.s.*

Theorem 1.12. *For a replicable claim* X *in the BS model, let* $\pi_t(X) = V_t(\phi)$ *be its price process, for some replicating strategy* ϕ*. This process does not depend on the choice of* ϕ*.*

Proof.
$$\frac{\pi_t(X)}{B_t} = V_t^*(\phi) = \mathbb{E}_{\mathbb{Q}}[V_T^*(\phi)|\mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}}[B_T^{-1}X|\mathcal{F}_t]. \qquad \square$$

This yields the risk-neutral valuation formula: the price of a replicable claim is the risk-neutral expectation of its discounted payoff.

$$\boxed{\pi_t(X) = B_t\mathbb{E}_{\mathbb{Q}}[B_T^{-1}X|\mathcal{F}_t] = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t], \quad 0 \le t \le T} \tag{1.1}$$

It is clear that $\pi_0(X)$ is the only arbitrage-free price for the claim at time 0. Indeed, if someone would pay $\hat{\pi} > \pi_0(X)$ for it, then sell it to him, pocket the difference $\hat{\pi} - \pi_0(X)$, and replicate the claim, using the initial capital $\pi_0(X)$.

If you can buy the claim for $\hat{\pi} < \pi_0(X)$, do so, and go short in the replicating strategy to *receive* $\pi_0(X) - \hat{\pi} > 0$ up front.

Note that (1.1) is linear in X, which is a natural property of a pricing rule.

1.2 Black-Scholes formula, Black-Scholes PDE

A call option is the right to buy at time T one share for the price K. Payoff at maturity is $(S_T - K)^+$. Define

$$d_{1,2}(s,\tau) := \frac{\log(s/K) + (r \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}.$$

Note that $d_2 = d_1 - \sigma\sqrt{\tau}$. With Φ the standard normal cdf, define $c : \mathbb{R}^+ \times [0,T] \to \mathbb{R}^+$ by

$$c(s,\tau) := s\Phi(d_1(s,\tau)) - Ke^{-r\tau}\Phi(d_2(s,\tau)).$$

For $\tau = 0$, take the limit $(s-K)^+$ of the right hand side.

Theorem 1.13 (BS formula)**.** *The call option is replicable in the BS model, and its price process is given by $C_t = c(S_t, T-t)$. Explicitly:*

$$C_t = S_t\Phi(d_1(S_t, T-t)) - Ke^{-r(T-t)}\Phi(d_2(S_t, T-t)).$$

A replicating strategy is given by

$$\begin{aligned} \phi_t^1 &= \frac{\partial c}{\partial s}(S_t, T-t) = \Phi(d_1(S_t, T-t)) \quad \text{(“delta hedging”)}, \\ \phi_t^2 &= e^{-rt}(c(S_t, T-t) - \phi_t^1 S_t). \end{aligned}$$

Intuitively, delta hedging is based on the Taylor approximation

$$c(S_{t+\Delta t}) - c(S_t) \approx \frac{\partial c(S_t)}{\partial s}(S_{t+\Delta t} - S_t).$$

First, we *assume* that there are smooth functions v, g, h with $C_t = v(S_t, t)$, $(\phi_t^1, \phi_t^2) = (g(S_t, t), h(S_t, t))$. By the self-financing property, with $V_t = V_t(\phi)$,

$$\begin{aligned} dV &= gdS + hdB \\ &= g(\mu S dt + \sigma S dW) + \underbrace{rBh}_{r(V-gS)}\, dt \\ &= (\mu - r)Sgdt + \sigma SgdW + rVdt. \end{aligned}$$

On the other hand, by Ito's formula (note that partial derivatives of functions are denoted by indices),

$$\begin{aligned} dv &= \frac{\partial v(S_t,t)}{\partial t}dt + \frac{\partial v(S_t,t)}{\partial s}dS_t + \frac{1}{2}\frac{\partial^2 v(S_t,t)}{\partial s^2}d[S,S]_t \\ &= v_t dt + v_s dS + \tfrac{1}{2}v_{ss}d[S,S] \\ &= v_t dt + v_s(\mu S dt + \sigma S dW) + \tfrac{1}{2}\sigma^2 v_{ss}S^2 dt. \end{aligned}$$

As we must have $v = V$, we can subtract dV from dv and obtain

$$0 = (v_t + \mu S v_s + \tfrac{1}{2}v_{ss}\sigma^2 S^2)dt + v_s S\sigma dW + (r-\mu)Sgdt - \sigma SgdW - rvdt.$$

Diffusion coefficient and drift must vanish:

$$\sigma S(v_s - g) = 0 \quad \Longrightarrow \quad g = v_s,$$

$$0 = v_t + \mu S v_s + \tfrac{1}{2}v_{ss}\sigma^2 S^2 + (r-\mu)S\underbrace{g}_{v_s} - rv$$

We obtain the Black-Scholes PDE for the value function:

$$\boxed{v_t(s,t) + \tfrac{1}{2}\sigma^2 s^2 v_{ss}(s,t) + rsv_s(s,t) - rv(s,t) = 0}$$

Note that the PDE does not depend on μ, and that we have not used the call payoff. The value function of the put, with payoff $(K - S_T)^+$, and those of other European, not path-dependent claims satisfy the same PDE, as we will prove in Section 1.4. After these motivating remarks, we give two proofs of the BS formula.

First proof of Theorem 1.13. Define $v(s,t) := c(s, T-t)$, with $c(\cdot,\cdot)$ from above. The function v satisfies the BS PDE (exercise). Define $\phi^1_t := v_s(S_t, t)$ and $\phi^2_t := B_t^{-1}(v(S_t,t) - v_s(S_t,t)S_t)$. By definition, the value process is $V_t(\phi) = v(S_t, t)$. By Ito's formula and the BS PDE,

$$\begin{aligned} dV &= v_s dS + \tfrac{1}{2}\sigma^2 S^2 v_{ss} dt + v_t dt \\ &= v_s dS + (rv - rSv_s - v_t + v_t)dt \\ &= \phi^1 dS + rB\frac{v - S\phi^1}{B}dt = \phi^1 dS + \phi^2 dB. \end{aligned}$$

Thus, ϕ is self-financing. Admissibility (cf. the proof of Theorem 1.4):

$$\begin{aligned} V_t^*(\phi) &= V_0^*(\phi) + \int_0^t v_s(S_u, u) dS_u^* \\ &= V_0^*(\phi) + \int_0^t \Phi(d_1(S_u, T-u))\sigma S_u^* dW_u^*. \end{aligned}$$

Since $\Phi \leq 1$ and $\int_0^t \mathbb{E}_{\mathbb{Q}}[(S_u^*)^2]du < \infty$ (consider the Gaussian mgf), V^* is a true martingale. $\square$

We now give a second proof of the BS formula. Consider again $X = (S_T - K)^+$, and $X^* := B_T^{-1}X$. Since $X^* \in L^2(\mathbb{Q})$, the martingale representation theorem implies that there is a predictable process θ such that

$$X^* = \mathbb{E}_{\mathbb{Q}}[X^*] + \int_0^T \theta_u dW_u^*,$$

with $\int_0^{\cdot} \theta_u dW_u^*$ a martingale. Since $dS^* = \sigma S^* dW^*$, a natural choice for the risky position is $h_t := \theta_t/(\sigma S_t^*)$. Define the candidate discounted value process

$$V_t^* := \mathbb{E}_{\mathbb{Q}}[X^*] + \int_0^t h_u dS_u^*,$$

and the candidate replication strategy

$$\phi^1 := h, \quad \phi^2 := B^{-1}(V - \phi^1 S),$$

where of course $V = BV^*$. Clearly, $V(\phi) = V = BV^* = X$ by definition.

We check that ϕ is self-financing:

$$\begin{aligned}
d(BV^*) &= BdV^* + V^* \underbrace{dB}_{rBdt} \\
&= BhdS^* + rVdt \\
&= Bh(B^{-1}dS + S \underbrace{dB^{-1}}_{-rB^{-1}dt}) + rVdt \\
&= \underbrace{h}_{\phi^1} dS + r(V - hS)B^{-1}Bdt \\
&= \phi^1 dS + \phi^2 dB.
\end{aligned}$$

We have thus shown that $X = (S_T - K)^+$ is replicable. By the risk-neutral valuation formula, its price process is independent of the replication strategy, and given by

$$C_t = B_t \mathbb{E}_{\mathbb{Q}}[B_T^{-1} X | \mathcal{F}_t].$$

As S is a time-homogeneous Markov process and X is a deterministic function of S_T, there must be a deterministic function $c(\cdot, \cdot)$ such that $C_t = c(S_t, T-t)$. Recall the following property of conditional expectations:

Lemma 1.14. *Let U, V be random variables, $g : \mathbb{R}^2 \to \mathbb{R}$ measurable with $\mathbb{E}[|g(U, V)|] < \infty$, and $\mathcal{G} \subseteq \mathcal{F}$ be a sub-σ-algebra. If V is $\mathcal{G}$-measurable, and U is independent of $\mathcal{G}$, then*

$$\mathbb{E}[h(U, V)|\mathcal{G}] = \mathbb{E}[h(U, v)]\big|_{v=V}.$$

We compute

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[(S_T - K)^+ | \mathcal{F}_t] &= \mathbb{E}_{\mathbb{Q}}[(S_t \frac{S_T}{S_t} - K)^+ | \mathcal{F}_t] \\
&= \mathbb{E}_{\mathbb{Q}}[(S_t e^{\sigma(W_T^* - W_t^*) + (r - \frac{1}{2}\sigma^2)(T-t)} - K)^+ | \mathcal{F}_t] \\
&= \mathbb{E}_{\mathbb{Q}}[(s e^{\sigma(W_T^* - W_t^*) + (r - \frac{1}{2}\sigma^2)(T-t)} - K)^+ | \mathcal{F}_t]|_{s=S_t} \\
&= \mathbb{E}_{\mathbb{Q}}[(s e^{\sigma W_{T-t}^* + (r - \frac{1}{2}\sigma^2)(T-t)} - K)^+]|_{s=S_t},
\end{aligned}$$

and see that it suffices to consider $t = 0$. Then,

$$\mathbb{E}_{\mathbb{Q}}[B_T^{-1}(S_T - K)^+] = \mathbb{E}_{\mathbb{Q}}[B_T^{-1} S_T \mathbf{1}_D] - \mathbb{E}_{\mathbb{Q}}[B_T^{-1} K \mathbf{1}_D] =: J_1 - J_2,$$

where $D := \{S_T > K\}$. For J_2, we have

$$\begin{aligned}
J_2 &= e^{-rT} K \mathbb{Q}[S_0 e^{\sigma W_T^* + (r - \frac{1}{2}\sigma^2)T} > K] \\
&= e^{-rT} K \mathbb{Q}\Big[\frac{W_T^*}{\sqrt{T}} > -\frac{\log(S_0/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\Big] \\
&= e^{-rT} K \mathbb{Q}\Big[-\frac{W_T^*}{\sqrt{T}} < \frac{\log(S_0/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\Big] \\
&= e^{-rT} K \Phi(d_2).
\end{aligned}$$

Define $\bar{\mathbb{P}}$ by

$$\frac{d\bar{\mathbb{P}}}{d\mathbb{Q}} = e^{\sigma W_T^* - \frac{1}{2}\sigma^2 T} = \frac{S_T^*}{S_0}.$$

This is sometimes called the *share measure*. By Girsanov, $\bar{W}_t := W_t^* - \sigma t$ is a standard BM under $\bar{\mathbb{P}}$. Then,

$$\begin{aligned}
J_1 &= \mathbb{E}_{\mathbb{Q}}[S_T^* \mathbf{1}_D] = \bar{\mathbb{E}}\Big[\frac{d\mathbb{Q}}{d\bar{\mathbb{P}}} S_T^* \mathbf{1}_D\Big] \\
&= S_0 \bar{\mathbb{E}}[\mathbf{1}_D] = S_0 \bar{\mathbb{P}}[S_T > K] = S_0 \bar{\mathbb{P}}[S_T^* > K B_T^{-1}] \\
&= S_0 \bar{\mathbb{P}}[e^{\sigma \bar{W}_T + \frac{1}{2}\sigma^2 T} > K B_T^{-1}] = S_0 \Phi(d_1).
\end{aligned}$$

This finishes the second proof of the BS formula; note that it provides a *derivation* of the formula, and not just a verification. On the other hand, it does not yield the delta hedge and the PDE.

Time-dependent coefficients: The BS model $dS/S = \mu dt + \sigma dW$ has just one single parameter that can be calibrated to market prices of options, the volatility σ. As a simple extension, consider the dynamics

$$dS_t/S_t = \mu(t, S_t)dt + \sigma(t)dW_t,$$

where μ and σ are deterministic functions. Exercise: Generalize the BS formula ($\sigma\sqrt{T-t}$ becomes $\sqrt{\int_t^T \sigma(u)^2 du}$).

Put-call parity: A put option is the right to sell one share at T for the price K. Clearly, the difference between call and put payoff is

$$(S_T - K)^+ - (K - S_T)^+ = S_T - K.$$

By risk-neutral valuation,

$$C_t - P_t = S_t - Ke^{-r(T-t)}, \tag{1.2}$$

which gives the BS formula for a put option:

$$p(s, \tau) = Ke^{-r\tau}\Phi(-d_2) - s\Phi(-d_1).$$

1.3 The Greeks

The Greeks are sensitivities of option prices, i.e. derivatives w.r.t. the parameters. Some of the most important ones are (with φ the standard normal pdf):

- The BS call delta $c_s = \Phi(d_1) > 0$
- The BS call gamma $c_{ss} = \frac{\varphi(d_1)}{s\sigma\sqrt{\tau}} > 0$
- The BS call theta $c_\tau = \frac{\sigma s}{2\sqrt{\tau}}\varphi(d_1) + Kre^{-r\tau}\Phi(d_2) > 0 \quad$ (if $r \geq 0$)
- The BS vega $c_\sigma = s\varphi(d_1)\sqrt{\tau} > 0$

We know already that the delta is the risky position of a hedge portfolio, i.e. a portfolio in risky and riskless asset that replicates the call.

A large gamma is an indication of a rapidly changing delta, which means in practice that the hedge of a call should be frequently adjusted. Also, at least for small maturity, the term $\log(s/K)$ in the definition of d_1 shows that gamma is largest if $s \approx K$, because of the thin tails of the Gaussian pdf.

Gamma appears also in the tracking error, if an asset evolving with a non-constant volatility is hedged with a BS hedge:

Proposition 1.15. *Assume $r = 0$, and that an asset $\hat{S}$ has dynamics*

$$d\hat{S}_t/\hat{S}_t = \mu dt + \sigma_t dW_t,$$

with some volatility process (σ_t). Consider some claim $h(\hat{S}_T)$. Let $v(s,t)$ be the BS value function arising from the payoff function h, and $V_0 = v(\hat{S}_0, 0)$ its BS price, using a constant volatility σ^{BS}. Then the tracking error of the BS hedge is given by

$$h(\hat{S}_T) - V_T = \frac{1}{2}\int_0^T (\hat{S}_t)^2 v_{ss}(t, \hat{S}_t)(\sigma_t^2 - (\sigma^{\mathrm{BS}})^2)dt.$$

Proof. By the BS PDE, we have $v_t + \frac{1}{2}(\sigma^{\mathrm{BS}})^2 s^2 v_{ss} = 0$, and $v(s,T) = h(s)$. The value process of the hedge satisfies

$$V_T = V_0 + \int_0^T v_s(\hat{S}_u, u)d\hat{S}_u.$$

By Ito's formula, $dv = v_s d\hat{S} + v_t dt + \frac{1}{2}v_{ss}d[\hat{S}]$, and so

$$\begin{aligned} h(\hat{S}_T) &= v(\hat{S}_T, T) \\ &= v(\hat{S}_0, 0) + \int_0^T v_s(\hat{S}_u, u)d\hat{S}_u + \int_0^T \Big(v_t(\hat{S}_u, u) + \frac{1}{2}\sigma_u^2(\hat{S}_u)^2 v_{ss}(\hat{S}_u, u)\Big)du. \end{aligned}$$

Now use the BS PDE. $\square$

Note that the convention ∂_t instead of ∂_τ with $\tau = T - t$ is also in use for defining theta, which changes the sign.

The theta of a *put* may be positive or negative (if $r \geq 0$). The following proof of the monotonicity of the *call* price is not restricted to the BS model.

Proposition 1.16. *For $r \geq 0$, the price of a call option increases w.r.t. time to maturity.*

Proof. The proof is based on the conditional Jensen inequality. For $0 < \tilde{T} < T$, we have

$$\begin{aligned} e^{-rT}\mathbb{E}_{\mathbb{Q}}[(S_T - K)^+] &= e^{-rT}\mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}[(S_T - K)^+ | \mathcal{F}_{\tilde{T}}]] \\ &\geq e^{-rT}\mathbb{E}_{\mathbb{Q}}[(\mathbb{E}_{\mathbb{Q}}[S_T | \mathcal{F}_{\tilde{T}}] - K)^+] \\ &= e^{-r\tilde{T}}\mathbb{E}_{\mathbb{Q}}[(S_{\tilde{T}} - e^{-r(T-\tilde{T})}K)^+] \\ &\geq e^{-r\tilde{T}}\mathbb{E}_{\mathbb{Q}}[(S_{\tilde{T}} - K)^+]. \end{aligned}$$

□

By the put-call parity (1.2), call and put have the same vega. As it is positive, both option prices increase w.r.t. volatility. See also the exercise class for more on the Greeks.

1.4 The Feynman-Kac formula for Brownian motion

Lemma 1.17. *Let W be a BM, $T > 0$ and $h : \mathbb{R} \to \mathbb{R}$ be such that*

$$\int_{-\infty}^{\infty} e^{-ax^2} |h(x)| dx < \infty, \quad \text{for all } a > 0.$$

Then the function

$$u(x,t) := e^{-r(T-t)}\mathbb{E}[h(W_T) | W_t = x]$$

is well defined and continuous on $\mathbb{R} \times [0,T]$, and smooth in the interior. It satisfies the PDE and final condition

$$-u_t = \tfrac{1}{2}u_{xx} - ru, \quad u(x,T) = h(x).$$

Sketch of the proof. The final condition follows from the definition. Moreover,

$$\begin{aligned} u(x,t) &= e^{-r(T-t)}\mathbb{E}[h(W_T - W_t + x) | W_t = x] \\ &= e^{-r(T-t)}\mathbb{E}[h(W_{T-t} + x)] \\ &= e^{-r(T-t)} \int_{-\infty}^{\infty} h(y) \frac{1}{\sqrt{2\pi(T-t)}} \exp\Big(-\frac{(y-x)^2}{2(T-t)}\Big) dy. \end{aligned}$$

Verifying the PDE is a straightforward calculation, applying the standard criterion for differentiation under the integral sign (involving an integrable bound for the derivative). □

The preceding lemma is a one-dimensional version of the Feynman-Kac formula. We will now use it to show that the price function of any European, not path-dependent claim in the BS model satisfies the BS PDE.

Theorem 1.18. *Let $T > 0$ and $g : \mathbb{R}^+ \to \mathbb{R}$ be such that*

$$\int_{-\infty}^{\infty} e^{-ax^2} |g(be^{\sigma x})| dx < \infty, \quad a, b > 0.$$

Then the arbitrage-free price of $X = g(S_T)$ in the BS model is $\pi_t(x) = v(S_t, t)$, where the function v satisfies

$$v_t + \tfrac{1}{2}\sigma^2 s^2 v_{ss} + rsv_s - rv = 0, \quad v(s, T) = g(s).$$

Proof. By the martingale representation theorem, X is replicable (same argument as for the call). The risk-neutral valuation formula and the Markov property of S show that there is a function v such that

$$\pi_t(X) = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}[g(S_T)|\mathcal{F}_t] = v(S_t, t).$$

Define

$$f(x,t) := S_0 e^{\sigma x + (r - \frac{1}{2}\sigma^2)t}$$

and

$$u(x,t) := e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}[g(\underbrace{f(W_T^*, T)}_{S_T})|W_t^* = x].$$

Clearly, we have $u(x,t) = v(f(x,t), t)$. By Lemma 1.17, $-u_t = \frac{1}{2}u_{xx} - ru$. The assertion follows easily from (write $s = f(x,t)$)

$$\begin{aligned} u_t &= v_s(\underbrace{f(x,t)}_{s}, t) \quad \underbrace{f_t(x,t)}_{(r-\frac{1}{2}\sigma^2)f(x,t)} \quad + v_t(s,t), \\ u_x &= v_s(s,t) \underbrace{f_x(x,t)}_{\sigma f(x,t)} \\ u_{xx} &= v_{ss}(s,t) f_x(x,t)^2 + v_s(s,t) \underbrace{f_{xx}(x,t)}_{\sigma^2 f(x,t)}. \end{aligned}$$ □

Depending on the option payoff, further boundary conditions will be needed to make the solution of the PDE unique. E.g., for a call we have the conditions $c(0,\tau)=0$ and $c(s,\tau)\sim s$ as $s\uparrow\infty$. The latter holds for any model of the form $S_t = S_0\exp(X_t)$, where $X_0=0$ and X does not depend on S_0. Indeed,

$$e^{-rT}\mathbb{E}_{\mathbb{Q}}[(S_T-K)^+] = e^{-rT}S_0\mathbb{E}_{\mathbb{Q}}[(e^{X_T}-K/S_0)^+],$$

and the expectation converges to $\mathbb{E}_{\mathbb{Q}}[e^{X_T}]=e^{rT}$ by dominated convergence.

1.5 Implied volatility

The implied volatility is the volatility parameter that makes the BS formula equal to a given call price, i.e. "the wrong number put into the wrong formula to get the correct result".

The quantity S_0/K is often called moneyness of a call, and $\log(S_0/K)$ the log-moneyness. In what follows, the *put* log-moneyness $k:=\log(K/S_0)$ is more convenient. By abbreviating $y=\sigma\sqrt{T}$, the BS call price can be written as

$$c(s,T)=s\Big(\Phi\Big(\frac{-k+rT}{y}+\frac{y}{2}\Big)-e^{k-rT}\Phi\Big(\frac{-k+rT}{y}-\frac{y}{2}\Big)\Big).$$

By positivity of vega, this is a strictly increasing function of y. Its limits are (distinguish cases according to the sign of $-k+rT$)

$$\lim_{y\downarrow 0} c(s,T)=s(1-e^{k-rT})^+,\quad \lim_{y\uparrow\infty} c(s,T)=s.$$

These correspond to the general no-arbitrage bounds

$$S_0(1-e^{k-rT})^+\le e^{-rT}\mathbb{E}_{\mathbb{Q}}[(S_T-K)^+]\le S_0, \tag{1.3}$$

which are based on Jensen's inequality and $(S_T-K)^+\le S_T$. Thus, for any given call price C_{Mkt} in the open interval

$$\Big(S_0(1-e^{k-rT})^+, S_0\Big),$$

we can compute a volatility σ such that the BS call price equals C_{Mkt}.

Remark 1.19. *The lower bound in* (1.3) *need not be strict, but this is of litte importance in practice. The upper bound is strict for $K>0$, because $(S_T-K)^+<S_T$ holds on the event $\{S_T>0\}$.*

Of course, this works not only for observed market prices, but also for call prices computed by other pricing models. This σ is called implied volatility, and is denoted e.g. by σ_{imp}, σ_{i}, σ_{BS}, σ^* or $\hat{\sigma}$.

By the put-call parity,

$$C_{\mathrm{BS}} - P_{\mathrm{BS}} = S_0 - e^{-rT}K,$$
$$C_{\mathrm{Mkt}} - P_{\mathrm{Mkt}} = S_0 - e^{-rT}K.$$

Therefore, put implied vol equals call implied vol (at least in theory).

Implied volatility is often used in practice to quote and compare option prices. Plotting $\sigma_{\mathrm{imp}}(k,T)$ as a function of k for fixed T often yields a "smile" or "skew" shape. Note that "skew" is also often used for the ATM *derivative* of implied vol. The map $(k,T) \mapsto \sigma_{\mathrm{imp}}(k,T)$ is called volatility surface.

The left wing of the smile ($k < 0$) often shows bigger volatilities than the right wing. Possible reasons:

- Asset prices may jump in reality, and downward jumps tend to be larger and more frequent than upside jumps. Downward movements often increase the volatility of asset prices.
- OTM puts are in demand by investors wishing to protect their portfolios against downward movements. Recall that both BS call and put price increase w.r.t. volatility.

To generate an implied volatility surface for all strikes and maturities of interest, quoted prices of liquid options usually need to be interpolated and extrapolated. For this, e.g. the SVI parametrization can be used ("stochastic volatility inspired"; see later):

$$T\sigma^{\mathrm{SVI}}(k)^2 = a + b\Big(\rho(k-m) + \sqrt{(k-m)^2 + \sigma^2}\Big).$$

In practice, delta hedging is sometimes done using the BS delta, with the volatility parameter fixed according to the point of the implied volatility surface corresponding to strike and maturity of the option. When reusing a volatility surface for readjusting a hedge, two rules of thumb are popular:

- Sticky strike: $\sigma_{\mathrm{imp}}(t+\Delta t, T-\Delta t, K) = \sigma_{\mathrm{imp}}(t,T,K)$
- Sticky delta: $\sigma_{\mathrm{imp}}(t+\Delta t, T-\Delta t, K) = \sigma_{\mathrm{imp}}(t,T,\hat{K})$, where $S_t/\hat{K} = S_{t+\Delta t}/K$ (same moneyness)

1.6 Integer constraints and scaling

As usual, we have assumed that assets can be traded in arbitrary quantities, and, in particular, that they can be divided into arbitrary fractions. This is not the case in practice, but is of little concern, by scaling. Suppose that a trading strategy (ϕ^1, ϕ^2) is used for hedging an option. A bank will usually not hedge one single option, but many identical ones, say N. If the risky asset can only be traded in full units – which might also correspond to round lots of 100 individual shares – then the theoretical strategy $(N\phi^1, N\phi^2)$ will be approximated by the rounded strategy $(\lfloor N\phi^1\rceil, N\phi^2)$. Let $V^{\mathbb{Z},N}(\phi)$ denote the corresponding value process. The resulting average hedge error, in L^2 sense, is, by Ito's isometry

$$\begin{aligned}
\frac{1}{N}\mathbb{E}\Big[\big(NV_T(\phi) - V_T^{\mathbb{Z},N}(\phi)\big)^2\Big] &= \frac{1}{N}\mathbb{E}\bigg[\Big(\int_0^T \big(N\phi_u^1 - \lfloor N\phi_u^1\rceil\big)dS_u\Big)^2\bigg] \\
&= \frac{1}{N}\mathbb{E}\bigg[\int_0^T \underbrace{\big(N\phi_u^1 - \lfloor N\phi_u^1\rceil\big)^2}_{\le 1/4} d[S]_u\bigg] \\
&\le \frac{\mathbb{E}\big[[S]_T\big]}{4N} = O\Big(\frac{1}{N}\Big).
\end{aligned}$$

1.7 Doubling strategy

The following example shows that the BS model would admit arbitrage, if the admissibility property in Definition 1.7 was dropped. Time interval $[0,1]$, $r = 0$, $t_n := 1 - 1/2^n$, $n \in \mathbb{N}_0$. For $b > 0$ and $n \in \mathbb{N}_0$, define the hitting time

$$\tau_n(b) := \inf\Big\{t \ge t_n : S_t = S_{t_n} + \frac{1}{b}\Big\}.$$

Fix $b_0 > 0$ (target wealth). We will construct a trading strategy with wealth process $V_0(\phi) = 0$, $V_1(\phi) = b_0$ a.s. Define $p := \mathbb{P}[\tau_0(b_0) \le t_1]$. Since

$$\{\tau_0(b_0) \le t_1\} \supseteq \Big\{S_{t_1} \ge S_0 + \frac{1}{b}\Big\}$$

and S_{t_1} is log-normal, we have $p > 0$. By continuity of paths and dominated convergence,

$$\lim_{b\downarrow 0} \mathbb{P}[\tau_n(b) \le t_{n+1}] = 0, \quad \lim_{b\uparrow\infty} \mathbb{P}[\tau_n(b) \le t_{n+1}] = 1.$$

For $n \in \mathbb{N}$, we can find $b_n > 0$ with $p = \mathbb{P}[\tau_n(b_n) \le t_{n+1}]$. (The required continuity w.r.t. b follows from Lemma 5.2.) By independence of the Brownian increments, the events $\{\tau_n(b_n) \le t_{n+1}\}$ are independent. Thus, for $m \in \mathbb{N}$,

$$\mathbb{P}\Big[\bigcap_{n\in\mathbb{N}_0}\{\tau_n(b_n) > t_{n+1}\}\Big] \le \mathbb{P}\Big[\bigcap_{n=0}^{m}\{\tau_n(b_n) > t_{n+1}\}\Big]$$
$$= \prod_{n=0}^{m} \mathbb{P}[\tau_n(b_n) > t_{n+1}] = (1-p)^{m+1} \to 0, \quad m \to \infty.$$

Therefore, a.s. there is $N = N(\omega)$ with $\tau_N(b_N) \le t_{N+1}$. We now define $v_0 := 0$, and, recursively,

$$v_{n+1} := b_n(b_0 - v_n)(S_{\tau_n(b_n)\wedge t_{n+1}} - S_{t_n}) + v_n, \quad n \in \mathbb{N}_0.$$

The trading strategy is

$$\phi_t^1 := \begin{cases} b_n(b_0 - v_n), & t \in (t_n, \tau_n(b_n) \wedge t_{n+1}], \quad n \in \mathbb{N}_0, \\ 0 & \text{otherwise}, \end{cases}$$

$$\phi_t^2 := \begin{cases} -b_n(b_0 - v_n)S_{t_n} + v_n, & t \in (t_n, \tau_n(b_n) \wedge t_{n+1}], \quad n \in \mathbb{N}_0, \\ 0 & t = 0 \\ b_0 & \text{otherwise}. \end{cases}$$

We will show that $\phi = (\phi^1, \phi^2)$ is a self-financing trading strategy with $V_{t_n}(\phi) = v_n$, $n \in \mathbb{N}$.

Claim 1: If $v_n = b_0$ for some $n \in \mathbb{N}$, then $v_m = b_0$ for $m \ge n$ and $\phi_t^1 = 0$, $\phi_t^2 = b_0$ for $t \ge t_n$. This claim easily follows from the definitions of v_n, ϕ^1, ϕ^2.

Claim 2: If $v_n < b_0$, then it follows from the definition of the strategy that $V_{t_n}(\phi) = v_n$. To prove this claim, consider the cases $\tau_{n-1}(b_{n-1}) < t_n$ and $\tau_{n-1}(b_{n-1}) \ge t_n$, and use the definitions to calculate V_{t_n} and v_n.

Claim 3: If $\tau_n(b_n) \le t_{n+1}$, then $v_{n+1} = b_0$. Indeed, by definition,

$$v_{n+1} = b_n(b_0 - v_n)\underbrace{(S_{\tau_n(b_n)} - S_{t_n})}_{1/b_n} + v_n = b_0.$$

In particular, for N defined above, we have $v_{N+1} = b_0$, and so the assumption of Claim 1 is satisfied if we take $n = N + 1$.

A.s., we trade only at finitely many times, and the strategy is piecewise constant, hence the required existence of integrals holds.

If $m > n$ and $\tau_n(b_n) \le t_{n+1}$, then Claim 1 and Claim 3 yield

$$V_{t_m}(\phi) = \underbrace{\phi_{t_m}^1}_{0} S_{t_m} + \phi_{t_m}^2 = b_0.$$

To show that ϕ is self-financing, it remains to check the case $v_n < b_0$, $\tau_n(b_n) > t_{n+1}$. Then,

$$\begin{aligned} V_{t_{n+1}}(\phi) &= b_n(b_0 - v_n)S_{t_{n+1}} - b_n(b_0 - v_n)S_{t_n} + v_n \\ &= \underbrace{b_n(b_0 - v_n)}_{\phi^1_{t_n}}(S_{t_{n+1}} - S_{t_n}) + \underbrace{v_n}_{=V_{t_n}(\phi)} . \end{aligned}$$

1.8 Bachelier model

Louis Bachelier: Dissertation 1900

$$B_t = e^{rt}, \quad S_t = S_0 + \mu t + \sigma W_t, \quad \mu \in \mathbb{R},\ \sigma > 0$$

Define $S^* := S/B$. By Ito's formula

$$dS^*_t = B_t^{-1}(\mu dt + \sigma dW_t - rS_t dt). \tag{1.4}$$

We look for an equivalent martingale measure, using Girsanov:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\Big(\int_0^T \gamma_t dW_t - \frac{1}{2}\int_0^T \gamma_t^2 dt\Big).$$

If the stochastic exponential is a martingale, then $dW^*_t = dW_t - \gamma_t dt$ defines a $\mathbb{Q}$-Brownian motion. For this, Novikov's condition is sufficient:

$$\mathbb{E}\Big[\exp\Big(\frac{1}{2}\int_0^T \gamma_t^2 dt\Big)\Big] \overset{!}{<} \infty. \tag{1.5}$$

Inserting dW_* into (1.4) gives the candidate process

$$\gamma_t = \frac{rS_t - \mu}{\sigma}.$$

In the literature, it is often assumed that $r = 0$. We assume $r < 1/T$. Clearly,

$$\sup_{0\le t\le T} \gamma_t^2 \le r^2\Big(\sup_{0\le t\le T} |W_t|\Big)^2 + const \sup_{0\le t\le T} |W_t|,$$

and so

$$\frac{1}{2}\int_0^T \gamma_t^2 dt \le \frac{1}{2}r^2 T\Big(\sup_{0\le t\le T} |W_t|\Big)^2 + const \sup_{0\le t\le T} |W_t|.$$

By the reflection principle (see (5.2)),

$$\begin{aligned} \mathbb{P}[\sup_{0\le t\le T} |W_t| \ge x] &= \mathbb{P}[\sup_{0\le t\le T} W_t \ge x] + \mathbb{P}[\inf_{0\le t\le T} W_t \le -x] \\ &= 2\,\mathbb{P}[\sup_{0\le t\le T} W_t \ge x] = 4\,\mathbb{P}[W_T \ge x]. \end{aligned}$$

To verify (1.5), is suffices to note that, since $r < 1/T$,

$$\int_{-\infty}^{\infty} e^{\frac{1}{2}r^2Tx^2+const\ x} e^{-\frac{x^2}{2T}} < \infty.$$

Call price: exercise.

2 Forwards and futures

2.1 Basics

A (long) forward contract is the obligation to buy the underlying at time T for the price K. Its payoff is $S_T - K$. Forwards are not exchange-traded, but OTC (over the counter). Collateral might be required, due to default risk. At inception, K is usually fixed such that the value of the forward contract is zero. This K is called the *forward price*. Since

$$0 = e^{r(T-t)}\mathbb{E}_{\mathbb{Q}}[S_T - K|\mathcal{F}_t],$$

the forward price at time t equals

$$K = e^{r(T-t)}S_t = \mathbb{E}_{\mathbb{Q}}[S_T|\mathcal{F}_t].$$

Futures contracts (deutsch: Terminkontrakt) are traded on exchanges. We write $f_S(t, T^*)$ for the futures price of the asset S with maturity (delivery date) T^*. (Later, we will consider options on futures with maturity $T < T^*$.) At maturity, we have

$$f_S(T^*, T^*) = S_{T^*}.$$

Before maturity, the futures price closely follows the spot price; below, we will rigorously define it, and see that it actually equals the forward price. For a financial future (as opposed to a commodity future), the main content of the contract – besides specifying the underlying instrument – is just the maturity T^*. The holder of a futures contract has a margin account.

- Initial margin: Percentage of the initial futures price (e.g., 5 or 10 percent). Depends mainly on volatility of the underlying.

- Maintenance margin: Dollar/Euro amount; lower than the initial margin amount. If the margin account falls below the maintenance margin, the broker or clearing house sends a margin call to the holder.

At $t + \Delta t$, the margin account for a long future changes by the amount

$$f_S(t + \Delta t, T^*) - f_S(t, T^*).$$

In theory, we let $\Delta t \downarrow 0$ (see below). At time T^*, no transaction is necessary, as this would involve paying $f_S(T^*, T^*) = S_{T^*}$ for the underlying. Gains and losses have been settled previously, by "marking to market". At any time, the futures contract has value zero.

For illustration, suppose that $r = 0$. Then we have:

$$\text{aggregate P\&L of long future} = \underbrace{(\text{futures price at } T^*)}_{S_{T^*}} - (\text{futures price at } 0)$$

Thus, a long future can be viewed as an obligation to buy the underlying at T^*, for the current futures price.

Let r be arbitrary again. We abbreviate $f_t := f_S(t, T^*)$, and the preparatory remarks lead to the mathematical definition of the futures price:

$$B_t \mathbb{E}_{\mathbb{Q}}\left[\int_t^{T^*} \frac{df_u}{B_u} \middle| \mathcal{F}_t \right] \stackrel{!}{=} 0.$$

It can be shown (exercise) that this implies that

$$f_t = f_S(t, T^*) = \mathbb{E}_{\mathbb{Q}}[S_{T^*} | \mathcal{F}_t] = e^{r(T^*-t)} S_t. \tag{2.1}$$

That is, in the BS model the futures price equals the T^*-forward price. This holds, more generally, for models with deterministic interest rates.

We focus on stock futures. Historically, futures arose from mitigating risks such as the following: A farmer sells a futures contract on crops or livestock, as a protection against falling prices. Essentially, this locks in the current price. While protecting against losses, it forfeits participation in rising market prices.

2.2 The Black model

Also known as Black-76. The model assumes that f_t is a geometric BM:

$$df_t = \mu_f f_t dt + \sigma_f f_t dW_t, \quad \mu_f \in \mathbb{R},\ \sigma_f > 0.$$

If we assume that the BS model is used for the underlying, (2.1) implies, by the Ito product rule,

$$df = \underbrace{(\mu - r)}_{\mu_f} f dt + \underbrace{\sigma}_{\sigma_f} f dW$$

Futures trading strategies are defined analogously to Definition 1.2. We write again $\phi_t = (\phi_t^1, \phi_t^2)$, but now ϕ^1 models the number of futures contracts we hold. The important difference is that the value process now is

$$V_t^f(\phi) = \phi_t^2 B_t,$$

as the futures contract has value zero. The self-financing condition is

$$V_t^f(\phi) = V_0^f(\phi) + \int_0^t \phi_u^1 df_u + \int_0^t \phi_u^2 dB_u.$$

Definition 2.1. $\tilde{\mathbb{P}} \sim \mathbb{P}$ *is an EMM, if, for any self-financing strategy,* $\tilde{V}^f(\phi) := V^f(\phi)/B$ *is a local martingale.*

Lemma 2.2. $\tilde{\mathbb{P}}$ *is an EMM* $\Longleftrightarrow$ f *is a local martingale.*

Proof.

$$d\tilde{V}^f(\phi) = B^{-1}(\phi^1 df + \phi^2 \underbrace{dB}_{rBdt}) - rB^{-1} \underbrace{V^f(\phi)}_{\phi^2 B} dt = B^{-1}\phi^1 df. \qquad \square$$

The following proposition is proven, of course, by Girsanov's theorem.

Proposition 2.3. *The unique EMM is given by*

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp\Big(-\frac{\mu_f}{\sigma}W_T - \frac{\mu_f^2}{2\sigma^2}T\Big).$$

$\tilde{W}_t = W_t + (\mu_f/\sigma)t$ *is a* $\tilde{\mathbb{P}}$*-BM, and* $df_t = \sigma f_t d\tilde{W}_t$.

Note that the futures price process is *not* the price process of a traded asset. Admissible strategies are defined by $\tilde{V}^f(\phi)$ being a $\tilde{\mathbb{P}}$-martingale. Define

$$c^f(f,\tau) := e^{-r\tau}(f\Phi(\tilde{d}_1(f,\tau)) - K\Phi(\tilde{d}_2(f,\tau))),$$

where

$$\tilde{d}_{1,2}(f,\tau) := \frac{\log(f/K) \pm \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}.$$

The payoff of a futures call option at time $T < T^*$ is

$$C_T^f = (f_S(T,T^*) - K)^+ = (f_T - K)^+.$$

We proceed to price this call. Note that the dependence on T^* in (2.2) is hidden in the value of f_t.

Theorem 2.4 (Black's formula). *The price of the futures call at time t is*

$$C_t^f = e^{-r(T-t)}(f_t\Phi(\tilde{d}_1(f_t, T-t)) - K\Phi(\tilde{d}_2(f_t, T-t))). \tag{2.2}$$

The replicating futures strategy is given by

$$\phi_t^1 = \frac{\partial c^f}{\partial f}(f_t, T-t), \quad \phi_t^2 = e^{-rt}c^f(f_t, T-t).$$

Proof. Above, we gave two proofs of the BS formula, which can be easily modified (exercise). If only the price is desired, we can argue as follows. The payoff can be written as

$$C_T^f = e^{r(T^*-T)}(S_T - Ke^{-r(T^*-T)})^+.$$

This is a constant multiple of a vanilla call payoff (i.e., a call with underlying S) with strike $\tilde{K} := Ke^{-r(T^*-T)}$. Its price at time t is

$$e^{r(T^*-T)}c_{\mathrm{BS}}(S_t, \tau = T-t, \tilde{K}),$$

and the BS formula easily yields (2.2). □

3 Foreign market derivatives

FX derivatives are very liquid. They are traded both on exchanges and OTC.

Example 3.1 (USD-EUR call option). *Suppose that the USD-EUR rate is $Q_0 = 0.9$. This means that the price of 1 USD is 0.9 EUR. For some fixed strike $K > 0$ and notional 100 USD, the holder of the call option has the right to buy 100 USD at $T > 0$ for $100 \cdot K$ EUR. If the exchange rate at T is $Q_T \leq K$, the option will not be exercised. If $Q_T > K$, then the holder pays $100 \cdot K$ EUR for a USD position worth $100Q_T$ EUR. Thus, the payoff is*

$$100(Q_T - K)^+.$$

Let $W = (W^1, \dots, W^d)$ be a standard d-dimensional BM. We consider deterministic interest rates r_{d} (domestic) and r_{f} (foreign). The bank accounts are

$$B_t^{\mathrm{d}} = e^{r_{\mathrm{d}}t}, \quad B_t^{\mathrm{f}} = e^{r_{\mathrm{f}}t}.$$

We write Q_t for the price of foreign unit of currency, in domestic currency (exchange rate, FX rate). Suppose that $Q_0 > 0$ and (Garman-Kohlhagen model, 1983)

$$dQ_t = Q_t(\mu_Q dt + \sigma_Q dW_t), \tag{3.1}$$

where σ_Q is a d-dimensional deterministic vector with positive components. Explicitly, we have

$$Q_t = Q_0 \exp(\sigma_Q W_t + (\mu_Q - \tfrac{1}{2}|\sigma_Q|^2)t).$$

Note that the one-dimensional Ito formula, applied with exp, shows that the latter satisfies (3.1). We write $\tilde{B}^{\mathrm{f}}_t := B^{\mathrm{f}}_t Q_t$ for the foreign bank account in domestic currency, and

$$Q^*_t := \tilde{B}^{\mathrm{f}}_t / B^{\mathrm{d}}_t = B^{\mathrm{f}}_t Q_t / B^{\mathrm{d}}_t$$

for the value of the foreign bank account in domestic currency, discounted by the domestic bank account.

Definition 3.2. $\mathbb{P}^* \sim \mathbb{P}$ *is a domestic EMM, if* (Q^*_t) *is a* $\mathbb{P}^*$*-martingale.*

For a trading strategy $\phi = (\phi^1, \phi^2)$, the value process (in domestic currency) is

$$V_t(\phi) := \phi^1_t \tilde{B}^{\mathrm{f}}_t + \phi^2_t B^{\mathrm{d}}_t.$$

The strategy is self-financing, if

$$dV_t(\phi) = \phi^1_t d\tilde{B}^{\mathrm{f}}_t + \phi^2_t dB^{\mathrm{d}}_t.$$

Lemma 3.3. *The discounted value process* $V^*_t(\phi) := (B^{\mathrm{d}}_t)^{-1} V_t(\phi)$ *of a self-financing strategy is a* $\mathbb{P}^*$*-martingale.*

Proof.

$$\begin{aligned}
dV^* &= V d(B^{\mathrm{d}})^{-1} + (B^{\mathrm{d}})^{-1} dV \\
&= (\phi^1 \tilde{B}^{\mathrm{f}} + \phi^2 B^{\mathrm{d}})(-r_{\mathrm{d}}(B^{\mathrm{d}})^{-1})dt + (B^{\mathrm{d}})^{-1}(\phi^1 d\tilde{B}^{\mathrm{f}} + \phi^2 \underbrace{dB^{\mathrm{d}}}_{r_{\mathrm{d}} B^{\mathrm{d}} dt}) \\
&= \phi^1(-r_{\mathrm{d}} \tilde{B}^{\mathrm{f}} (B^{\mathrm{d}})^{-1} dt + (B^{\mathrm{d}})^{-1} d\tilde{B}^{\mathrm{f}} \\
&= \phi^1 d(\tilde{B}^{\mathrm{f}} / B^{\mathrm{d}}) = \phi^1 d(B^{\mathrm{f}} Q_t / B^{\mathrm{d}}) = \phi^1 dQ^*.
\end{aligned}$$

□

By the definition of Q^*_t, it satisfies

$$dQ^* = Q^*((\mu_Q + r_{\mathrm{f}} - r_{\mathrm{d}})dt + \sigma_Q dW).$$

If we look for a measure change of the form

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \mathcal{E}(\int_0^{\cdot} \lambda_s dW_s)_T = \exp(\int_0^T \lambda_s dW_s - \tfrac{1}{2} \int_0^T |\lambda_s|^2 ds),$$

the process λ should satisfy

$$\mu_Q + r_{\mathrm{f}} - r_{\mathrm{d}} + \sigma_Q \lambda = 0.$$

Thus, in general we have infinitely many EMMs. General principle for a market driven by d factors (BMs):

- If there are at most $d+1$ traded assets, the market is free of arbitrage
- If there are at least $d+1$ traded assets, the market is complete.

For example, here we could consider a two-dimensional BM, and a complete, arbitrage-free market consisting of the two bank accounts and a domestic stock (see Chapter 4 in [8]). To handle the basic example of an FX option, we assume $d = 1$, i.e. we trade only in the two bank accounts. Note that the FX rate Q_t is not a traded asset.

For a claim X, in domestic currency, the risk-neutral valuation formula gives the price

$$\pi_t(X) = e^{-r_{\mathrm{d}}(T-t)}\mathbb{E}_{\mathbb{P}^*}[X|\mathcal{F}_t].$$

We can now price the FX call from Example 3.1. Define

$$h_{1,2}(q,\tau) := \frac{\log(q/K) + (r_{\mathrm{d}} - r_{\mathrm{f}} \pm \frac{1}{2}\sigma_Q^2)\tau}{\sigma_Q\sqrt{\tau}}$$

Theorem 3.4. *Recall that we assume $d = 1$ (one-dimensional BM). Then the price at time t of the FX call (currency call) with payoff $(Q_T - K)^+$ is given by*

$$Q_t e^{-r_{\mathrm{f}}(T-t)}\Phi(h_1(Q_t, T-t)) - Ke^{-r_{\mathrm{d}}(T-t)}\Phi(h_2(Q_t, T-t)).$$

Proof. Since $Q^* = B^{\mathrm{f}}Q/B^{\mathrm{d}}$ and $\tilde{B}^{\mathrm{f}} = B^{\mathrm{f}}Q$,

$$dQ = Q((r_{\mathrm{d}} - r_{\mathrm{f}})dt + \sigma_Q dW^*),$$

and

$$d\tilde{B}^{\mathrm{f}} = \tilde{B}^{\mathrm{f}}(r_{\mathrm{d}}dt + \sigma_Q dW^*).$$

By risk-neutral valuation, the price is

$$\begin{aligned} e^{-r_{\mathrm{d}}(T-t)}\mathbb{E}^*[(Q_T - K)^+|\mathcal{F}_t] &= e^{-r_{\mathrm{f}}T}e^{-r_{\mathrm{d}}(T-t)}\mathbb{E}^*[(\tilde{B}^{\mathrm{f}}_T - e^{r_{\mathrm{f}}T}K)^+|\mathcal{F}_t] \\ &= e^{-r_{\mathrm{f}}T}c(\tilde{B}^{\mathrm{f}}_t, T-t, \underbrace{e^{r_{\mathrm{f}}T}K}_{\text{strike}}, \underbrace{r_{\mathrm{d}}}_{\text{interest rate}}, \underbrace{\sigma_Q}_{\text{vola}}), \end{aligned}$$

where the last line follows from the standard BS formula. □

4 American options

4.1 Arbitrage-free price

American options can be exercised at any time before or at the expiry date. In practice, options with a single stock as underlying are usually American. We assume that the underlying follows the BS model.

Definition 4.1. *A consumption process is a progressively measurable, increasing process* $(A_t)_{0\le t\le T}$ *with* $A_0 = 0$. *A self-financing trading strategy with consumption is a pair* (ϕ, A) *such that* A *is a consumption process,* $\phi = (\phi^1, \phi^2)$ *is adapted, and the wealth process* $V_t(\phi, A) := \phi^1_t S_t + \phi^2_t B_t$ *satisfies*

$$V_t(\phi, A) = V_0(\phi, A) + \int_0^t \phi^1_u dS_u + \int_0^t \phi^2_u dB_u - A_t.$$

By applying the product rule to $V^* = B^{-1}V$, it is easy to see that

$$V_t = B_t\Big(V_0 - \int_0^t B_u^{-1} dA_u + \int_0^t \sigma\zeta_u B_u^{-1} dW^*_u\Big),$$

where $\zeta_t := \phi^1_t S_t$ is the value of the risky position of the portfolio. Clearly,

$$V^*_t + \int_0^t B_u^{-1} dA_u = V_0 + \int_0^t \sigma\zeta_u B_u^{-1} dW^*_u$$

is a local $\mathbb{Q}$-martingale. It is a $\mathbb{Q}$-martingale, if the following holds:

Definition 4.2. *A self-financing trading strategy with consumption* (ϕ, A) *is admissible, if*

$$\mathbb{E}_{\mathbb{Q}}\Big[\int_0^T \zeta_u^2 du\Big] < \infty.$$

We assume that our American claim is defined by a payoff function $g : \mathbb{R}^+ \times [0, T] \to \mathbb{R}$ such that $|g(s,t)| \le K_1 s + K_2$ for some constants K_1, K_2. For example, $g(s,t) = (K - s)^+$ yields the American put, with payoff $(K - S_t)^+$ when exercised at time t. We will see that the price process is given by

$$V_t := \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[t,T]}} \mathbb{E}_{\mathbb{Q}}[e^{-r(\tau-t)} g(S_\tau, \tau)|\mathcal{F}_t]. \tag{4.1}$$

Lemma 4.3. *There is an admissible strategy with consumption* (ϕ, A) *such that* $V(\phi, A) = V$.

Proof. Define $U_t := e^{-rt}V_t$. Thus,

$$U_t = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[t,T]}} \mathbb{E}_{\mathbb{Q}}[e^{-r\tau} g(S_\tau, \tau)|\mathcal{F}_t]$$

is the Snell envelope of the discounted payoff process $e^{-rt}g(S_t, t)$. By applying Doob's L^p inequality for $p = 2$, and $L^2 \subset L^1$, we find an integrable r.v. Y such that

$$e^{-rt}g(S_t, t) \le Y, \quad 0 \le t \le T.$$

Thus we have

$$\sup_{\tau \in \mathcal{T}_{[0,T]}} \mathbb{E}_{\mathbb{Q}}[e^{-r\tau} g(S_\tau, \tau) \mathbf{1}_{\{e^{-r\tau} g(S_\tau,\tau) \geq x\}}] \leq \mathbb{E}_{\mathbb{Q}}[Y \mathbf{1}_{\{Y \geq x\}}] \overset{x \to \infty}{\to} 0,$$

and so $\{e^{-r\tau} g(S_\tau, \tau) : \tau \in \mathcal{T}_{[0,T]}\}$ is uniformly integrable. It can be shown that $\{U_\tau : \tau \in \mathcal{T}_{[0,T]}\}$ is uniformly integrable as well. By the Doob-Meyer decomposition, we have $U = M - H$, where M is a square-integrable martingale, and H is a continuous increasing process with $H_0 = 0$. By the martingale representation theorem, there is a progressively measurable process ξ with $M_t = M_0 + \int_0^t \xi_u dW_u^*$. Define

$$\phi_t^1 := B_t \xi_t \sigma^{-1} S_t^{-1}, \quad \phi_t^2 := U_t - \xi_t \sigma^{-1}, \quad A_t := \int_0^t e^{ru} dH_u.$$

Clearly, we have

$$V_t(\phi, A) = B_t \xi_t \sigma^{-1} + (U_t - \xi_t \sigma^{-1}) B_t = U_t B_t = V_t.$$

The strategy is self-financing, since

$$\begin{aligned} \phi^1 dS + \phi^2 dB - dA &= B_t \xi_t \sigma^{-1} S_t^{-1} \underbrace{dS_t}_{rS_t dt + \sigma S_t dW_t^*} + (U_t - \xi_t \sigma^{-1}) dB_t - B_t dH_t \\ &= b \underbrace{\xi dW^*}_{dM = dU + dH} + U dB - B dH \\ &= B dU + U dB = d(BU). \end{aligned}$$

□

It is intuitively clear, and true, that the optimizer in the definition of U_0 is given by

$$\tau_0 := \inf\{u \in [0, T] : U_u = e^{-ru} g(S_u, u)\},$$

and that

$$V_{\tau_0} = g(S_{\tau_0}, \tau_0).$$

Lemma 4.4. *The stopped process $U^{\tau_0} := (U_{t \wedge \tau_0})_{0 \leq t \leq T}$ is a martingale. In particular, the consumption process A found in Lemma 4.3 vanishes on* $[0, \tau_0]$.

Proof. Write $Z_t^* := e^{-rt} g(S_t, t)$ for the discounted payoff, and

$$\tau_t := \inf\{u \in [t, T] : U_u = Z_u^*\}$$

for the optimizer in the definition of U_t. Note that

$$\tau_{t \wedge \tau_0} = \begin{cases} \tau_t = \tau_0, & t \leq \tau_0 \\ \tau_0, & t \geq \tau_0 \end{cases} = \tau_0,$$

and so, by definition of τ_t,

$$U_{t\wedge\tau_0} = \mathbb{E}_{\mathbb{Q}}[Z^*_{t\wedge\tau_0}|\mathcal{F}_{t\wedge\tau_0}] = \mathbb{E}_{\mathbb{Q}}[Z^*_{\tau_0}|\mathcal{F}_{t\wedge\tau_0}].$$

Let $s \le t$. On $\{s \le \tau_0\}$, we have

$$\mathbb{E}_{\mathbb{Q}}[U_{t\wedge\tau_0}|\mathcal{F}_s] = \mathbb{E}_{\mathbb{Q}}[Z^*_{\tau_0}|\mathcal{F}_s] = \mathbb{E}_{\mathbb{Q}}[Z^*_{\tau_0}|\mathcal{F}_{s\wedge\tau_0}] = U_{s\wedge\tau_0},$$

and on $\{s \ge \tau_0\}$, we have

$$U_{t\wedge\tau_0} = U_{\tau_0} = U_{s\wedge\tau_0}. \qquad \square$$

Definition 4.5. *A semi-static trading strategy for the American claim consists of:*

- $c \in \mathbb{R}$ *(the number of American claims bought at time zero. We will only need* $c = \pm 1$.*)*
- $\tau \in \mathcal{T}_{[0,T]}$ *(exercise time)*
- $(\phi_t, A_t)_{0\le t<\tau}$ *(an admissible trading strategy with consumption, possibly a specialization of some strategy* $(\phi_t, A_t)_{0\le t\le T}$*)*

The value of such a strategy, depending on a price $\pi \in \mathbb{R}$ *of the American claim at time zero, is*

- *For* $t = 0$*:* $\phi^1_0 S_0 + \phi^2_0 + c\pi$
- *For* $t \in [\tau, T]$*:* $(\phi^2_\tau + \phi^1_\tau S_\tau B_\tau^{-1} + cg(S_\tau, \tau)B_\tau^{-1})B_t$

As we do not model dynamic trading in the claim, the value for $t \in (0, \tau)$ is irrelevant. The interpretation of the definition is as follows: At time zero, c American claims are bought. For $t \in [0, \tau)$, we invest and consume according to (ϕ, A), and keep the position of c American claims. At τ, the American claims are exercised and the risky position is closed. The riskless position is adjusted accordingly, and no transactions happen for $t \in (\tau, T]$.

We now argue that the arbitrage-free price of the American claim at t is given by (4.1). By the Markov property, it suffices to consider $t = 0$. We actually prove that any price $\pi \ne V_0$ leads to semi-static arbitrage. Thus, if we *assume* that an arbitrage-free price exists, it must be V_0. Proving that dynamic trading in the American claim yields no arbitrage is more involved; see Section 11.2.2 in [4].

Theorem 4.6. *Suppose that the American claim is traded at* $\pi \ne V_0$*. Then there exists a semi-static arbitrage strategy.*

Proof. The proof is based on the following observation: Since U dominates the discounted payoff Z^*, we have

$$V_\tau \geq g(S_\tau, \tau)$$

for any stopping time τ. By Lemma 4.3, we can replicate V, and the value of this replication strategy at time τ is

$$\phi^1_\tau S_\tau + \phi^2_\tau B_\tau = V_\tau \geq g(S_\tau, \tau).$$

Assume $\pi > V_0$. Put $c = -1$ (sell one claim), let τ be an arbitrary stopping time (the exercise time of the buyer), and let (ϕ, A) replicate V. The value of this semi-static strategy at $t = 0$ is

$$\phi^1_0 S_0 + \phi^2_0 - \pi = V_0 - \pi < 0.$$

The value for $t \in [\tau, T]$ is

$$(\phi^2_\tau + \phi^1_\tau S_\tau B_\tau^{-1} - g(S_\tau, \tau) B_\tau^{-1}) B_t = (\phi^2_\tau B_\tau + \phi^1_\tau S_\tau - g(S_\tau, \tau)) B_t / B_\tau \geq 0.$$

Now assume $\pi < V_0$. Put $c = 1$ (buy one claim), let $\tau = \tau_0$ be optimal, and let (ϕ, A) replicate V. The trading strategy we choose is $(-\phi, 0)$, which replicates $-V$ on $[0, \tau_0]$, by Lemma 4.4. The value of this semi-static strategy at $t = 0$ is

$$-\phi^1_0 S_0 - \phi^2_0 + \pi = -V_0 + \pi < 0.$$

The value for $t \in [\tau_0, T]$ is

$$(\phi^2_{\tau_0} + \phi^1_{\tau_0} S_{\tau_0} B_{\tau_0}^{-1} - g(S_{\tau_0}, \tau_0) B_{\tau_0}^{-1}) B_t = (\underbrace{\phi^2_{\tau_0} B_{\tau_0} + \phi^1_{\tau_0} S_{\tau_0}}_{= V_{\tau_0}} - g(S_{\tau_0}, \tau_0)) B_t / B_{\tau_0} = 0.$$

□

4.2 Calls and puts

The discounted payoff of a call in the BS model is

$$Z^*_t = (S^*_t - B_t^{-1} K)^+ = (B_t^{-1} S_t - B_t^{-1} K)^+.$$

Lemma 4.7. *For $r \geq 0$, the discounted call payoff Z^*_t is a $\mathbb{Q}$-submartingale.*

Proof. We have $Z^*_t = h(S^*_t, t)$, where $h(s, t) := (s - B_t^{-1} K)^+$ is a convex function of s. Using the conditional Jensen inequality, we calculate for $u \geq t$

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[Z^*_u | \mathcal{F}_t] &= \mathbb{E}_{\mathbb{Q}}[h(S^*_u, u) | \mathcal{F}_t] \\ &\geq h(\mathbb{E}_{\mathbb{Q}}[S^*_u | \mathcal{F}_t], u) \\ &= h(S^*_t, u) \geq h(S^*_t, t) = Z^*_t. \end{aligned}$$

Note that the last inequality uses that $r \geq 0$. □

Analogously, for $r \leq 0$ the discounted put payoff is a submartingale.

Theorem 4.8. *For $r \geq 0$, it is optimal to exercise the American call at maturity T. Its price process equals the European price process.*

Proof. By considering the exercise time $\tau = T$, it is clear that the American price is bounded below by the European price. For a stopping time $\tau \in \mathcal{T}_{[0,T]}$, recall that $\mathcal{F}_\tau$ is defined by

$$\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t, \ \forall t \geq 0\}.$$

For $\tau_1 \leq \tau_2$, we have $\mathcal{F}_{\tau_1} \subseteq \mathcal{F}_{\tau_2}$. By the preceding lemma and the optional sampling theorem, we have $\mathbb{E}_\mathbb{Q}[Z_T^*|\mathcal{F}_\tau] \geq Z_\tau^*$. Thus, the price process satisfies

$$\begin{aligned}\operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{[t,T]}} \mathbb{E}_\mathbb{Q}[e^{rt} Z_\tau^* | \mathcal{F}_t] &\leq \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{[t,T]}} \mathbb{E}_\mathbb{Q}[e^{rt} \mathbb{E}_\mathbb{Q}[Z_T^*|\mathcal{F}_\tau]|\mathcal{F}_t] \\ &= e^{rt} \mathbb{E}_\mathbb{Q}[Z_T^*|\mathcal{F}_t]. \qquad \square\end{aligned}$$

The American put with time to expiry $T - t$ has price function

$$P^{\mathrm{a}}(s, T-t) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{[t,T]}} \mathbb{E}_\mathbb{Q}[e^{-r(\tau - t)}(K - S_\tau)^+ | S_t = s].$$

We can decompose $\mathbb{R}^+ \times [0, T]$ into the continuation region

$$\mathcal{C} := \{(s,t) : P^{\mathrm{a}}(s, T-t) > (K-s)^+\}$$

and the stopping region

$$\mathcal{D} := \{(s,t) : P^{\mathrm{a}}(s, T-t) = (K-s)^+\}.$$

They are separated by the critical stock price

$$b^*(T-t) := \sup\{s \geq 0 : P^{\mathrm{a}}(s, T-t) = (K-s)^+\}.$$

The function $c^*(t) := b^*(T-t)$ is smooth, increasing and satisfies $\lim_{t \uparrow T} c^*(t) = K$. More precisely, it can be shown that

$$K - c^*(t) \sim K\sigma \sqrt{(T-t) \log \frac{1}{T-t}}, \quad t \uparrow T.$$

As for c^* being increasing, an intuitive argument can be based on the fact that the price of any American claim increases w.r.t. maturity, which is obvious from the optimal stopping representation.

The following theorem, stated without proof, characterizes the put price as solution of a free boundary problem. We use the standard notation $\tau = T - t$ for the time to expiry, not to be confused with the stopping times considered above.

Theorem 4.9. *P^{a} is a smooth function on $\mathcal{C}$, and satisfies the BS PDE*

$$P^{\mathrm{a}}_\tau(s,\tau) = \tfrac{1}{2}\sigma^2 s^2 P^{\mathrm{a}}_{ss}(s,\tau) + rsP^{\mathrm{a}}_s(s,\tau) - rP^{\mathrm{a}}(s,\tau)$$

on $\mathcal{C}$, as well as

$$-1 \le P^{\mathrm{a}}_s(s,\tau) \le 0.$$

Moreover, we have

$$\begin{aligned}
\lim_{\tau\downarrow 0} P^{\mathrm{a}}(s,\tau) &= (K-s)^+, \\
\lim_{s\downarrow b^*(\tau)} P^{\mathrm{a}}(s,\tau) &= K - b^*(\tau), \\
\lim_{s\uparrow\infty} P^{\mathrm{a}}(s,\tau) &= 0, \qquad (4.2)\\
P^{\mathrm{a}}(s,\tau) &\ge (K-s)^+, \\
\lim_{s\downarrow b^*(\tau)} P^{\mathrm{a}}_s(s,\tau) &= -1. \qquad (4.3)
\end{aligned}$$

The equality (4.3) is called the smooth fit principle. Note that, on $\mathcal{D}$, we have $P^{\mathrm{a}}(s,\tau) = K - s$. Under appropriate growth conditions, the free boundary problem has a unique solution (P^{a}, b^*), and P^{a}_s defines a hedging strategy. We refer to [9] for details.

An explicit solution is available for the perpetual American put ($T=\infty$). The critical stock price becomes a constant $b_\infty \le K$, to be determined. The price $v(s) := P^{\mathrm{a}}(s,\infty)$ satisfies the ODE

$$0 = \tfrac{1}{2}\sigma^2 s^2 v''(s) + rsv'(s) - rv(s),$$

which has the general solution $c_1 s^{-2r/\sigma^2} + c_2 s$. By (4.2), we have $c_2 = 0$. From $v(b_\infty) = K - b_\infty$, we conclude

$$v(s) = (K - b_\infty)\Big(\frac{s}{b_\infty}\Big)^{-\frac{2r}{\sigma^2}}.$$

Since

$$v'(s) = -(K-b_\infty)\frac{2r}{\sigma^2}\Big(\frac{s}{b_\infty}\Big)^{-\frac{2r}{\sigma^2}-1}\frac{1}{b_\infty},$$

the smooth fit condition (4.3) implies

$$b_\infty = \frac{K}{1+\frac{\sigma^2}{2r}}.$$

5 Exotic options

5.1 Forward start options

Fix $0 < T_0 < T$. The holder of a forward start call receives at T_0 a call with strike KS_{T_0}, where $K > 0$ is fixed. Thus, its payoff at maturity is

$$FS_T = (S_T - KS_{T_0})^+.$$

In the BS model, the value at T_0 is

$$\begin{aligned}FS_{T_0} &= C(S_{T_0}, T - T_0, KS_{T_0}) = C(s, T - T_0, Ks)|_{s=S_{T_0}} \\ &= e^{-r(T-T_0)}\mathbb{E}_{\mathbb{Q}}[(se^{\sigma W^*_{T-T_0}+(r-\frac{1}{2}\sigma^2)(T-T_0)} - Ks)^+]|_{s=S_{T_0}} \\ &= S_{T_0}C(1, T - T_0, K).\end{aligned}$$

Therefore, the price of the foward start call is

$$\begin{aligned}FS_0 &= e^{-rT_0}\mathbb{E}_{\mathbb{Q}}[S_{T_0}]C(1, T - T_0, K) \\ &= S_0C(1, T - T_0, K) = C(S_0, T - T_0, S_0K).\end{aligned}$$

5.2 Barrier options

If W is a BM and τ a stopping time, then

$$\tilde{W} = (W_{t+\tau} - W_\tau)_{t\geq 0}$$

is a BM w.r.t. its natural filtration, which is independent of $\mathcal{F}_\tau$. We write

$$M_t^W := \max_{0\leq u\leq t} W_u, \qquad m_t^W := \min_{0\leq u\leq t} W_u$$

for the running maximum resp. minimum of W.

Lemma 5.1 (Reflection principle)**.** *Let $t > 0$, $y > 0$, $x \leq y$. Then we have*

$$\mathbb{P}[W_t \leq x, M_t^W \geq y] = \mathbb{P}[W_t \geq 2y - x].$$

Proof. Define the hitting time

$$\tau_y := \inf\{t > 0 : W_t = y\}.$$

We define $\tilde{W}$ as above, using τ_y, and note that $\hat{W} := -\tilde{W}$ is a BM as well.

$$\begin{aligned}
\mathbb{P}[W_t \le x, M_t^W \ge y] &= \mathbb{P}[\tau_y \le t, W_{\tau_y + t - \tau_y} - \underbrace{W_{\tau_y}}_{=y} \le x - y] \\
&= \mathbb{P}[\tau_y \le t, \tilde{W}_{t-\tau_y} \le x - y] \\
&= \mathbb{P}[\tau_y \le t, \hat{W}_{t-\tau_y} \ge y - x] \\
&= \mathbb{P}[\tau_y \le t, W_{\tau_y + t - \tau_y} - \underbrace{W_{\tau_y}}_{=y} \ge y - x] \\
&= \mathbb{P}[\tau_y \le t, W_t \ge 2y - x] = \mathbb{P}[W_t \ge 2y - x]. \qquad \square
\end{aligned}$$

We now consider the max and min of BM with drift.

Lemma 5.2. *Let $X_t = \nu t + \sigma W_t$, where $\sigma > 0$ and $\nu \in \mathbb{R}$. Then, for $x, s > 0$ we have*

$$\mathbb{P}[\max_{0 \le u \le s} X_u \le x] = \Phi\Big(\frac{x - \nu s}{\sigma\sqrt{s}}\Big) - e^{2\nu x/\sigma^2}\Phi\Big(\frac{-x - \nu s}{\sigma\sqrt{s}}\Big), \tag{5.1}$$

and for $x < 0$ and $s > 0$ we have

$$\mathbb{P}[\min_{0 \le u \le s} X_u \le x] = \Phi\Big(\frac{-x + \nu s}{\sigma\sqrt{s}}\Big) - e^{2\nu x/\sigma^2}\Phi\Big(\frac{x + \nu s}{\sigma\sqrt{s}}\Big).$$

In particular, for $\sigma = 1$ and $\nu = 0$ we obtain

$$\begin{aligned}
\mathbb{P}[\max_{0 \le u \le s} W_u \ge x] &= 1 - \Phi(x/\sqrt{s}) + \Phi(-x/\sqrt{s}) \\
&= 2(1 - \Phi(x/\sqrt{s})) = 2\mathbb{P}[W_s \ge x].
\end{aligned} \tag{5.2}$$

Proof of Lemma 5.2. It suffices to consider (5.1), as

$$\mathbb{P}[\min_{0 \le u \le s}(\nu u + \sigma W_u) \le x] = \mathbb{P}[\max_{0 \le u \le s}(-\nu u + \sigma W_u) \ge -x].$$

Moreover, we may assume $\sigma = 1$, since

$$\mathbb{P}[\max_{0 \le u \le s}(\nu u + \sigma W_u) \le x] = \mathbb{P}[\max_{0 \le u \le s}(\nu\sigma^{-1}u + W_u) \le x\sigma^{-1}].$$

Define $\mathbb{P}^*$ by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = e^{-\nu W_s - \frac{1}{2}\nu^2 s}.$$

By Girsanov, $W^* = X$ is a $\mathbb{P}^*$-BM. With $\tau_x := \inf\{t \ge 0 : W_t^* = x\}$,

$$\tilde{W}_t := W_t^* \mathbf{1}_{\{\tau_x \ge t\}} + (2x - W_t^*)\mathbf{1}_{\{\tau_x < t\}}$$

is a BM. We have

$$\{\max_{0\le u\le s} \tilde{W}_s \ge x, \tilde{W}_s \le x\} = \{W_s^* \ge x\}.$$

This implies the third equality in

$$\begin{aligned}
\mathbb{P}[\max_{0\le u\le s} X_s > x, X_s \le x] &= \mathbb{E}^*[e^{\nu W_s^* - \frac{1}{2}\nu^2 s}\mathbf{1}_{\{\max_{0\le u\le s} W_u^* > x, W_s^* \le x\}}] \\
&= \mathbb{E}^*[e^{\nu \tilde{W}_s - \frac{1}{2}\nu^2 s}\mathbf{1}_{\{\max_{0\le u\le s} \tilde{W}_u > x, \tilde{W}_s \le x\}}] \\
&= \mathbb{E}^*[e^{\nu(2x - W_s^*) - \frac{1}{2}\nu^2 s}\mathbf{1}_{\{W_s^* \ge x\}}] \\
&= e^{2\nu x}\mathbb{E}^*[e^{\nu W_s^* - \frac{1}{2}\nu^2 s}\mathbf{1}_{\{W_s^* \le -x\}}] = e^{2\nu x}\mathbb{P}[W_s^* \le -x].
\end{aligned}$$

The claim follows from

$$\mathbb{P}[\max_{0\le u\le s} X_s \le x] = \mathbb{P}[X_s \le x] - \mathbb{P}[\max_{0\le u\le s} X_s > x, X_s \le x]. \qquad \square$$

Define

$$M_s^X := \max_{0\le u\le s} X_u, \qquad m_s^X := \min_{0\le u\le s} X_u.$$

Lemma 5.3. *Let $s > 0$, $y \ge 0$, $x \le y$. Then*

$$\mathbb{P}[X_s \le x, M_s^X \ge y] = e^{2\nu y/\sigma^2}\mathbb{P}[X_s \ge 2y - x + 2\nu s].$$

If $x \le y$ and $y \ge 0$, then

$$\mathbb{P}[X_s \le x, M_s^X \le y] = \Phi\Big(\frac{x - \nu s}{\sigma\sqrt{s}}\Big) - e^{2\nu y/\sigma^2}\Phi\Big(\frac{x - 2y - \nu s}{\sigma\sqrt{s}}\Big) \tag{5.3}$$

Proof. We omit the proof of the first assertion, which is very similar to the previous lemma. The second one follows from the first one and

$$\mathbb{P}[X_s \le x, M_s^X \le y] + \mathbb{P}[X_s \le x, M_s^X > y] = \mathbb{P}[X_s \le x]. \qquad \square$$

Lemma 5.4. *Let $y \le 0$ and $y \le x$. Then*

$$\mathbb{P}[X_s \ge x, m_s^X \ge y] = \Phi\Big(\frac{-x + \nu s}{\sigma\sqrt{s}}\Big) - e^{2\nu y/\sigma^2}\Phi\Big(\frac{2y - x + \nu s}{\sigma\sqrt{s}}\Big) \tag{5.4}$$

and

$$\mathbb{P}[m_s^X \ge y] = \Phi\Big(\frac{-y + \nu s}{\sigma\sqrt{s}}\Big) - e^{2\nu y/\sigma^2}\Phi\Big(\frac{y + \nu s}{\sigma\sqrt{s}}\Big).$$

Proof. The first assertion follows easily from (5.3), and the second one by putting $x = y$. $\square$

We now apply these results to FX barrier options. Recall that Q_t denotes the FX rate, and $\mathbb{P}^*$ the domestic martingale measure. Among eight standard types of payoff, we focus on the down-and-out barrier call. Its payoff is

$$(Q_T - K)^+ \mathbf{1}_{\{\min_{0\le t\le T} Q_t > H\}},$$

where $H, K > 0$. As in Section 3, we assume that

$$Q_t = Q_0 e^{\sigma W_t^* + \lambda t} =: Q_0 e^{X_t},$$

where $\lambda = r_\mathrm{d} - r_\mathrm{f} - \frac{1}{2}\sigma^2$.

Case 1: $H < K$. With

$$D := \{X_T \ge \log \frac{K}{Q_0}, \min X_t \ge \log \frac{H}{Q_0}\},$$

the payoff becomes (a.s. equal to)

$$Q_0 e^{X_T} \mathbf{1}_D - K \mathbf{1}_D.$$

The price is

$$Q_0 e^{-r_\mathrm{d} T} \mathbb{E}^*[e^{X_T} \mathbf{1}_D] - K e^{-r_\mathrm{d} T} \underbrace{\mathbb{P}^*[D]}_{\text{use (5.4)}}.$$

We calculate the expectation with Girsanov's theorem. Define

$$\frac{d\bar{\mathbb{P}}}{d\mathbb{P}^*} = e^{\sigma W_T^* - \frac{1}{2}\sigma^2 T},$$

so that $\bar{W}_t := W_t^* - \sigma t$ is a $\bar{\mathbb{P}}$-BM, and

$$X_t = \sigma \bar{W}_t + (r_\mathrm{d} - r_\mathrm{f} + \frac{1}{2}\sigma^2)t.$$

Thus,

$$\mathbb{E}^*[e^{X_T} \mathbf{1}_D] = e^{(r_\mathrm{d} - r_\mathrm{f})T} \bar{\mathbb{P}}[D]$$

can be expressed using (5.4).

Case 2: $H \ge K$. If the barrier is not hit, then the call matures ITM. Thus, its payoff is $(Q_T - K)\mathbf{1}_D$, with

$$D := \{m_T^X \ge \log \frac{H}{Q_0}\}.$$

Its price is computed analogosly to case 1, using Lemma 5.2.

5.3 Asian options

Asian options are options on some average value of the underlying. E.g., a call on the arithmetic average has payoff

$$C_T^{\mathrm{A}} = \left(\frac{1}{T - T_0} \int_{T_0}^{T} S_u du - K \right)^+ .$$

It is more robust w.r.t. market manipulations than a vanilla call. The distribution of the integral of geometric BM is not availabe in closed form, and so Asian options are tricky to price even in the BS model.

An easy special case occurs if it is known at time t that $\frac{1}{T-T_0} \int_{T_0}^{t} S_u du \geq K$, because then the option will be ITM at maturity, and thus its value is

$$e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \Big[\frac{1}{T - T_0} \int_{T_0}^{T} S_u du - K \Big| \mathcal{F}_t \Big]. \tag{5.5}$$

Since

$$e^{rT} S_T^* - e^{rt} S_t^* = \int_t^T d(e^{ru} S_u^*) = \int_t^T r S_u du + \int_t^T e^{ru} dS_u^*,$$

we have

$$\int_t^T \mathbb{E}_{\mathbb{Q}}[S_u | \mathcal{F}_t] du = \frac{1}{r} S_t (e^{r(T-t)} - 1),$$

and thus (5.5) becomes

$$\frac{S_t(1 - e^{-r(T-t)})}{r(T - T_0)} + \frac{e^{-r(T-t)}}{T - T_0} \int_{T_0}^{t} S_u du - e^{-r(T-t)} K.$$

In general, Asian options are often priced by Monte Carlo. In the BS model, see [8], Proposition 6.8.1 for an expression as a double integral.

A geometric Asian option is based on the mean

$$\Big(\prod_{i=1}^{n} S_{T_i} \Big)^{1/n} = \exp \Big(\frac{1}{n} \sum_{i=1}^{n} \log S_{T_i} \Big) \quad \text{(discrete monitoring)},$$

or on the continuous geometric mean:

$$C_T^{\mathrm{gA}} = \left(\exp \Big(\frac{1}{T - T_0} \int_{T_0}^{T} \log S_u du \Big) - K \right)^+ .$$

They can be priced by a variant of the BS formula (use Lemma 6.9.1 in [8]). Note that the integral of a continuous Gaussian process is Gaussian, and that the variance of $\int_0^t W_u du$ can be calculated by applying the product rule to $d(tW_t)$.

Geometric Asian options can be used as control variates in the Monte Carlo pricing of arithmetic Asian options. This means that we are computing the expectation of

$$Z := C_T^{\mathrm{A}} + \alpha(C_T^{\mathrm{gA}} - \mathbb{E}_{\mathbb{Q}}[C_T^{\mathrm{gA}}])$$

by Monte Carlo simulation, where α is a parameter to be chosen. A good α makes

$$\mathrm{Var}[Z] = \mathrm{Var}[C_T^{\mathrm{A}}] + \alpha^2 \mathrm{Var}[C_T^{\mathrm{gA}}] + 2\alpha \mathrm{Cov}[C_T^{\mathrm{A}}, C_T^{\mathrm{gA}}]$$

small. The optimal α is

$$\alpha^* = -\frac{\mathrm{Cov}[C_T^{\mathrm{A}}, C_T^{\mathrm{gA}}]}{\mathrm{Var}[C_T^{\mathrm{gA}}]},$$

leading to

$$\mathrm{Var}[Z] = \mathrm{Var}[C_T^{\mathrm{A}}] - \frac{\mathrm{Cov}[C_T^{\mathrm{A}}, C_T^{\mathrm{gA}}]^2}{\mathrm{Var}[C_T^{\mathrm{gA}}]} = \mathrm{Var}[C_T^{\mathrm{A}}]\big(1 - \mathrm{Corr}[C_T^{\mathrm{A}}, C_T^{\mathrm{gA}}]^2\big).$$

In practice, α^* can be approximated in the simulation using the empirical variance and covariance.

5.4 Basket options

Basket options are written on several underlyings S^i, $i = 1, \dots, k$. With weights $w_i > 0$ summing to 1, a basket call has payoff

$$C_T^{\mathrm{B}} = \Big(\sum_{i=1}^k w_i S_T^i - K\Big)^+.$$

For example, the sum could model a stock index. We assume a multidemsional BS model,

$$dS_t^i = S_t^i(rdt + \hat{\sigma}_i \cdot dW_t^*),$$

where $\hat{\sigma}_i \in (\mathbb{R}^+)^k$, and $W^* = (W^{1*}, \dots, W^{k*})$ is a $\mathbb{Q}$-BM. This can be rewritten using correlated BMs: Define

$$\sigma_i := |\hat{\sigma}_i|, \quad \tilde{W}^i := \frac{1}{\sigma_i}\hat{\sigma}_i W^*.$$

Then,

$$dS_t^i = S_t^i(rdt + \sigma_i d\tilde{W}^i),$$

and

$$[\tilde{W}^i, \tilde{W}^j]_t = \frac{\sum_{l=1}^k \hat{\sigma}_{il}\hat{\sigma}_{jl}}{\sigma_i\sigma_j} t = \frac{\hat{\sigma}_i \cdot \hat{\sigma}_j}{\sigma_i\sigma_j} t =: \rho_{ij} t.$$

Thus, $\tilde{W}$ is a correlated k-dimensional BM with instantaneous correlations ρ_{ij}. Using the forward price $F_i(t,T) := S_t^i e^{r(T-t)}$ of the ith asset and the modified weights

$$\hat{w}_i := \frac{w_i S_t^i}{\sum_{j=1}^k w_j S_t^j} = \frac{w_i F_i(t,T)}{\sum_{j=1}^k w_j F_j(t,T)},$$

we have

$$C_T^{\mathrm{B}} = \sum_{j=1}^k w_j F_j(t,T) \Bigg(\underbrace{\sum_{i=1}^k \hat{w}_i \underbrace{\frac{S_T^i}{F_i(t,T)}}_{:=\tilde{S}_T^i}}_{=:\tilde{A}_T} - \underbrace{\frac{K}{\sum_{j=1}^k w_j F_j(t,T)}}_{=:\tilde{K}} \Bigg)^+.$$

We approximate the weighted arithmetic mean $\tilde{A}_T = \sum_{i=1}^k \hat{w}_i \tilde{S}_T^i$ by the weighted geometric mean $\tilde{G}_t := \prod_{i=1}^k (\tilde{S}_T^i)^{\hat{w}_i}$. W.l.o.g. consider $t = 0$. We define

$$\hat{C}_0^{\mathrm{B}} := \sum_{j=1}^k w_j S_0^j \mathbb{E}_{\mathbb{Q}}[(\tilde{G}_T - \hat{K})^+], \tag{5.6}$$

where

$$\hat{K} := \tilde{K} + \mathbb{E}_{\mathbb{Q}}[\tilde{G}_T - \tilde{A}_T].$$

Note that

$$\mathbb{E}_{\mathbb{Q}}[\tilde{A}_T - \tilde{K}] = \mathbb{E}_{\mathbb{Q}}[\tilde{G}_T - \hat{K}],$$

i.e. replacing $\tilde{K}$ by $\hat{K}$ can be motivated by assuming a linear payoff function. As for the geometric mean $\tilde{G}_T$ in (5.6), note that

$$\tilde{S}_T^i = \frac{S_T^i}{e^{rT} S_0^i} = e^{\hat{\sigma}_i W_T^* - \frac{1}{2}\sigma_i^2 T},$$

and so

$$\tilde{G}_T = \prod_{i=1}^k e^{\hat{w}_i \hat{\sigma}_i W_T^* - \frac{1}{2}\hat{w}_i \sigma_i^2 T} = \exp\Big(\underbrace{\sum_{i=1}^k \hat{w}_i \hat{\sigma}_i W_T^*}_{=:\eta_T} - \tfrac{1}{2} T \sum_{i=1}^k \hat{w}_i \sigma_i^2 \Big).$$

Thus, (5.6) can be evaluated analogously to the BS formula (use Lemma 6.9.1 in [8]). The required variance is

$$\begin{aligned}
\mathrm{Var}[\eta_T] &= \mathrm{Var}[\sum_{i=1}^{k} \hat{w}_i \hat{\sigma}_i W_T^*] = \mathbb{E}_{\mathbb{Q}}\Big[\Big(\sum_{i,l=1}^{k} \hat{w}_i \hat{\sigma}_{il} W_T^{*l}\Big)\Big(\sum_{i,l=1}^{k} \hat{w}_i \hat{\sigma}_{il} W_T^{*l}\Big)\Big] \\
&= \sum_{i,l,j,m=1}^{k} \hat{w}_i \hat{w}_j \hat{\sigma}_{il} \hat{\sigma}_{jm} \underbrace{\mathbb{E}_{\mathbb{Q}}[W_T^{*l} W_T^{*m}]}_{=\delta_{lm} T} \\
&= \sum_{i,j=1}^{k} \hat{w}_i \hat{w}_j (\hat{\sigma}_i \cdot \hat{\sigma}_j) T.
\end{aligned}$$

As for $\hat{K}$, note that $\tilde{G}_T$ is log-normal, and that

$$\mathbb{E}_{\mathbb{Q}}[\tilde{A}_T] = \mathbb{E}_{\mathbb{Q}}\Big[\sum_{i=1}^{k} \hat{w}_i \frac{S_T^i}{e^{rT} S_0^i}\Big] = \sum_{i=1}^{k} \hat{w}_i = 1.$$

6 Stochastic volatility models

6.1 Local volatility

In a local volatility model, the volatility is a function of time and the spot price:

$$dS_t = S_t(r dt + \sigma(t, S_t) dW_t^*)$$

In general, there are no explicit formulas for option prices. The call price $C(K,T)$ satisfies the Dupire equation (exercise):

$$-C_T(K,T) + \tfrac{1}{2}\sigma(K,T)^2 K^2 C_{KK}(K,T) - rKC_K(K,T) = 0, \quad C(K,0) = (S_0 - K)^+.$$

Solving for σ yields the Dupire formula

$$\sigma(K,T)^2 = \frac{rKC_K(K,T) + C_T(K,T)}{\frac{1}{2}K^2 C_{KK}(K,T)}.$$

Thus, if call prices for all strikes and maturity are given, the model can be calibrated to all of them. Exotic options can be priced by Monte Carlo or PDE methods. In practice, market data has to be interpolated. In particular, the second derivative w.r.t. strike can cause numerical problems.

6.2 Two-factor models

Now consider a two-dimensional Brownian motion $W = (W^1, W^2)$ under $\mathbb{P}$, and let $\rho \in [-1, 1]$. Defining

$$\bar{W}^1 := W^1, \quad \bar{W}^2 := \rho W^1 + \underbrace{\sqrt{1-\rho^2}}_{=:\bar{\rho}} W^2$$

yields a two-dimensional BM $(\bar{W}^1, \bar{W}^2)$ with instantaneous correlation ρ, since

$$d[\bar{W}^2] = \rho^2 dt + \bar{\rho}^2 dt = dt, \quad d[\bar{W}^1, \bar{W}^2] = \rho dt.$$

We define the stochastic volatility model

$$\begin{aligned} dS_t/S_t &= \mu(S_t, \sigma_t, t)dt + \sigma_t d\bar{W}_t^1 \\ d\sigma_t &= \tilde{a}(\sigma_t)dt + b(\sigma_t)d\bar{W}_t^2, \end{aligned}$$

where it is assumed that the given functions μ, $\tilde{a}$, b are sufficiently regular. Let $\lambda = (\lambda_u^1, \lambda_u^2)_{u\in[0,T]}$ be a process such that

$$\exp\Big(\int_0^t \lambda_u \cdot dW_u - \frac{1}{2}\int_0^t |\lambda_u|^2 du\Big) \tag{6.1}$$

is a $\mathbb{P}$-martingale. According to Girsanov, define $\hat{W}_t^i := W_t^i - \int_0^t \lambda_u du$. This yields

$$\begin{aligned} dS_t/S_t &= (\mu + \sigma_t\lambda^1)dt + \sigma_t d\hat{W}^1 \\ d\sigma_t &= (\tilde{a} + b(\rho\lambda^1 + \bar{\rho}\lambda^2))dt + b(\rho d\hat{W}^1 + \bar{\rho} d\hat{W}^2). \end{aligned}$$

$(\hat{W}^1, \rho\hat{W}^1 + \bar{\rho}\hat{W}^2)$ is a correlated BM under $\mathbb{Q}$. We want $S_t^* = e^{-rt}S_t$ to be a local $\mathbb{Q}$-martingale. This leads to

$$\lambda_t^1 = \frac{r - \mu(S_t, \sigma_t, t)}{\sigma_t},$$

but does not fix λ^2. Therefore, a stochastic volatility model typically admits infinitely many EMMs, and is free of arbitrage, but incomplete. Several mathematical questions arise: Do the SDEs defining the model admit a solution? In what sense (weak or strong)? Is (6.1) a true martingale? Is S^* a true martingale? How can option prices be computed? For some answers, see e.g. [3, 5]. One of the best known SV models is the Heston model (see also the exercise class): $X = \log S$ satisfies

$$\begin{aligned} dX_t &= -\tfrac{1}{2}V_t dt + \sqrt{V_t}dW_t^{*1}, \\ dV_t &= \kappa(\theta - V_t)dt + \xi\sqrt{V_t}dW_t^{*2}, \end{aligned}$$

with $d[W^{*1}, W^{*2}] = \rho dt$.

Theorem 6.1 (Feynman-Kac)**.** *Let $\mu, \Sigma : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous, Σ bounded, $f, g : \mathbb{R} \to \mathbb{R}$ be smooth and bounded, and assume that $v : [0,T] \times \mathbb{R} \to \mathbb{R}$ is solves the PDE*

$$v_t(t,x) = \tfrac{1}{2}\Sigma(x)^2 v_{xx}(t,x) + \mu(x)v_x(t,x) - g(x)v(t,x), \quad v(0,x) = f(x) \tag{6.2}$$

and that v_x is bounded. Let X^x be the solution of the SDE

$$dX_t^x = \mu(X_t^x)dt + \Sigma(X_t^x)dW_t, \quad X_0^x = x.$$

Then we have

$$v(t,x) = \mathbb{E}\Big[f(X_t^x)\exp\Big(-\int_0^t g(X_u^x)du\Big)\Big].$$

Proof. Fix $t > 0$. By Ito's formula (exercise),

$$M_s := v(t-s, X_s^x)\exp\Big(-\int_0^s g(X_u^x)du\Big)$$

satisfies

$$dM_s = \exp\Big(-\int_0^s g(X_u^x)du\Big)\Sigma(X_t^x)v_x(t-s, X_s^x)dW_s,$$

and so M is a local martingale. By assumption, it is bounded by a deterministic constant, and so it is a true martingale. We conclude $v(t,x) = M_0 = \mathbb{E}[M_t]$. $\square$

With the standard notation

$$\mathcal{A}f(x) = \frac{1}{2}\Sigma(x)^2\frac{\partial^2 f}{\partial x^2} + \mu(x)\frac{\partial f}{\partial x},$$

the PDE (6.2) can be written as $v_t = \mathcal{A}v - gv$. Now let W be a d-dimensional BM. Let $\mu : \mathbb{R}^k \to \mathbb{R}^k$, $\Sigma : \mathbb{R}^k \to \mathbb{R}^{k\times d}$, $g, f : \mathbb{R}^k \to \mathbb{R}$ be sufficiently regular (see [6] for details). Define the diffusion matrix $a(x) := \Sigma(x)\Sigma(x)^{\mathrm{T}}$,

$$\mathcal{A} := \sum_{i=1}^k \mu_i(x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^k a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j},$$

suppose

$$v_t = \mathcal{A}v - gv, \quad v(0,x) = f(x), \tag{6.3}$$

$$dX_t = \mu(X_t)dt + \Sigma(X_t)dW_t.$$

Then, the multidimensional version of Feynman-Kac asserts

$$v(t,x) = \mathbb{E}\Big[f(X_t^x)\exp\Big(-\int_0^t g(X_u^x)du\Big)\Big].$$

This can be used to compute option prices numerically, and for finding explicit formulas for the characteristic function for certain SV models (see the exercise class for the Heston model). Consider a SV model

$$\begin{aligned} dS_t &= rS_t dt + \sigma_t S_t \underbrace{d\hat{W}_t^1}_{dW_t^{*1}}, \\ d\sigma_t &= \hat{a}(\sigma_t)dt + b(\sigma_t)(\underbrace{\rho d\hat{W}_t^1 + \bar{\rho} d\hat{W}_t^2}_{dW_t^{*2}}). \end{aligned} \tag{6.4}$$

Thus, $d = k = 2$, $X_t = (S_t, \sigma_t)^{\mathrm{T}}$, $\mu(x_1,x_2) = (rx_1, \hat{a}(x_2))^{\mathrm{T}}$, $\Sigma(x_1,x_2) = \begin{pmatrix} x_1x_2 & 0 \\ b(x_2)\rho & b(x_2)\bar{\rho} \end{pmatrix}$, $g = r$. We have

$$a(x) = \begin{pmatrix} x_1x_2 & 0 \\ b(x_2)\rho & b(x_2)\bar{\rho} \end{pmatrix}\begin{pmatrix} x_1x_2 & b(x_2)\rho \\ 0 & b(x_2)\bar{\rho} \end{pmatrix} = \begin{pmatrix} x_1^2x_2^2 & b(x_2)\rho x_1x_2 \\ b(x_2)\rho x_1 x_2 & b(x_2)^2 \end{pmatrix}.$$

If h is some payoff function and $f(x_1,x_2) = h(x_1)$, then

$$\mathbb{E}_{\mathbb{Q}}[f(X_t^x)e^{-rt}] = e^{-rt}\mathbb{E}_{\mathbb{Q}}[h(S_t)].$$

The PDE (6.3) becomes (with $s = x_1$)

$$\underbrace{\tfrac{1}{2}s^2\sigma^2 v_{ss} + rsv_s - v_t - rv}_{\text{BS PDE}} + \hat{a}(\sigma)v_\sigma + \rho s\sigma b(\sigma)v_{s\sigma} + \tfrac{1}{2}b(\sigma)^2 v_{\sigma\sigma} = 0.$$

Solving this with initial condition $v(0,s,\sigma) = h(s)$ yields the option price $v(T, S_0, \sigma_0) = e^{-rt}\mathbb{E}[h(S_T)]$.

6.3 The Hull-White formula

In the SV model (6.4), the discounted stock can be written as ($W^{1*} = \hat{W}^1$)

$$S_t^* = S_0 \exp\Big(\int_0^t \sigma_u dW_u^{*1} - \frac{1}{2}\int_0^t \sigma_u^2 du\Big).$$

Assume now that $\rho = 0$, i.e. the driving BMs are uncorrelated. Then, S^* is a $\mathbb{Q}$-martingale, since for $u \le t$

$$\mathbb{E}_{\mathbb{Q}}[S_t^*|\mathcal{F}_u] = \mathbb{E}_{\mathbb{Q}}[\underbrace{\mathbb{E}_{\mathbb{Q}}[[S_t^*|\mathcal{F}_u \vee \sigma(\sigma_s : u \le s \le t)]}_{=S_u^*}|\mathcal{F}_u] = S_u^*.$$

Consider a claim with payoff $h(S_T)$. For a continuous function g, we write $A_g := \int_0^T g(u)^2 du$, and F_T for the cdf of the aggregated variance $\int_0^T \sigma_u^2 du$. The price of the claim is

$$\begin{aligned}
e^{-rT}\mathbb{E}_{\mathbb{Q}}[h(S_T)] &= e^{-rT}\mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}\Big[h(S_0 e^{rT+\int_0^T \sigma_u dW_u^{*1} - \frac{1}{2}\int_0^T \sigma_u^2 du})\Big|(\sigma_u)_{u\le T}\Big]\Big] \\
&= e^{-rT}\mathbb{E}_{\mathbb{Q}}\Big[\mathbb{E}_{\mathbb{Q}}\Big[h(S_0 e^{rT+\int_0^T g(u) dW_u^{*1} - \frac{1}{2}\int_0^T g(u)^2 du})\Big]\Big|_{g=\sigma}\Big] \\
&= e^{-rT}\mathbb{E}_{\mathbb{Q}}\Big[\int_{-\infty}^{\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}} h(S_0 e^{rT+\sqrt{A_g}y-\frac{1}{2}A_g})dy\Big|_{g=\sigma}\Big] \\
&= e^{-rT}\int_0^{\infty} F_T(dw)\int_{-\infty}^{\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}} h(S_0 e^{rT+\sqrt{w}y-\frac{1}{2}w})dy.
\end{aligned}$$

7 Lévy models

7.1 Lévy processes

Definition 7.1. *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\ge 0})$ be a filtered probability space. An $\mathbb{R}^d$-valued process X with $X_0 = 0$ is called a Lévy process, if*

(i) For $0 \le s \le t$, $X_t - X_s$ is independent of $\mathcal{F}_s$

(ii) $X_t - X_s \overset{\mathrm{d}}{=} X_{t-s}$, $0 \le s \le t$

(iii) X is stochastically continuous: For all $\varepsilon > 0$ and $t \ge 0$,

$$\lim_{h\downarrow 0} \mathbb{P}\big[|X_{t+h} - X_t| \ge \varepsilon\big] = 0.$$

We assume that X has càdlàg paths, as it can be shown that X always has such a modification, which is again a Lévy process [10, Theorem I.30]. We may assume that $(\mathcal{F}_t)$ is the natural filtration of X, i.e. the generated filtration augmented by the null sets. It can be shown that it is right-continuous [10, Theorem I.31]. The characteristic function of X_t is

$$\varphi_{X_t}(\xi) = \mathbb{E}[e^{i\xi\cdot X_t}], \quad t \ge 0,\ \xi \in \mathbb{R}^d.$$

Recall that a random variable Y is infinitely divisible, if for all $n \in \mathbb{N}$ there exist iid random variables $Y_1^{(1)}, \dots, Y_n^{(n)}$ such that $Y \overset{\mathrm{d}}{=} Y_1^{(1)} + \dots + Y_n^{(n)}$. For example, if $Y \sim \mathcal{N}(\mu, \sigma^2)$, we may take $Y_i^{(n)} \sim \mathcal{N}(\frac{\mu}{n}, \frac{\sigma^2}{n})$ iid.

Lemma 7.2. *For a Lévy process X and $t \ge 0$, X_t is infinitely divisible, and $\varphi_{X_t}(\xi) = \varphi_{X_{t/n}}(\xi)^n$ for $n \in \mathbb{N}$.*

Proof. Put

$$Y_k^{(n)} := X_{kt/n} - X_{(k-1)t/n} \sim X_{t/n}, \quad k = 1, \dots, n. \qquad \square$$

Lemma 7.3. *Let X be a Lévy process (stochastically continuous and càdlàg would suffice). Then, $(t,\xi) \mapsto \varphi_{X_t}(\xi)$ is continuous.*

Proof. Fix $\hat{\xi} \in \mathbb{R}^d$, $t \geq 0$, $\varepsilon > 0$. Let δ_ε be such that

$$\sup_{\substack{|y|\leq\delta_\varepsilon \\ |\xi-\hat{\xi}|\leq 1}} |e^{i\xi y} - 1| \leq \frac{\varepsilon}{4}.$$

Then there is $\delta'_\varepsilon > 0$ such that

$$|s-t| < \delta'_\varepsilon \Longrightarrow \mathbb{P}[|X_t - X_s| > \delta_\varepsilon] \leq \frac{\varepsilon}{8}.$$

Hence, for such s and $|\xi - \hat{\xi}| \leq 1$, we have

$$\begin{aligned} &|\varphi_{X_t}(\xi) - \varphi_{X_s}(\xi)| \leq \mathbb{E}[|e^{i\xi(X_t-X_s)} - 1|] \\ &= \mathbb{E}[\underbrace{|e^{i\xi(X_t-X_s)} - 1|}_{\leq 2}\mathbf{1}_{\{|X_t-X_s|>\delta_\varepsilon\}}] + \mathbb{E}[\underbrace{|e^{i\xi(X_t-X_s)} - 1|}_{<\varepsilon/4}\underbrace{\mathbf{1}_{\{|X_t-X_s|\leq\delta_\varepsilon\}}}_{\leq 1}] \leq \varepsilon/2. \end{aligned}$$

For fixed ω, $X_\cdot(\omega)$ is bounded on any finite interval, as it is càdlàg. By dominated convergence,

$$\lim_{\xi\to\hat{\xi}} \mathbb{E}[\sup_{s:|s-t|\leq\delta'_\varepsilon} |e^{i(\xi-\hat{\xi})X_s} - 1|] = 0.$$

Let $\tilde{\delta}_\varepsilon \in (0,1]$ be such that the latter expectation is $\leq \varepsilon/2$ for $|\xi - \hat{\xi}| \leq \tilde{\delta}_\varepsilon$. For such ξ and s with $|s-t| < \delta'_\varepsilon$, we have

$$\begin{aligned} |\varphi_{X_t}(\xi) - \varphi_{X_s}(\hat{\xi})| &\leq |\varphi_{X_t}(\xi) - \varphi_{X_s}(\xi)| + |\varphi_{X_s}(\xi) - \varphi_{X_s}(\hat{\xi})| \\ &\leq \varepsilon/2 + \mathbb{E}[|e^{i(\xi-\hat{\xi})X_s} - 1|] \leq \varepsilon. \end{aligned} \qquad \square$$

The following two lemmas are proved in [12] (Lemmas 7.5 and 7.6).

Lemma 7.4. *For $\xi \in \mathbb{R}^d$, $t \geq 0$, we have $\varphi_{X_t}(\xi) \neq 0$.*

Proof. We give a different proof than [12]. If $\varphi_{X_t}(\xi) = 0$, then Lemma 7.2 implies that $\varphi_{X_{t/n}}(\xi) = 0$ for all $n \in \mathbb{N}$. Since $\varphi_{X_0}(\xi) = 1$, Lemma 7.3 yields a contradiction. $\square$

Lemma 7.5. *Let $\varphi \in C(\mathbb{R}^d, \mathbb{C})$ with $\varphi(0) = 1$, and without any zeros. Then there is a unique $g \in C(\mathbb{R}^d, \mathbb{C})$ with $g(0) = 0$ and $\varphi(\xi) = e^{g(\xi)}$ for all $\xi \in \mathbb{R}^d$.*

This function is called "distinguished logarithm" and is sometimes written as "$g = \log \varphi$". This is *not* a composition of functions though: The example $d = 1$, $\varphi(\xi) = e^{i\xi}$, $g(\xi) = i\xi$, $\varphi(0) = \varphi(2\pi) = 1$ shows that $\varphi(\xi) = \varphi(\xi')$ does not imply $g(\xi) = g(\xi')$.

Theorem 7.6. *Let X be a Lévy process. Then there is a unique $\psi \in C(\mathbb{R}^d, \mathbb{C})$ with $\psi(0) = 0$ and $\varphi_{X_t}(\xi) = e^{t\psi(\xi)}$ for all t, ξ. It is called characteristic exponent or Lévy exponent of X.*

Proof. Let g_t be the distinguished logarithm of φ_{X_t}. For $m, n \in \mathbb{N}$, we have

$$e^{g_m(\xi)} = \varphi_{X_m}(\xi) = \left(\varphi_{X_{m/n}}(\xi)\right)^n = \exp\left(ng_{n/m}(\xi)\right),$$

and so

$$g_m(\xi) = ng_{m/n}(\xi) + 2\pi i k(\xi)$$

for some $\mathbb{Z}$-valued function k. Since g_m and $g_{m/n}$ are continuous, and $k(0) = 0$, k must vanish identically. Therefore,

$$g_{m/n}(\xi) = \frac{1}{n} g_m(\xi) = \frac{m}{n} g_1(\xi).$$

For arbitrary $t \geq 0$, let (q_n) be a sequence of rational numbers converging to t. Then we have

$$g_t(\xi) = \lim_{n\to\infty} g_{q_n}(\xi) = \lim_{n\to\infty} q_n g_1(\xi) = tg_1(\xi).$$

Thus, we can take $\psi = g_1$. $\square$

Example 7.7. *(i) For one-dimensional Brownian motion with drift, $\mu t + \sigma W_t$, we have*

$$\varphi_{X_t}(\xi))\mathbb{E}[e^{i\xi X_t}] = e^{i\mu t\xi}\mathbb{E}[e^{i\xi\sigma W_t}] = e^{i\mu t\xi + \frac{1}{2}(i\xi\sigma)^2 t} = e^{i\mu t\xi - \frac{1}{2}\sigma^2 t\xi^2},$$

hence $\psi(\xi) = i\mu\xi - \frac{1}{2}\sigma^2\xi^2$.

(ii) For $d, n \geq 1$, $\mu \in \mathbb{R}^d$, $\sigma \in \mathbb{R}^{d\times n}$, covariance matrix $C = \sigma\sigma^{\mathrm{T}}$, and $X_t = \mu + \sigma W_t$, we have

$$\psi(\xi) = i\mu \cdot \xi - \frac{1}{2} C\xi \cdot \xi$$

(iii) For a Poisson process with intensity λ, we have

$$\varphi_{N_t}(\xi) = \mathbb{E}[e^{i\xi N_t}] = \sum_{n=0}^{\infty} e^{i\xi n} e^{-\lambda t} \frac{(\lambda t)^n}{n!} = e^{\lambda t(e^{i\xi} - 1)},$$

hence $\psi(\xi) = \lambda(e^{i\xi} - 1)$.

(iv) Let $X_t = \sum_{n=1}^{N_t} Z_n$ *be a compound Poisson process, i.e.* N_t *is a Poisson process and* (Z_n) *an independent iid d-dimensional sequence with cdf* η. *If*

$$\hat{\eta}(z) = \mathbb{E}[e^{i\xi Z_1}] = \int_{\mathbb{R}^d} e^{i\xi x}\eta(dx)$$

denotes the characteristic function of η, *then*

$$\varphi_{X_t}(\xi) = \sum_{n=0}^{\infty} \hat{\eta}(\xi)^n \mathbb{P}[N_t = n] = e^{\lambda t(\hat{\eta}(\xi)-1)},$$

hence $\psi(\xi) = \lambda(\hat{\eta}(\xi) - 1)$. *For the compensated compound Poisson process* $\tilde{X}_t = X_t - \lambda t m$, *where* $m = \int_{\mathbb{R}^d} x\eta(dx)$, *we have*

$$\varphi_{\tilde{X}_t}(\xi) = \varphi_{X_t}(\xi)e^{-i\lambda t\xi m},$$

hence

$$\psi(\xi) = \lambda(\hat{\eta}(\xi) - 1 - i\xi m) = \int_{\mathbb{R}^d} (e^{i\xi x} - 1 - i\xi x)\lambda\eta(dx).$$

If X and Y are independent Lévy processes with Lévy exponents ψ_X and ψ_Y, an easy calculation shows that $\psi_{X+Y} = \psi_X + \psi_Y$.

Example 7.8. *A jump diffusion is a Lévy process of the form*

$$X_t = \mu t + B_t + \sum_{n=1}^{N_t} Z_n, \tag{7.1}$$

where $\mu \in \mathbb{R}^d$, B *is a d-dimensional Browian motion with covariance matrix* C, *and the sum is an independent compound Poisson process. The Lévy exponent is*

$$\psi(\xi) = i\mu\xi - \tfrac{1}{2}C\xi\cdot\xi + \int_{\mathbb{R}^d} (e^{i\xi x} - 1)\lambda\eta(dx).$$

For $d = 1$, $Z_1 \sim \mathcal{N}(\alpha, \delta^2)$ *and* $C = \sigma^2$ *we obtain the famous Merton jump diffusion, with*

$$\psi(\xi) = i\mu\xi - \frac{1}{2}\sigma^2\xi^2 + \lambda(e^{i\alpha\xi - \frac{1}{2}\delta^2\xi^2} - 1).$$

Let $(T_n)_{n\geq 1}$ be the jump times of the d-dimensional jump diffusion (7.1). For $I \times H \in \mathcal{B}([0,\infty) \times \mathbb{R}^d)$, define

$$J(I \times H) := \sum_{n=1}^{\infty} \delta_{T_n}(I)\delta_{Z_n}(H)$$

Since $N_t = \#\{n \geq 1 : T_n \leq t\}$, we have

$$J([0,t] \times H) = \sum_{n=1}^{N_t} \delta_{Z_n}(H).$$

This defines a random measure, which is called the jump measure of X. We have

$$\begin{aligned}
\mathbb{E}[J([0,t] \times H)] &= \mathbb{E}[\sum_{n=1}^{N_t} \delta_{Z_n}(H)] = \sum_{n=1}^{\infty} \mathbb{E}[\sum_{k=1}^{n} \delta_{Z_k}(H) \mathbf{1}_{\{N_t = n\}}] \\
&= \sum_{n=1}^{\infty} \mathbb{P}[N_t = n] \sum_{k=1}^{n} \mathbb{P}[Z_k \in H] = \eta(H) \underbrace{e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} n}_{=\lambda t} \\
&= \lambda t \eta(H),
\end{aligned}$$

and thus

$$\mathbb{E}[J([0,t] \times H)] = t \mathbb{E}[J([0,1] \times H)] =: t\nu(H).$$

Definition 7.9. *The finite measure*

$$\nu(H) = \mathbb{E}[J([0,1] \times H)] = \lambda \eta(H),$$

which satisfies $\eta(\mathbb{R}^d) = \lambda$, is called Lévy measure of the jump diffusion X, and (μ, C, ν) is the characteristic triplet (Lévy triplet) of X.

For a càdlàg function f, we write

$$\Delta f(t) := f(t) - f(t-)$$

for the jump in t.

Theorem 7.10. *Let X be a jump diffusion, and $f : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ be measurable. Then*

$$\sum_{\substack{0 < s \leq t \\ \Delta X_s \neq 0}} f(s, \Delta X_s) = \int_0^t \int_{\mathbb{R}^d} f(s,x) J(ds, dx).$$

Define $\tilde{J}(dt, dx) := J(dt, dx) - dt\nu(dx)$ (compensated jump measure). For $f \in L^1(\mathbb{R}^+ \times \mathbb{R}^d, ds \otimes \nu)$, define

$$M_t := \int_0^t \int_{\mathbb{R}^d} f(s,x) \tilde{J}(ds, dx).$$

Then M is a martingale with $M_0 = 0$ and

$$\mathrm{Var}[M_t] = \int_0^t \int_{\mathbb{R}^d} f(s,x)^2 \nu(dx) ds.$$

The proof is straightforward, and omitted.

Example 7.11. *For a compound Poisson process, the theorem implies*

$$X_t = \sum_{n=1}^{N_t} Z_n = \int_0^t \int_{\mathbb{R}^d} xJ(ds, dx).$$

For the variation, the theorem implies

$$\mathbb{V}_{[0,t]}(X) = \sum_{n=1}^{N_t} |Z_n| = \int_0^t \int_{\mathbb{R}^d} |x|J(ds, dx).$$

Therefore,

$$\mathbb{E}[\mathbb{V}_{[0,t]}(X)] = t \int_{\mathbb{R}^d} |x|\nu(dx).$$

For the quadratic variation, we have

$$\mathbb{E}[\mathbb{V}^2_{[0,t]}(X)] = t^2 \int_{\mathbb{R}^d} |x|^2\nu(dx).$$

Now consider a general Lévy process X. For $H \in \mathcal{B}(\mathbb{R}^d)$ with $0 \notin \bar{H}$ (closure of H), and $T > 0$, a.s. there are only finitely many jump times with jump in H ("large jumps"). For such H, we again define the jump measure

$$J(I \times H) = \#\{t \in I : \Delta X_t \in H\}.$$

It can be shown that J can be extended to a random measure on $\mathbb{R}^+ \times \mathbb{R}^d$ with $J(I \times \{0\}) = 0$, and that

$$\nu(H) = \mathbb{E}[J([0,1] \times H)], \quad H \in \mathcal{B}(\mathbb{R}^d),$$

defines a σ-finite measure on $\mathbb{R}^d$. If $0 \notin \bar{H}$, it can be shown that

$$J([0,t] \times H) = \#\{s \le t : \Delta_s \in H\}$$

is a Poisson process with intensity $\nu(H)$.

Theorem 7.12 (Lévy-Ito decomposition)**.** *Let X be a Lévy process. The Lévy measure satisfies $\int_{|x|\ge 1} \nu(dx) < \infty$ and $\int_{|x|<1} |x|^2\nu(dx) < \infty$. There is a Brownian motion B such that for arbitrary $R > 0$ there exist $\mu_R \in \mathbb{R}^d$ and processes X^R and M^R such that B, X^R, M^R are independent,*

$$X_t = \mu_R t + B_t + X_t^R + M_t^R,$$

$$X_t^R = \int_0^t \int_{|x|\geq R} xJ(ds, dx) = \sum_{0<s\leq t} \Delta X_s \mathbf{1}_{\{|\Delta X_s|\geq R\}}$$

is a compound Poisson process, and

$$\begin{aligned} M_t^R &= \int_0^t \int_{|x|<R} x\tilde{J}(ds, dx) \\ &= \lim_{\varepsilon\downarrow 0} \underbrace{\int_0^t \int_{\varepsilon<|x|<R} x(J(ds, dx) - \nu(dx)ds)}_{=:\tilde{X}_t^{R,\varepsilon}} \end{aligned}$$

is a martingale. If C is the covariance matrix of B, then (μ_R, C, ν) is called characteristic triplet (Lévy triplet) of X.

Theorem 7.13 (Lévy-Khintchine representation)**.** *Let X be a Lévy process with Lévy triplet (μ_1, C, ν). Then the characteristic function is given by*

$$\varphi_{X_t}(\xi) = e^{t\psi_X(\xi)},$$

where

$$\psi_X(\xi) := i\mu_1\xi - \frac{1}{2}C\xi\cdot\xi + \int_{\mathbb{R}^d} (e^{i\xi x} - 1 - i\xi x\mathbf{1}_{\{|x|<1\}})\nu(dx).$$

Proof. Since

$$X_t = \mu_1 t + B_t + X_t^1 + \lim_{\varepsilon\downarrow 0} \tilde{X}_t^{1,\varepsilon},$$

by independence and Example 7.7 we obtain

$$\begin{aligned} &\psi_{\mu_1 t+B_t+X_t^1+\tilde{X}_t^{1,\varepsilon}}(\xi) = \psi_{\mu_1 t+B_t}(\xi) + \psi_{X_t^1}(\xi) + \psi_{\tilde{X}_t^{1,\varepsilon}}(\xi) \\ &\quad = i\mu_1\xi - \frac{1}{2}C\xi\cdot\xi + \int_{|x|\geq 1} (e^{i\xi x} - 1)\nu(dx) + \int_{\varepsilon\leq|x|\leq 1} \underbrace{(e^{i\xi x} - 1 - i\xi x)}_{=O(|x|^2)}\nu(dx). \end{aligned}$$

Now take the limit $\varepsilon \downarrow 0$. □

If we assume $\int_{|x|\leq 1} |x|\nu(dx) < \infty$ (finite activity), then

$$\psi_X(\xi) = i\xi\mu_0 - \frac{1}{2}C\xi\cdot\xi + \int_{\mathbb{R}^d} (e^{i\xi x} - 1)\nu(dx),$$

where $\mu_0 = \mu_1 - \int_{|x|\leq 1} x\nu(dx)$. If $\int_{|x|\geq 1} |x|\nu(dx) < \infty$, then it follows from Theorem 7.12 that

$$\mathbb{E}[X_1] = \mu_1 + \int_{|x|\geq 1} x\nu(dx).$$

More generally, it can be shown for $n \in \mathbb{N}$ that $\mathbb{E}[|X_1|^n] < \infty$ if and only if $\int_{|x|\geq 1} |x|^n \nu(dx) < \infty$. For $d = 1$ and $\xi \in \mathbb{R}$, $\mathbb{E}[e^{\xi X_1}] < \infty$ if and only of $\int_{|x|\geq 1} e^{\xi x} \nu(dx) < \infty$.

Theorem 7.14. *Let X be a 1-dimensional Lévy process with characteristic triplet (μ_1, σ^2, ν).*

(i) If $\mathbb{E}[|X_1|] < \infty$, then $X_t - \mathbb{E}[X_t]$ is a martingale.

(ii) X is a martingale $\iff \int_{|x|\geq 1} |x|\nu(dx) < \infty$ and $\mu_1 + \int_{|x|\geq 1} x\nu(dx) = 0$.

(iii) For $\xi \in \mathbb{R}$ with $\mathbb{E}[e^{\xi X_1}] < \infty$, the process $e^{\xi X_t}/\mathbb{E}[e^{\xi X_t}]$ is a martingale.

(iv) e^{X_t} is a martingale $\iff \int_{|x|\geq 1} e^x \nu(dx) < \infty$ and $0 = \mu_1 + \frac{\sigma^2}{2} + \int_{\mathbb{R}}(e^x - 1 - x\mathbf{1}_{\{|x|\leq 1\}})\nu(dx)$.

Proof. (i) By independence of the increments,

$$\mathbb{E}[X_t - X_s|\mathcal{F}_s] = \mathbb{E}[X_t - X_s], \quad s \leq t.$$

(ii) This follows from (i) and the remarks above.

(iii) Define

$$M_t := e^{\xi X_t}/\mathbb{E}[e^{\xi X_t}] = e^{\xi X_t - t\psi_X(-i\xi)}.$$

Then we have

$$\begin{aligned}
\mathbb{E}[M_t - M_s|\mathcal{F}_s] &= \mathbb{E}[e^{\xi X_t - t\psi(-i\xi)} - e^{\xi X_s - s\psi(-i\xi)}|\mathcal{F}_s] \\
&= \mathbb{E}[e^{\xi(X_t - X_s) - (t-s)\psi(-i\xi)} - 1]e^{\xi X_s - s\psi(-i\xi)} \\
&= \mathbb{E}[e^{\xi X_{t-s} - (t-s)\psi(-i\xi)} - 1]M_s \\
&= M_s(e^{-(t-s)\psi(-i\xi)}e^{(t-s)\psi(-i\xi)} - 1)) = 0.
\end{aligned}$$

(iv) Since $\mathbb{E}[e^{X_t}] = e^{t\psi(-i)} = 0$, the Lévy-Khintchine formula implies $\psi(-i) = 0$. □

Example 7.15. *An α-stable Lévy process has Lévy measure*

$$\nu(dx) = \frac{c_1}{x^{1+\alpha}}\mathbf{1}_{\{x>0\}}dx + \frac{c_2}{(-x)^{1+\alpha}}\mathbf{1}_{\{x<0\}}dx.$$

with constants $c_1, c_2 > 0$. By the integrability properties of the Lévy measure (see Theorem 7.12),

$$\int_{0<x<1} \frac{x^2}{x^{1+\alpha}}dx < \infty \quad \text{and} \quad \int_{x>1} x^{-1-\alpha}dx < \infty,$$

we must have $\alpha \in (0, 2)$. Recall that a random variable Y is strictly stable, if for all $n \in \mathbb{N}$ there are c_n and iid $Y_1^{(n)}, \ldots, Y_n^{(n)} \sim Y$ such that $c_m Y \overset{\mathrm{d}}{=} Y_1^{(n)} + \cdots + Y_n^{(n)}$. Then, there is $\alpha \in (0, 2]$ with $c_n = n^{1/\alpha}$. For our ν, X_1 s strictly α-stable. The Lévy exponent can be calculated explicitly.

7.2 Option pricing

We consider a bank account $B_t = e^{rt}$ and a risky asset $S_t = S_0 e^{X_t}$, where X is a Lévy process. In general, there are infinitely many EMMs. If $\mathbb{Q}$ is an EMM, and g the payoff function of a contingent claim, then

$$e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[g(S_T)|\mathcal{F}_t]$$

is an arbitrage free price process.

Proposition 7.16. *Let X be a Lévy process under an equivalent probability measure $\mathbb{Q}$ with characteristic exponent $\psi_{\mathbb{Q},X}$. Then the discounted process $S_t^* = e^{-rt}S_t$ is a $\mathbb{Q}$-martingale on $[0,T]$ if and only if $\mathbb{E}_{\mathbb{Q}}[e^{X_T}] < \infty$ and $\psi_{\mathbb{Q},X}(-i) = r$.*

Proof. By Theorem 7.14, $S_t^* = S_0 e^{X_t - rt}$ is a $\mathbb{Q}$-martingale if and only if

$$S_0 = \mathbb{E}_{\mathbb{Q}}[S_0 e^{X_t - rt}] = S_0 e^{\psi_{\mathbb{Q},X}(-i)t - rt}. \qquad \square$$

Definition 7.17. *Let Y be a random variable, and $\theta \in \mathbb{R}$ with $\mathbb{E}[e^{\theta Y}] < \infty$. Then the probability measure $\mathbb{P}^\theta$ defined by*

$$\frac{d\mathbb{P}^\theta}{d\mathbb{P}} = \frac{e^{\theta Y}}{\mathbb{E}[e^{\theta Y}]}$$

is called Esscher transform of $\mathbb{P}$ with parameter θ.

For a Lévy process X, we write $\ell_X(z) = \psi_X(-iz)$, which is called Laplace exponent.

Theorem 7.18. *Let X be a Lévy process and $\theta \in \mathbb{R}$, $T > 0$ such that $\mathbb{E}[e^{\theta X_T}] < \infty$. Then*

$$Z_t^\theta := e^{\theta X_t - t\ell_X(\theta)}$$

is a martingale. Under $\mathbb{P}_T^\theta$, $(X_t)_{0 \le t \le T}$ is a Lévy process with Laplace exponent $\ell_\theta(z) = \ell_X(z+\theta) - \ell_X(\theta)$.

Proof.

$$\mathbb{E}\Big[\frac{Z_t^\theta}{Z_s^\theta}\Big|\mathcal{F}_s\Big] = \underbrace{\mathbb{E}[e^{\theta(X_t - X_s)}]}_{=e^{(t-s)\ell_X(\theta)}} e^{-(t-s)\ell_X(\theta)} = 1.$$

Moreover,

$$\begin{aligned}
\mathbb{E}_T^\theta[e^{z(X_t - X_s)}|\mathcal{F}_s] &= \frac{\mathbb{E}[e^{z(X_t - X_s)}Z_T^\theta|\mathcal{F}_s]}{\mathbb{E}[Z_T^\theta|\mathcal{F}_s]} \\
&= \frac{1}{Z_s^\theta}\mathbb{E}\Big[e^{z(X_t - X_s)}\mathbb{E}[Z_T^\theta|\mathcal{F}_t]\Big|\mathcal{F}_s\Big] \\
&= \mathbb{E}[e^{(z+\theta)(X_t - X_s)}|\mathcal{F}_s]e^{-(t-s)\ell_X(\theta)} \\
&= e^{(t-s)(\ell_X(z+\theta) - \ell_X(\theta))}.
\end{aligned} \qquad \square$$

Lemma 7.19. *Let (μ, σ^2, ν) be the characteristic triplet of X, and θ, T as in Theorem 7.18. Then, the triplet $(\mu_\theta, \sigma_\theta^2, \nu_\theta)$ under $\mathbb{P}_T^\theta$ is given by*

$$\begin{aligned}\mu_\theta &= \mu + \theta\sigma^2 + \int_{\mathbb{R}} (e^{\theta x} - 1) x \mathbf{1}_{\{|x|<1\}} \nu(dx), \\ \sigma_\theta^2 &= \sigma^2, \\ \nu_\theta(dx) &= e^{\theta x}\nu(dx).\end{aligned}$$

Proof. A straightforward calculation, using the Lévy-Khintchine formula. □

Theorem 7.20. *Consider the market model $(B_t, S_0 e^{X_t})_{0 \le t \le T}$. If θ^* satisfies $\mathbb{E}[e^{\theta^* X_T}] < \infty$, $\mathbb{E}[e^{(\theta^*+1)X_T}] < \infty$ and $\ell(\theta^* + 1) - \ell(\theta^*) = r$, then $\mathbb{P}_T^{\theta^*}$ is an EMM.*

Proof. By assumption, $\mathbb{P}_T^{\theta^*}$ is well-defined, and $\mathbb{E}_T^{\theta^*}[S_T] < \infty$. Moreover, we have

$$\begin{aligned}\mathbb{E}_T^{\theta^*}[S_t^* | \mathcal{F}_s] = \mathbb{E}_T^{\theta^*}[S_0 e^{X_t - rt} | \mathcal{F}_s] &= e^{-r(t-s)} e^{X_s - rs} S_0 \underbrace{\mathbb{E}_T^{\theta^*}[e^{X_t - X_s} | \mathcal{F}_s]}_{= e^{(t-s)\ell_{\theta^*}(1)}} \\ &= S_s^* e^{(t-s)(\ell_{\theta^*}(1) - r)}.\end{aligned}$$

By assumption, we have

$$\ell_{\theta^*}(1) = \ell(\theta^* + 1) - \ell(\theta^*) = r.$$ □

7.3 Option pricing by Fourier transform

For $f \in L^1(\mathbb{R})$, we write

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{i\xi x} f(x) dx$$

for the Fourier transform of f.

Theorem 7.21 (Inversion formula)**.** *Suppose that f and $\hat{f}$ are in L^1. Then we have*

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ix\xi} \hat{f}(\xi) d\xi.$$

We consider a market model $(e^{rt}, S_t)_{0 \le t \le T}$ with $S_t = S_0 e^{X_t}$, with characteristic function

$$\varphi_{X_t}(\xi) = \mathbb{E}[e^{i\xi X_T}] = \int_{\mathbb{R}} e^{i\xi x} Q_T(dx),$$

where Q_T is the distribution of X_T. We assume that we work under a pricing measure, i.e. we dispense with the notation $\mathbb{Q}$ and $\mathbb{E}_{\mathbb{Q}}$. Let f be the payoff function in terms of the log-price, i.e. the option payoff is $f(\log S_t) = f(\log S_0 + X_T)$.

We compute the price of the claim, for ease of notation with $S_0 = 0$ and $r = 0$:

$$\begin{aligned}
\mathbb{E}[f(X_T)] &= \int_{\mathbb{R}} f(x) Q_T(dx) \\
&= \int_{\mathbb{R}} \frac{1}{2\pi} \Big(\int_{\mathbb{R}} e^{-ix\xi} \hat{f}(\xi) d\xi \Big) Q_T(dx) \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi) \Big(\int_{\mathbb{R}} e^{-ix\xi} Q_T(dx) \Big) d\xi \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi) \varphi_{X_T}(-\xi) d\xi.
\end{aligned}$$

With general r and S_0, the price becomes

$$e^{rT} \mathbb{E}[f(X_T)] = \frac{e^{-rT}}{2\pi} \int_{\mathbb{R}} e^{-i\xi \log S_0} \hat{f}(\xi) \varphi_{X_T}(-\xi) d\xi.$$

For instance, $f(x) = (e^x - K)^+$ for a call. As this function is not in L^1, we consider the damped payoff function $f_\alpha(x) := e^{-\alpha x} f(x)$, $\alpha \in \mathbb{R}$. For $\alpha > 1$,

$$f_{\mathrm{Call},\alpha}(x) = e^{-\alpha x}(e^x - K)^+$$

is in L^1. The damped put payoff

$$f_{\mathrm{Put},\alpha}(x) = e^{-\alpha x}(K - e^x)^+$$

is in L^1 for $\alpha < 0$.

Proposition 7.22. *Let $\alpha \in \mathbb{R}$, $f_\alpha, \hat{f}_\alpha \in L^1$, $\mathbb{E}[S_T^\alpha] < \infty$ (w.r.t. the pricing measure). Then the price of the claim with payoff f at time zero is*

$$\frac{e^{-rT} S_0^\alpha}{\pi} \int_0^\infty e^{-i\xi \log S_0} \varphi_{X_T}(-(\xi + i\alpha)) \hat{f}(\xi + i\alpha) d\xi.$$

Proof. We have

$$\hat{f}(\xi + i\alpha) = \int_{\mathbb{R}} e^{i(\xi + i\alpha)x} f(x) dx = \hat{f}_\alpha(\xi).$$

The computation is then very similar to the case $\alpha = 0$. □

The Fourier transform of the payoff function is usually easy to calculate. For $\alpha > 1$, we have

$$\hat{f}_{\text{Call}}(\xi + i\alpha) = \hat{f}_{\text{Call},\alpha}(\xi) = \int_{\log K}^{\infty} e^{(i\xi-\alpha)x}(e^x - K)dx = \frac{K^{1-\alpha}e^{i\xi \log K}}{(i\xi - \alpha)(i\xi - \alpha + 1)},$$

which is in L^1. For $\alpha < 0$, the same expression gives the Fourier transform of the damped put.

8 Fundamental theorem of asset pricing

Consider a filtered probability space satisfying the usual conditions, and k assets modeled by a k-dimensional semimartingale

$$(Z_t^1, \dots, Z_t^k)_{t \in [0,T]},$$

with Z^k positive. By the Girsanov theorem for semimartingales [10, p. 132], the semimartingale property is preserved under equivalent measure changes. Thus, to achieve freeness of arbitrage, it is standard to assume that prices are semimartingales under the physical measure.

For an appropriate set of trading strategies, the following definitions are analogous to the BS model: discounted price process, value process, self-financing, admissible strategy, arbitrage, EMM. For instance, the value process of a strategy ϕ is

$$V_t(\phi) = \sum_{i=1}^{k} \phi_t^i Z_t,$$

and

$$V_t(\phi) = V_0(\phi) + \sum_{i=1}^{k} \int_0^t \phi_u^i dZ_u^i$$

if ϕ is self-financing.

Let $\mathcal{K}_0$ denote the claims that can be replicated with zero initial capital. We write

$$\mathcal{K} := \mathcal{K}_0 \cap L^\infty$$

for the (essentially) bounded attainable claims. With

$$L_+^\infty := \{X \in L^\infty : X \geq 0 \text{ a.s.}\},$$

we define the cone

$$\mathcal{C} := \mathcal{K} - L_+^\infty = \{X - Y : X \in \mathcal{K}, Y \in L_+^\infty\} \subseteq \mathcal{K}.$$

It can be shown that freeness of arbitrage is equivalent to $\mathcal{C} \cap L^\infty_+$ being trivial:

$$NA \iff \mathcal{C} \cap L^\infty_+ = \{0\}.$$

Thus, the convex sets $\mathcal{C}$ and L^∞_+ intersect only in one point. To achieve this, it would be convenient to find $0 \neq F \in L^1$ such that

$$\begin{aligned} \mathbb{E}[FX] &\geq 0, \quad X \in L^\infty_+, \\ \mathbb{E}[FX] &\leq 0, \quad X \in \mathcal{C}. \end{aligned} \tag{8.1}$$

If $F \in L^1$ satisfying (8.1) exists, it satisfies $F \geq 0$, by considering $X = \mathbf{1}_{\{F<0\}}$. W.l.o.g., $\mathbb{E}[F] = 1$, and we can define $d\mathbb{Q} = Fd\mathbb{P}$. Since $\mathcal{K} \subseteq \mathcal{C}$, we have

$$\mathbb{E}[FX] \leq 0, \quad X \in \mathcal{K},$$

and since $\mathcal{K}$ is a vector space

$$\mathbb{E}[FX] = 0, \quad X \in \mathcal{K}.$$

Two Problems arise: 1. As we mentioned, the existence of F is not trivial. 2. We must have $F > 0$, because otherwise $\mathbb{Q}$ might only be absolutely continuous.

For $F \in L^1$, the map

$$\begin{aligned} L^\infty &\to \mathbb{R} \\ X &\mapsto \mathbb{E}[FX] \end{aligned}$$

is linear and continuous, and in fact L^∞ is the dual space of L^1. On the other hand, the dual of L^∞ is in general strictly larger than L^1, and so we cannot apply the Hahn-Banach theorem.

The following condition is a bit stronger than NA.

Definition 8.1. *The market model satisfies NFLVR (no free lunch with vanishing risk) if $\bar{\mathcal{C}} \cap L^\infty_+ = \{0\}$, where the closure of $\mathcal{C}$ is taken w.r.t. $\|\cdot\|_\infty$.*

The negation of NFLVR is a bit weaker than arbitrage, but “close”: It implies that there is $0 \neq X \in L^\infty_+$ and $X_n \in \mathcal{C}$ with $|X_n - X| \leq 1/n$ a.s., $n \in \mathbb{N}$. In particular, we have $X_n \geq -1/n$.

Theorem 8.2 (Kreps-Yan, 1980, 1981, Delbaen–Schachermayer 1994, 1998)**.** *NFLVR implies that there is $F \in L^1$, $F > 0$ with $\mathbb{E}[FX] \leq 0$, $X \in \mathcal{C}$.*

Corollary 8.3. *If NFLVR holds, then there is an EMM.*

Proof. Let F be as in the theorem, and w.l.o.g. $\mathbb{E}[F] = 1$. Define $d\mathbb{Q}/d\mathbb{P} = F$. We have already shown that $\mathbb{E}_{\mathbb{Q}}[X] = 0$ for $X \in \mathcal{K}$. W.l.o.g., assume $Z^k = 1$ (zero interest rate). Let $s \le t$, $A \in \mathcal{F}_s$, and $i \in \{1, \dots, k\}$. Consider the following strategy: At time s, take a loan and buy $\mathbf{1}_A$ units of Z^i. At time t, sell the units and put the result in the bank account. Then

$$V_t = \mathbf{1}_A(Z_t^i - Z_s^i) = V_T.$$

As the strategy has zero initial cost, we have $V_T \in \mathcal{K}$, and so $\mathbb{E}_{\mathbb{Q}}[\mathbf{1}_A(Z_t^i - Z_s^i)] = 0$. By definition of the conditional expectation, this implies $\mathbb{E}_{\mathbb{Q}}[Z_t^i|\mathcal{F}_s] = Z_s^i$. Thus, $\mathbb{Q}$ is an EMM. $\square$

9 Variance-optimal hedging

The goal is to minimize the variance of the hedging error for some non-replicable claim. We write $\mathcal{M}^2$ for the real-valued martingales on $[0, T]$ with $\mathbb{E}[[M]_T] < \infty$. By the Burkholder-Davis-Gundy inequality,

$$C_1\mathbb{E}[[M]_T] \le \mathbb{E}\Big[\sup_{0\le t\le T} M_t^2\Big] \le C_2\mathbb{E}[[M]_T],$$

this is equivalent to $\mathbb{E}[\sup_{0\le t\le T} M_t^2] < \infty$. With the inner product $(M, N) = \mathbb{E}[M_T N_T]$, $\mathcal{M}^2$ becomes a Hilbert space, isomorphic to $L^2(\mathbb{P})$. M, N are orthogonal if $\mathbb{E}[M_T N_T] = 0$. We define them to be *strongly orthogonal*, if MN is a martingale. For $M, N \in \mathcal{M}_0^2$ (i.e., starting in 0), strong orthogonality implies orthogonality.

Recall that the quadratic covariation of two semimartingales X, Y is defined by the limit in probability

$$[X, Y]_t = \lim \sum_{k=1}^{n} (X_{t_k} - X_{t_{k-1}})(Y_{t_k} - Y_{t_{k-1}})$$

taken over partitions of $[0, t]$ with mesh tending to zero, and satisfies

$$d(XY) = X_- dY + Y_- dX + d[X, Y].$$

Theorem 9.1 (Kunita-Watanabe inequality)**.** *Let X, Y be semimartingales, and θ, ψ be measurable processes. Then we have*

$$\int_0^T |\theta_s||\psi_s| d\mathbb{V}_{[0,s]}([X, Y]) \le \left(\int_0^T \theta_s^2 d[X]_s\right)^{1/2} \left(\int_0^T \psi_s^2 d[Y]_s\right)^{1/2}.$$

Proposition 9.2 (Strong orthogonality). *Let $M, N \in \mathcal{M}^2$ be strongly orthogonal. Then we have*

(i) $[M, N]$ *is a martingale,*

(ii) $\mathbb{E}[M_\tau N_\tau] = M_0 N_0$, *where* $\tau \leq T$ *is a stopping time,*

Conversely, (ii) implies strong orthogonality.

Proof. Since

$$d(MN) = M_- dN + N_- dM + d[M, N],$$

$[M, N]$ is a local martingale. We have

$$|[M, N]_t| \leq \mathbb{V}_{[0,T]}([M, N]) \leq \sqrt{[M]_T [N]_T},$$

where the second inequality follows from Kunita-Watanabe with $\theta = \psi = 1$. By Cauchy-Schwarz,

$$\mathbb{E}\Big[\sqrt{[M]_T [N]_T}\Big] \leq \mathbb{E}\big[[M]_T\big]^{1/2} \mathbb{E}\big[[N]_T\big]^{1/2} < \infty,$$

and so $\sup_{0 \leq t \leq T} |[M, N]|$ is integrable, which implies (i).

(ii) By the optional sampling theorem.

Finally, suppose that (ii) holds. For $t \in [0, T]$ and $\Gamma \in \mathcal{F}_t$, define

$$\tau_\Gamma := \begin{cases} t & \omega \in \Gamma \\ T & \omega \notin \Gamma. \end{cases}$$

Then we have

$$\mathbb{E}[M_t N_t \mathbf{1}_\Gamma] + \mathbb{E}[M_T N_T \mathbf{1}_{\Omega \setminus \Gamma}] = \mathbb{E}[M_{\tau_\Gamma} N_{\tau_\Gamma}] \overset{(ii)}{=} M_0 N_0,$$

and thus

$$\mathbb{E}[M_t N_t \mathbf{1}_\Gamma = \underbrace{M_0 N_0 - \mathbb{E}[M_T N_T]}_{=0} + \mathbb{E}[M_T N_T \mathbf{1}_\Gamma]. \qquad \square$$

Definition 9.3. *For $M \in \mathcal{M}^2_{\text{loc}}$, define*

$$L^2(M) := \Big\{\theta \textit{ predictable}, \mathbb{E}\Big[\int_0^T \theta_s^2 d[M]_s\Big] < \infty\Big\}.$$

This is a Hilbert space w.r.t. to the inner product

$$(\theta, \psi) := \mathbb{E}\Big[\int_0^T \theta_s dM_s \int_0^T \psi_s dM_s\Big].$$

By the Ito isometry, the corresponding norm is

$$\|\theta\|_{L^2(M)} = \mathbb{E}\Big[\int_0^T \theta_s^2 d[M]_s\Big]^{1/2}.$$

The stable subspace generated by M is

$$\mathcal{S}(M) := \Big\{\int_0^\cdot \theta dM : \theta \in L^2(M)\Big\}.$$

Proposition 9.4. *$\mathcal{S}(M)$ is a closed subspace of $\mathcal{M}^2$. For $\int \theta dM \in \mathcal{S}(M)$ and a stopping time $\tau \leq T$, the stopped process $(\int \theta dM)^\tau$ is in $\mathcal{S}(M)$.*

Proof. For $\int \theta dM \in \mathcal{S}(M)$, $[\int \theta dM]_T = \int \theta^2 d[M]$ is integrable, and so $\mathcal{S}(M) \subseteq \mathcal{M}^2$. The map $\theta \mapsto \int \theta dM$ is an isometry from $L^2(M)$ to $\mathcal{S}(M)$, since

$$\Big\|\int_0^\cdot \theta dM\Big\|_{\mathcal{M}^2}^2 = \mathbb{E}\Big[\Big(\int_0^T \theta dM\Big)^2\Big] = \mathbb{E}\Big[\int_0^T \theta_s^2 d[M]_s\Big] = \|\theta\|_{L^2(M)}^2.$$

Thus, $\mathcal{S}(M)$ is isometrically isomorphic to $L^2(M)$, and in particular it is complete.

If τ is a stopping time, then

$$\Big(\int_0^\cdot \theta dM\Big)^\tau = \int_0^\cdot \mathbf{1}_{[0,\tau]}\theta dM. \qquad \square$$

Theorem 9.5 (Kunita-Watanabe decomposition)**.** *Let $M \in \mathcal{M}^2_{\text{loc}}$ and $N \in \mathcal{M}^2$. Then there is a unique representation $N = N_0 + \int \theta dM + L$ with $\theta \in L^2(M)$ with $L \in \mathcal{M}_0^2$ strongly orthogonal to $\mathcal{S}(M)$.*

Proof. $\mathcal{S}(M)$ is a closed subspace of $\mathcal{M}^2$. By orthogonal projection, we find $\theta \in L^2(M)$ with

$$N = \underbrace{\int \theta dM}_{\in \mathcal{S}(M)} + \underbrace{\tilde{L}}_{\in \mathcal{S}(M)^\perp},$$

where the orthogonal complement is w.r.t. $(\cdot,\cdot)_{\mathcal{M}^2}$. With $L := \tilde{L} - N_0$, we have $N = N_0 + \int \theta dM + L$. To show strong orthogonality, we use Proposition 9.2. Let $\theta \in L^2(M)$ and $\tau \leq T$ be a stopping time. Then,

$$\mathbb{E}\Big[L_\tau \int_0^\tau \theta_t dM_t\Big] = \mathbb{E}\Big[L_T \int_0^\tau \theta_t dM_t\Big] = \mathbb{E}\Big[L_T \int_0^T \underbrace{\mathbf{1}_{[0,\tau]}\theta_t}_{\in L^2(M)} dM_t\Big] = 0.$$

The first equality holds because L is a martingale, and the last one by $L \in \mathcal{S}(M)^\perp$. $\square$

Consider now an asset with price process $S \in \mathcal{M}^2_{\mathrm{loc}}$. For simplicity, we assume zero interest rate. For a claim $H \in L^2(\mathcal{F}_T)$, let $V_t := \mathbb{E}[H|\mathcal{F}_t]$, where the expectation is w.r.t. to the pricing measure. For an initial capital c and a strategy $\theta \in L^2(S)$, we obtain the value process $c + \int_0^\cdot \theta dS$. The goal is to find c, θ such that

$$\mathbb{E}\Big[\Big(H - c - \int_0^T \theta_t dS_t\Big)^2\Big] \to \min . \tag{9.1}$$

Theorem 9.6. *Let $V = \mathbb{E}[H] + \int_0^\cdot \theta^H dS + L$ be the Kunita-Watanabe decomposition of V. Then the optimization problem is solved by $c^* = \mathbb{E}[H]$ and $\theta^* = \theta^H$.*

Proof. Define

$$\mathcal{U} := \Big\{c + \int_0^T \theta dS : c \in \mathbb{R}, \theta \in L^2(S)\Big\} \subseteq L^2(\mathcal{F}_T).$$

The solution of (9.1) is the projection of H to $\mathcal{U}$. We have

$$V_T = H = \underbrace{\mathbb{E}[H] + \int_0^T \theta^H dS}_{\in \mathcal{U}} + L_T.$$

Since $L_T \in \mathcal{S}(S)^\perp$, we have $L_T \in \mathcal{U}^\perp$. □

For Lévy models, the quadratic hedging strategy of standard options can be explicitly calculated (See Chapter 6 in [11]).

10 Duality and semi-static hedging

Throughout this section, we consider a price process S with $S_0 = 1$. For most of the results, we assume zero interest rate.

Definition 10.1. *Let $S = e^X$ be a positive $\mathbb{P}$-martingale. Define the share measure by $d\mathbb{P}'/d\mathbb{P} = S_T$. Then the dual process is defined by $S' := 1/S$ (resp. $X' := -X$).*

If S' is integrable, then it is a $\mathbb{P}'$-martingale:

$$\mathbb{E}'[S'_t|\mathcal{F}_s] = \frac{\mathbb{E}[S_T S'_t|\mathcal{F}_s]}{\mathbb{E}[S_T|\mathcal{F}_s]} = \frac{\mathbb{E}[\mathbb{E}[S_T|\mathcal{F}_t] S'_t|\mathcal{F}_s]}{S_s} = \frac{1}{S_s} = S'_s.$$

Theorem 10.2 (Girsanov-Meyer)**.** *Let $\mathbb{P} \sim \mathbb{Q}$, with density process Z, and X be a semimartingale with decomposition $X = M + A$ under $\mathbb{P}$. Then the decomposition of X under $\mathbb{Q}$ is*

$$X = \underbrace{\Big(M - \int \frac{1}{Z} d[Z,M]\Big)}_{\text{local } \mathbb{Q}-\text{martingale}} + \underbrace{\Big(A + \int \frac{1}{Z} d[Z,M]\Big)}_{\text{finite variation}}.$$

Proposition 10.3. *Let $S = e^X$ be a continuous martingale with $S_0 = 1$. Then the canonical decomposition of $X' = -X$ under $\mathbb{P}'$ is given by*

$$X' = (-X + \tfrac{1}{2}[X]) - \tfrac{1}{2}[X].$$

Proof. We have to show that $-X + \frac{1}{2}[X]$ is a local $\mathbb{P}'$-martingale. The martingale S can be written as the stochastic exponential $S = \mathcal{E}(Y)$, where $Y := X + \frac{1}{2}[X]$. We have $dS = SdY$, $d[S,Y] = Sd[Y]$, $dY = (1/S)dS$. By Girsanov, $Y - \int (1/S) d[S,Y]$ is a local $\mathbb{P}'$-martingale. We compute

$$Y - \int \frac{1}{S} d[S,Y] = Y - \int \frac{1}{S} S d[Y] = Y - [Y] = X - \tfrac{1}{2}[X]. \qquad \square$$

In the BS model, we have $X_t = \sigma W_t - \frac{1}{2}\sigma^2 t$ and $Y = \sigma W$.

$$-X + \tfrac{1}{2}[X] = -\sigma W + \tfrac{1}{2}\sigma^2 t + \tfrac{1}{2}\sigma^2 t = -\sigma W + \sigma^2 t$$

is a local $\mathbb{P}'$-martingale. By Girsanov, $\hat{W}_t := -W_t + \sigma t$ is a $\mathbb{P}'$-BM. Since

$$S'_t = e^{-\sigma W_t + \frac{1}{2}\sigma^2 t} = e^{\sigma \hat{W}_t - \frac{1}{2}\sigma^2 t},$$

the dual model is the BS model itself.

In general, we write $C(S,K)$ for the call price, and $C'(S',K)$ for the call price in the dual model.

Proposition 10.4 (put-call duality)**.** *With similar notation for the put, we have*

$$\frac{1}{K} C(S,K) = P'(S', 1/K).$$

Proof.

$$\begin{aligned} C(S,K) &= \mathbb{E}[(S_T - K)^+] = \mathbb{E}'\Big[\frac{1}{S_T}(S_T - K)^+\Big] \\ &= \mathbb{E}'[(1 - KS'_T)^+] = K\mathbb{E}'\Big[\Big(\frac{1}{K} - S_T\Big)^+\Big] = KP'(S', 1/K). \qquad \square \end{aligned}$$

Under some additional assumptions, duality can be used to simplify the prices of exotic options.

Definition 10.5. $(X'_t)_{0\le t\le T}$ *satisfies the dual reflection principle, if* $\sup_{0\le t\le T} X'_t - X'_T$ *and* $-\inf_{0\le t\le T} X'_t$ *have the same law under* $\mathbb{P}'$.

For BM, the dual reflection principle follows from an easy calculation using Lemmas 5.1 and 5.2. More generally, it can be shown that it holds if X' is a $\mathbb{P}'$-Lévy process (Lemma 3.5 in [7]).

Proposition 10.6. *Suppose that* X' *satisfies the dual reflection principle. Then the price of a lookback call with floating strike can be expressed by a simpler lookback option in the dual model.*

Proof. Let $\alpha > 0$. We have

$$\begin{aligned}\mathbb{E}[(S_T - \alpha \inf_{t\le T} S_t)^+] &= \mathbb{E}'[(1-\alpha e^{\inf_{t\le T} X_t - X_T})^+] \\ &= \mathbb{E}'[(1-\alpha e^{X'_T - \sup_{t\le T} X'_t})^+] = \alpha\mathbb{E}'\Big[\Big(\frac{1}{\alpha} - e^{\inf_{t\le T} X'_t}\Big)^+\Big] \\ &= \alpha\mathbb{E}'\Big[\Big(\frac{1}{\alpha} - \inf_{t\le T} S'_t\Big)^+\Big]. \qquad \square\end{aligned}$$

Definition 10.7. $(X'_t)_{0\le t\le T}$ *satisfies the time inversion principle, if the processes* $X'_T - X'_{(T-t)-}$ *and* X'_t *have the same law under* $\mathbb{P}'$.

Again, this holds for Lévy processes.

Proposition 10.8. *If* X' *satisfies the time inversion principle, then the price of an Asian put with floating strike can be expressed by the price of an Asian put with fixed price in the dual model.*

Proof. We have

$$\begin{aligned}\mathbb{E}\Big[\Big(S_T - \frac{1}{T}\int_0^T S_t dt\Big)^+\Big] &= \mathbb{E}'\Big[\Big(1 - \frac{1}{T}\int_0^T \underbrace{\frac{S_t}{S_T}}_{=e^{-(X'_t - X'_T)}} dt\Big)^+\Big] \\ &= \mathbb{E}'\Big[\Big(1 - \frac{1}{T}\int_0^T e^{X'_T - X'_{T-u}} du\Big)^+\Big] \\ &= \mathbb{E}'\Big[\Big(1 - \frac{1}{T}\int_0^T e^{X'_u} du\Big)^+\Big] = \mathbb{E}'\Big[\Big(1 - \frac{1}{T}\int_0^T S'_u du\Big)^+\Big]. \qquad \square\end{aligned}$$

Definition 10.9. *An adapted positive process* S *is quasi-selfdual of order* α *(selfdual if* $\alpha = 1$*), if for every non-negative Borel function* f *and stopping time* τ

$$\mathbb{E}\Big[f\Big(\frac{S_T}{S_\tau}\Big)\Big|\mathcal{F}_\tau\Big] = \mathbb{E}\Big[\Big(\frac{S_T}{S_\tau}\Big)^\alpha f\Big(\frac{S_\tau}{S_T}\Big)\Big|\mathcal{F}_\tau\Big].$$

In particular, $f \equiv 1$ shows that S^α is a martingale.

Proposition 10.10. *S is quasi-selfdual if and only if there is $\alpha \neq 0$ with S^α selfdual.*

Proof. Let S be quasi-selfdual. We show that S^α is selfdual. Let f be as in the definition, and $g(x) := f(x^\alpha)$. We have

$$\begin{aligned}\mathbb{E}\Big[\frac{S_T^\alpha}{S_\tau^\alpha} f\Big(\frac{S_\tau^\alpha}{S_T^\alpha}\Big)\Big|\mathcal{F}_\tau\Big] &= \mathbb{E}\Big[\Big(\frac{S_T}{S_\tau}\Big)^\alpha g\Big(\frac{S_\tau}{S_T}\Big)\Big|\mathcal{F}_\tau\Big] \\ &= \mathbb{E}\Big[g\Big(\frac{S_T}{S_\tau}\Big)\Big|\mathcal{F}_\tau\Big] = \mathbb{E}\Big[f\Big(\frac{S_T^\alpha}{S_\tau^\alpha}\Big)\Big|\mathcal{F}_\tau\Big].\end{aligned}$$

Converse: consider $f(x^{1/\alpha})$. □

We now consider a knock-in barrier option with barrier $H \neq S_0$. Define $\eta := \operatorname{sgn}(H - S_0)$ and the hitting time

$$\tau_H := \inf\{t : \eta S_t \geq \eta H\}.$$

The payoff is $g(S_T)\mathbf{1}_{\{\tau_H \leq T\}}$.

Theorem 10.11. *Let the martingale S be quasi-selfdual of order α, and suppose that $S_{\tau_H} = H$ holds on $\{\tau_H \leq T\}$ (e.g., this holds if S is continuous). Moreover, assume $\mathbb{P}[S_T = H] = 0$. Then, the knock-in barrier option can be hedged semi-statically as follows:*

(1) At time 0, buy the European claim

$$G(S_T) := \left(g(S_T) + \Big(\frac{S_T}{H}\Big)^\alpha g\Big(\frac{H^2}{S_T}\Big)\right)\mathbf{1}_{\{\eta S_T > \eta H\}}$$

(2) If S hits the barrier, exchange the option against the claim $g(S_T)$ (at no cost).

Proof. If $\tau_H > T$, then both options are worthless at time T. Consider $\tau_H \leq T$. We have to show that $\mathbb{E}[g(S_T)|\mathcal{F}_{\tau_H}] = \mathbb{E}[G(S_T)|\mathcal{F}_{\tau_H}]$. We have

$$\begin{aligned}\mathbb{E}[g(S_T)|\mathcal{F}_{\tau_H}] &= \mathbb{E}[g(S_T)\mathbf{1}_{\{\eta S_T > \eta H\}}|\mathcal{F}_{\tau_H}] + \mathbb{E}[g(S_T)\mathbf{1}_{\{\eta S_T < \eta H\}}|\mathcal{F}_{\tau_H}] \\ &= \mathbb{E}[g(S_T)\mathbf{1}_{\{\eta S_T > \eta H\}}|\mathcal{F}_{\tau_H}] + \mathbb{E}\Big[g\Big(\frac{S_T}{H}H\Big)\mathbf{1}_{\{\eta S_T/H < \eta\}}|\mathcal{F}_{\tau_H}\Big] \\ &= \mathbb{E}[g(S_T)\mathbf{1}_{\{\eta S_T > \eta H\}}|\mathcal{F}_{\tau_H}] + \mathbb{E}\Big[\Big(\frac{S_T}{H}\Big)^\alpha g\Big(\frac{H}{S_T}H\Big)\mathbf{1}_{\{\eta H/S_T < \eta\}}|\mathcal{F}_{\tau_H}\Big] \\ &= \mathbb{E}[G(S_T)|\mathcal{F}_{\tau_H}].\end{aligned}$$ □

Definition 10.12. *Let $(M_t)_{0\le t\le T}$ be an adapted process. It is called symmetric, if*

$$\mathbb{E}[f(M_T - M_\tau)|\mathcal{F}_\tau] = \mathbb{E}[f(M_\tau - M_T)|\mathcal{F}_\tau]$$

for every Borel function $f \ge 0$ and stopping time $0 \le \tau \le T$.

If M is integrable, consider the functions $f_1(x) = x^+$, $f_2(x) = x^-$. Then $f_1(x) - f_2(x) = x$, and $\mathbb{E}[M_T - M_\tau|\mathcal{F}_\tau] = \mathbb{E}[M_\tau - M_T|\mathcal{F}_\tau]$ implies $\mathbb{E}[M_T|\mathcal{F}_\tau] = M_\tau$. Thus, M is a martingale.

Theorem 10.13 (Tehranchi)**.** *Let Y be a continuous martingale with $Y_0 = 0$ and such that $S = \mathcal{E}(Y)$ is a martingale. Then S is selfdual if and only if Y is symmetric.*

Proposition 10.14 (symmetric stochastic volatility models)**.** *Suppose*

$$\begin{aligned} dY_t &= \sigma(Y_t, V_t)dB_t \\ dV_t &= \alpha(Y_t, V_t)dt + \beta(Y_t, V_t)dW_t \end{aligned}$$

where B, W are independent BMs, σ, α, β are even functions of Y. Assume that Y and $\mathcal{E}(Y)$ are martingales with $Y_0 = 0$. Moreover, assume uniqueness in law. Then, $S = \mathcal{E}(Y)$ is selfdual.

Proof. We have $d(-Y) = \sigma(-Y, V)d(-B)$ and $dV = \alpha(-Y, V)dt + \beta(-Y, V)dW$. Thus, the weak solutions (Y, V, B, W) and $(-Y, V, -B, W)$ have the same law. Therefore,

$$\mathcal{L}(Y_T - Y_\tau|\mathcal{F}_\tau) = \mathcal{L}(Y_\tau - Y_T|\mathcal{F}_\tau).$$

We conclude that Y is symmetric, and the assertion follows from Theorem 10.13. □

A The Snell envelope

We fix a filtered probability space satisfying the usual conditions. (The filtration will only be needed from Lemma A.6 onwards.) The sup of an uncountable family of random variables need not be measurable, and if it is, it might have undesirable properties:

Example A.1. *consider Lebesgue measure on $[0, 1]$ and the family $X_i = \mathbf{1}_{\{i\}}$, $i \in [0, 1]$. Then $\sup_{i\in I} X_i = 1$, but each X_i is a.s. zero.*

Theorem A.2. *Let $\{X_i : i \in I\}$ be a family of real random variables. Then there exists a random variable X^*, called the essential supremum (*$\operatorname{ess\,sup}$*) of the family, such that* *(i)* $X^* \overset{a.s.}{\geq} X_i$ *for all* $i \in I$,

(ii) If a random variable $\hat{X}$ *satisfies* $\hat{X} \overset{a.s.}{\geq} X_i$ *for all* $i \in I$*, then* $\hat{X} \overset{a.s.}{\geq} X^*$,

(iii) If $I \supseteq J$ *is countable, then* $\sup_{i \in J} X_i \overset{a.s.}{\leq} X^*$.

Proof. By applying a strictly increasing map $\mathbb{R} \to [0,1]$, we may assume that all X_i take values in $[0,1]$.

For $I \supseteq J$ countable, define the random variable $X_J := \sup_{i \in J} X_i$. We claim that the upper bound

$$c := \sup\{\mathbb{E}[X_J] : J \subseteq I \text{ countable}\}$$

is attained. Indeed, take a sequence J_n with $\mathbb{E}[X_{J_n}] \to c$ and define the countable set $J^* := \bigcup_n J_n$. Then

$$\mathbb{E}[X_{J_n}] \leq \mathbb{E}[X_{J^*}] \leq c,$$

and so $\mathbb{E}[X_{J^*}] = c$.

Define $X^* := X_{J^*}$. If (i) does not hold, then there exists $i \in I$ with $\mathbb{P}[X_i > X^*] > 0$. Thus, $J' := J^* \cup \{i\}$ satisfies

$$\mathbb{E}[X_{J'}] > \mathbb{E}[X_{J^*}] = c,$$

contradicting the definition of c.

Now let $\hat{X}$ be as in (ii). Then $\{\hat{X} \geq X^*\} = \bigcap_{i \in J^*} \{\hat{X} \geq X_i\}$ has probability 1.

(iii) follows from (i), since $\{X^* \geq X_J\} = \bigcap_{i \in J} \{X^* \geq X_i\}$. □

Definition A.3. *A family* $\{X_i,\ i \in I\}$ *of real random variables is directed upwards, if*

$$\forall i, j \in I\ \exists k \in I :\ X_i \vee X_j \overset{a.s.}{\leq} X_k$$

Note that the family in Example A.1 is *not* directed upwards. We will often omit "a.s." in the following equalities and inequalities.

Lemma A.4. *If* $\{X_i\}$ *is directed upwards, then there is a sequence* (i_k) *in* I *such that* $X_{i_k} \leq X_{i_{k+1}}$ *and*

$$\operatorname*{ess\,sup}_{i \in I} X_i = \sup_{k \in \mathbb{N}} X_{i_k} = \lim_{k \in \mathbb{N}} X_{i_k}.$$

Proof. In the proof of Theorem A.2, let (j_k) be an enumeration of J^*. Let $i_1 = j_1$, then pick i_2 such that $X_{i_2} \geq X_{i_1} \vee X_{j_2}$, pick i_3 such that $X_{i_3} \geq X_{i_2} \vee X_{j_3}$ and so on. □

Lemma A.5. *Let $\mathcal{G} \subseteq \mathcal{F}$ be an arbitrary sub-σ-algebra.*

(a) We have

$$\mathbb{E}[\operatorname*{ess\,sup}_{i\in I} X_i|\mathcal{G}] \geq \operatorname*{ess\,sup}_{i\in I} \mathbb{E}[X_i|\mathcal{G}].$$

(b) Assume that $\{X_i\}$ is directed upwards. If $X_i \geq 0$ for all $i \in I$, or $\mathbb{E}[\operatorname{ess\,sup}_{i\in I} |X_i|] < \infty$, then

$$\mathbb{E}[\operatorname*{ess\,sup}_{i\in I} X_i|\mathcal{G}] = \operatorname*{ess\,sup}_{i\in I} \mathbb{E}[X_i|\mathcal{G}].$$

Proof. (a) By Theorem A.2 (i), $\mathbb{E}[X^*|\mathcal{G}] \geq \mathbb{E}[X_i|\mathcal{G}]$, $i \in I$, and the assertion follows from Theorem A.2 (ii).

(b) By (a), it suffices to show $\leq$. We have

$$\mathbb{E}[X^*|\mathcal{G}] = \sup_{k\in\mathbb{N}} \mathbb{E}[X_{i_k}|\mathcal{G}] \leq \operatorname*{ess\,sup}_{i\in I} \mathbb{E}[X_i|\mathcal{G}],$$

where the equality follows Lemma A.4 and conditional monotone resp. dominated convergence, and the inequality from Theorem A.2 (iii). $\square$

We write $\mathcal{T}_J$ for the set of stopping times with values in the interval $J \subseteq [0, \infty)$.

Lemma A.6. *Let $(Z_t)_{t\geq 0}$ be adapted, non-negative, and right-continuous, with $\mathbb{E}[\sup_{t\geq 0} Z_t] < \infty$. For any $t \geq 0$, the family $\{\mathbb{E}[Z_\tau|\mathcal{F}_t] : \tau \in \mathcal{T}_{[t,\infty)}\}$ is directed upwards.*

Proof. Note that the integrability of the sup ensures integrability of Z_τ for any random time τ. Let $\tau_1, \tau_2 \in \mathcal{T}_{[t,\infty)}$ and define $X_i := \mathbb{E}[Z_{\tau_i}|\mathcal{F}_t]$, $i = 1, 2$. Define

$$\tau := \tau_1 \mathbf{1}_{\{X_1 \geq X_2\}} + \tau_2 \mathbf{1}_{\{X_1 < X_2\}}.$$

This is a stopping time, since for $u \geq t$

$$\{\tau \leq u\} = \underbrace{\{\tau_1 \leq u\}}_{\in\mathcal{F}_u} \cap \underbrace{\{X_1 \geq X_2\}}_{\in\mathcal{F}_t} \cup \underbrace{\{\tau_2 \leq u\}}_{\in\mathcal{F}_u} \cap \underbrace{\{X_1 < X_2\}}_{\in\mathcal{F}_t} \in \mathcal{F}_u.$$

For $A \in \mathcal{F}_t$, we calculate

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[Z_\tau|\mathcal{F}_t]\mathbf{1}_A] &= \mathbb{E}[Z_\tau \mathbf{1}_A] \\
&= \mathbb{E}[Z_{\tau_1} \mathbf{1}_{A\cap\{X_1 \geq X_2\}} + Z_{\tau_2} \mathbf{1}_{A\cap\{X_1 < X_2\}}] \\
&= \mathbb{E}[\underbrace{\mathbb{E}[Z_{\tau_1}|\mathcal{F}_t]}_{=X_1} \mathbf{1}_{A\cap\{X_1 \geq X_2\}} + \underbrace{\mathbb{E}[Z_{\tau_2}|\mathcal{F}_t]}_{=X_2} \mathbf{1}_{A\cap\{X_1 < X_2\}}] \\
&= \mathbb{E}[(X_1 \vee X_2)\mathbf{1}_A] \geq \mathbb{E}[X_i \mathbf{1}_A] \\
&= \mathbb{E}[\mathbb{E}[Z_{\tau_i}|\mathcal{F}_t]\mathbf{1}_A], \quad i = 1, 2.
\end{aligned}$$

This shows that $\mathbb{E}[Z_\tau|\mathcal{F}_t] \geq X_i$ for $i = 1, 2$. $\square$

We now define the Snell envelope of the process Z.

Theorem A.7. *Let Z be as in Lemma A.6.*

(a) The process

$$U_t := \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{[t,\infty)}} \mathbb{E}[Z_\tau | \mathcal{F}_t]$$

is a supermartingale.

(b) $\mathbb{E}[U_t] = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{[t,\infty)}} \mathbb{E}[Z_\tau]$

(c) Under the usual conditions (which we assume), U has a right-continuous modification. This is the Snell envelope of Z.

Proof. (a) For $0 \le s \le t$, we have

$$\begin{aligned}\mathbb{E}[U_t | \mathcal{F}_s] &= \mathbb{E}[\operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{[t,\infty)}} \mathbb{E}[Z_\tau | \mathcal{F}_t] | \mathcal{F}_s] \\ &= \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{[t,\infty)}} \mathbb{E}[\mathbb{E}[Z_\tau | \mathcal{F}_t] | \mathcal{F}_s] \\ &\le \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{[s,\infty)}} \mathbb{E}[Z_\tau | \mathcal{F}_s] = U_s.\end{aligned}$$

The second equality follows from Lemma A.5 (b), and the inequality from $\mathcal{T}_{[t,\infty)} \subseteq \mathcal{T}_{[s,\infty)}$.

(b) Use Lemma A.5 (b).

(c) It suffices to show that $t \mapsto \mathbb{E}[U_t]$ is right continuous (see Section I.2 in [10]). Let $t \ge 0$ and $t_n \downarrow t$. Since $\mathbb{E}[U_{t_n}] = \mathbb{E}[\mathbb{E}[U_{t_n} | \mathcal{F}_t]] \le \mathbb{E}[U_t]$ by (a), we have $\limsup \mathbb{E}[U_{t_n}] \le \mathbb{E}[U_t]$. For the converse inequality, let $\tau \in \mathcal{T}_{[t,\infty)}$ and $\tau_n := \tau \vee t_n$. Then $Z_\tau = \lim_{n\to\infty} Z_{\tau_n}$. Indeed, on $\{\tau > t\}$ we have $Z_{\tau_n} = Z_\tau$ for large n, and on $\{\tau = t\}$ we have $\tau_n = t_n$ and can use the right continuity of Z. By Fatou's lemma and Theorem A.2 (iii),

$$\mathbb{E}[Z_\tau] \le \liminf_{n\to\infty} \mathbb{E}[Z_{\tau_n}] \le \liminf_{n\to\infty} \operatorname*{ess\,sup}_{\hat\tau \in \mathcal{T}_{[t_n,\infty)}} \mathbb{E}[Z_{\hat\tau}] \overset{\text{(b)}}{=} \liminf_{n\to\infty} \mathbb{E}[U_{t_n}].$$

As this holds for all $\tau \in \mathcal{T}_{[t,\infty)}$, part (b) implies that $\mathbb{E}[U_t] \le \liminf \mathbb{E}[U_{t_n}]$. $\square$

Theorem A.8. *Let $T > 0$ and $(Z_t)_{t\in[0,T]}$ be adapted, non-negative, and right-continuous, with $\mathbb{E}[\sup_t Z_t] < \infty$. Then the Snell envelope of Z is the smallest right continuous supermartingale dominating Z.*

Proof. By definition, U dominates Z. Let V be a right continuous supermartingale such that $V_t \overset{\text{a.s.}}{\ge} Z_t$ for $t \ge 0$. By right continuity, we have a.s. for all $t \ge 0$ $V_t \ge Z_t$. By the optional sampling theorem, for $t \ge 0$,

$$U_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{[t,T]}} \mathbb{E}[\underbrace{Z_\tau}_{\le V_\tau} | \mathcal{F}_t] \le V_t. \qquad \square$$

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