Approximation and Aggregation of Risks by Variants of Panjer's Recursion

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   - Panjer's recursion complemented by weighted convolution
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   - Equivalent stochastic representation of claim numbers
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The collective model of risk theory

Task: Calculate (fast and in a numerically stable way if possible) the distribution of the random sum

$$S = \sum_{n=1}^{N} X_n$$

where

- $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of $\mathbb{N}_0$-valued i.i.d. random variables,
- N is an $\mathbb{N}_0$-valued random variable independent of $\{X_n\}_{n\in\mathbb{N}}$.

Applications:

- N insurance claims with sizes $X_1, X_2, \ldots$
- N credit losses, X_n equals the loss given default minus recovery
- N operational losses with sizes $X_1, X_2, \ldots$

The simple-minded solution

Start with $S_1 := X_1$ and calculate the distribution of

$$S_k := X_1 + \cdots + X_k = S_{k-1} + X_k \qquad \text{for } k \geq 2$$

recursively by convolution (due to independence of S_{k-1} and X_k)

$$\mathbb{P}(S_k = n) = \sum_{j=0}^{n} \mathbb{P}(S_{k-1} = n - j)\,\mathbb{P}(X_k = j), \quad n \in \mathbb{N}_0.$$

For the distribution of $S = \sum_{k=1}^{N} X_k = S_N$, due to independence of N and $\{S_k\}_{k\in\mathbb{N}}$, just sum up

$$\mathbb{P}(S = n) = \sum_{k=0}^{\infty} \mathbb{P}(N = k)\,\mathbb{P}(S_k = n), \quad n \in \mathbb{N}_0.$$

This is numerically stable but **very** time consuming.

More sophisticated approaches

- **Approximations** based on clever use of limit theorems (cf. textbooks on risk theory).
- **Fast Fourier Transform (FFT):** Can be problematic for heavy-tailed distributions (see later).
- **FFT with exponential tilting:** Critical choice of tilting parameter, numerical instabilities are possible (see later).

I will concentrate on:

- **Recursive methods**, in particular variants and extensions involving Panjer's recursion.
  $\rightarrow$ Requires restrictions on the distribution of claim number N.

Panjer class distributions (for claim number N)

Definition

A probability distribution $\{q_n\}_{n\in\mathbb{N}_0}$ is in the Panjer(a, b, k) class with $a, b \in \mathbb{R}$ and $k \in \mathbb{N}_0$ if $q_0 = q_1 = \cdots = q_{k-1} = 0$ and

$$q_n = \left(a + \frac{b}{n}\right) q_{n-1} \qquad \text{for all } n \in \mathbb{N} \text{ with } n \geq k+1.$$

Note: Same distribution on both sides, see later ...

Determination of all distributions:

- $k = 0$: Sundt and Jewell (1981)
- $k = 1$: Willmot (1988)
- General $k \in \mathbb{N}_0$: Hess, Liewald and Schmidt (2002)

Basic Panjer class distributions

- $\mathrm{Bin}(m, p) \in \mathrm{Panjer}(\frac{p}{q}, -\frac{m+1}{q}p, 0)$ with $m \in \mathbb{N}$ and $p \in [0, 1)$
- $\mathrm{Poisson}(\lambda) \in \mathrm{Panjer}(0, \lambda, 0)$ with $\lambda \geq 0$
- $\mathrm{NegBin}(\alpha, p) \in \mathrm{Panjer}(q, (\alpha - 1)q, 0)$ with $\alpha > 0$ and $p \in (0, 1)$
- $\mathrm{Log}(q) \in \mathrm{Panjer}(q, -q, 1)$ with $q \in (0, 1)$ and $q_n = -\frac{q^n}{n \log(1-q)}$ for all $n \in \mathbb{N}$
- Extended logarithmic distribution: Given $k \in \mathbb{N} \setminus \{1\}$ and $q \in (0, 1]$, define $q_0 = \cdots = q_{k-1} = 0$ and

$$q_n = \frac{\binom{n}{k}^{-1} q^n}{\sum_{l=k}^{\infty} \binom{l}{k}^{-1} q^l} \quad \text{for } n \geq k.$$

  $\mathrm{ExtLog}(k, q)$ is in $\mathrm{Panjer}(q, -kq, k)$, has heavy tails for $q = 1$. Closed-form expression for the series is available in our paper.

Basic Panjer class distributions (cont.)

- Extended Negative Binomial Distribution: For $k \in \mathbb{N}$, $\alpha \in (-k, -k+1)$ and $p \in [0,1)$ define $q = 1-p$, $q_0 = \cdots = q_{k-1} = 0$ and

$$q_n = \frac{\binom{\alpha+n-1}{n} q^n}{p^{-\alpha} - \sum_{j=0}^{k-1} \binom{\alpha+j-1}{j} q^j} \quad \text{for } n \geq k.$$

ExtNegBin(α, k, p) is in Panjer$(q, (\alpha-1)q, k)$. It has heavy tails for $q = 1$, which is good for reinsurance companies.

Theorem (Hess, Liewald and Schmidt, 2002)

Let $Q = \{q_n\}_{n\in\mathbb{N}_0}$ be non-degenerate. Then are equivalent:

- *Q is in* Panjer(a, b, k).
- *Q is the k-truncation of a basic* Panjer(a, b, k') *distribution $Q' = \{q'_n\}_{n\in\mathbb{N}_0}$ with $k' \leq k$ and $c := \sum_{n=k}^{\infty} q'_n > 0$, i.e., $q_n = 0$ for $n \in \{0, 1, \ldots, k-1\}$ and $q_n = q'_n / c$ for all $n \geq k$.*

(Extended) Panjer recursion

Theorem (Panjer, 1981; Hess, Liewald and Schmidt, 2002)

Assume that the probability distribution $\{q_n\}_{n\in\mathbb{N}_0}$ *of* N *belongs to the* Panjer(a,b,k) *class and* $a\,\mathbb{P}(X_1=0)\neq 1$. *Then the distribution* $\{p_n\}_{n\in\mathbb{N}_0}$ *of* $S=X_1+\cdots+X_N$ *can be calculated by*

$$p_0=\varphi_N(\mathbb{P}(X_1=0))=\begin{cases} q_0 & \text{if } \mathbb{P}(X_1=0)=0,\\ \mathbb{E}\big[(\mathbb{P}(X_1=0))^N\big] & \text{otherwise,}\end{cases}$$

where $\varphi_N(s)=\sum_{n\in\mathbb{N}_0} q_n s^n$ *is the probability generating function of* N, *and the recursion formula*

$$p_n=\frac{1}{1-a\,\mathbb{P}(X_1=0)}\Bigg(\mathbb{P}(S_k=n)q_k+\sum_{j=1}^{n}\underbrace{\Big(a+\frac{bj}{n}\Big)}_{\geq 0\,?}\mathbb{P}(X_1=j)\underbrace{p_{n-j}}_{\substack{\text{same}\\ \text{dist.}}}\Bigg)$$

for all $n\in\mathbb{N}$, *where* $S_k:=X_1+\cdots+X_k$.

Historical comment on Panjer's recursion

For $\alpha \in \mathbb{R}$ and a power series $f(s) = \sum_{k=0}^{\infty} a_k s^k$ with $a_0 \neq 0$, the coefficients $\{b_n\}_{n\in\mathbb{N}_0}$ of the power series $f^{-\alpha}(s)$ satisfy the recursion

$$b_n = \frac{1}{n a_0} \sum_{k=1}^{n} \bigl((1-\alpha)k - n\bigr) a_k b_{n-k}, \qquad n \in \mathbb{N}.$$

Gould (1974) has traced this remarkable, often rediscovered recurrence back to Euler (1748). Using the probability generating functions of the binomial, negative binomial, and extended negative binomial claim number distributions and $\varphi_S = \varphi_N \circ \varphi_{X_1}$, the above formula applied to $f(s) = 1 - q\varphi_{X_1}(s)$ gives the corresponding Panjer recursions.

Panjer (1981) introduced the recursion to actuarial science.

Numerical stability of Panjer's recursion

Panjer's recursion is certainly numerically stable when

$$a + \frac{bj}{n} \geq 0 \quad \text{for all } j \in \{1, \ldots, n\}.$$

This is the case when $a \geq 0$ and $b \geq -a$, hence for

- Poisson distribution,
- Negative binomial distribution,
- Logarithmic distribution,
- Trancations of the above.

It is potentially unstable for

- Binomial distribution,
- Extended negative binomial distribution,
- Extended logarithmic distribution.

Example for numerical instability due to cancellation

Take $N \sim \mathrm{ExtNegBin}(\alpha, k, p)$ with $k \in \mathbb{N}$, $\varepsilon, p \in (0,1)$ and $\alpha = -k + \varepsilon$. Consider the loss distribution $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = l) = 1/2$ with $l \geq 3$. Then

$$p_{k+l} = q\frac{k(l-1)+\varepsilon k}{k+l}\Bigl(\frac{q_k}{2^{k+1}} + \frac{q_{k+l-1}}{k2^{k+l}}\Bigr) - q\frac{k(l-1)-\varepsilon l}{k+l}\frac{q_k}{2^{k+1}}.$$

With $\varepsilon = 1/10\,000$, $k = 1$, $l = 5$, $p = 1/10$:

$$p_6 = 0.1499926 - 0.1499701 = 0.0000225\,.$$

Panjer's recursion with five significant digits gives

$$p_6 = 0.0000400\ldots$$

Panjer's recursion complemented by weighted convolution

Theorem (Gerhold, S., Warnung, 2010)

Fix $l \in \mathbb{N}$, consider $N \sim \{q_n\}_{n\in\mathbb{N}_0}$ and $\tilde{N}_i \sim \{\tilde{q}_{i,n}\}_{n\in\mathbb{N}_0}$ such that $\tilde{q}_{i,0} = \cdots = \tilde{q}_{i,k+l-i-1} = 0$ for all $i \in \{1,\dots,l\}$ and one $k \in \mathbb{N}_0$. Assume that there exist $a_1,\dots,a_l, b_1,\dots,b_l \in \mathbb{R}$ such that

$$q_n = \sum_{i=1}^{l} \Bigl(a_i + \frac{b_i}{n}\Bigr)\tilde{q}_{i,n-i} \quad \text{for } n \ge k+l.$$

Define $S = X_1 + \cdots + X_N \sim \{p_n\}_{n\in\mathbb{N}_0}$ and $\tilde{S}_{(i)} = X_1 + \cdots + X_{\tilde{N}_i} \sim \{\tilde{p}_{i,n}\}_{n\in\mathbb{N}_0}$ for $i \in \{1,\dots,l\}$. Then $p_0 = \varphi_N(\mathbb{P}(X_1 = 0))$ and, for $n \in \mathbb{N}$,

$$p_n = \sum_{j=1}^{k+l-1} \mathbb{P}(S_j = n)\, q_j + \sum_{i=1}^{l}\sum_{j=0}^{n} \Bigl(a_i + \frac{b_i j}{in}\Bigr)\mathbb{P}(S_i = j)\,\tilde{p}_{i,n-j}.$$

Example: Combination of truncated distributions

Lemma (Gerhold, S., Warnung, 2010)

Fix $k \in \mathbb{N}_0$, $l \in \mathbb{N}$. *For all* $i \in \{1, \dots, l\}$ *assume that* $\alpha_i \geq 0$, $\beta_i \geq -i\alpha_i$ *(at least one* $\neq$*) and that the* $\mathbb{N}_0$*-valued* $\tilde{N}_i$ *satisfies* $\mathbb{P}(\tilde{N}_i < k + l - i) = 0$. *Consider* $q_0, \dots, q_{k+l-1} \geq 0$ *with* $q_0 + \cdots + q_{k+l-1} \leq 1$. *Define*

$$q_n = c \sum_{i=1}^{l} \Bigl(\alpha_i + \frac{\beta_i}{n}\Bigr) \mathbb{P}(\tilde{N}_i = n - i) \quad \text{for } n \geq k + l,$$

$$c = \Bigl(1 - \sum_{n=0}^{k+l-1} q_n\Bigr) \Big/ \sum_{i=1}^{l} \Bigl(\alpha_i + \beta_i \, \mathbb{E}\Bigl[\frac{1}{i + \tilde{N}_i}\Bigr]\Bigr).$$

Then $\{q_n\}_{n \in \mathbb{N}_0}$ *is a probability distribution satisfying the recursion condition of the theorem with* $a_i = c\alpha_i$ *and* $b_i = c\beta_i$ *and the calculation of* $\{p_n\}_{n \in \mathbb{N}_0}$ *is numerically stable.*

Weighted convolution for extended logarithmic dist.

Corollary

Let $k \in \mathbb{N}$ *and* $q \in (0,1)$. *Let* $N \sim \operatorname{ExtLog}(k+1, q)$ *and* $\tilde{N} \sim \operatorname{ExtLog}(k, q)$, *where* $\operatorname{ExtLog}(1, q)$ *means* $\operatorname{Log}(q)$.
Define $S = X_1 + \cdots + X_N$ *and* $\tilde{S} = X_1 + \cdots + X_{\tilde{N}}$.
Then, with an explicit $b_1 > 0$,

$$\mathbb{P}(S = n) = \frac{b_1}{n} \sum_{j=1}^{n} j \, \mathbb{P}(X_1 = j) \, \mathbb{P}(\tilde{S} = n - j), \quad n \in \mathbb{N}.$$

Algorithm (for $\operatorname{ExtLog}(k, q)$, numerically stable, $q \neq 1$)

- *Panjer's recursion for* $N \sim \operatorname{Log}(q)$
- $k - 1$ *weighted convolutions:* $\operatorname{Log}(q) \to \operatorname{ExtLog}(2, q) \to \cdots \to \operatorname{ExtLog}(k-1, q) \to \operatorname{ExtLog}(k, q)$

Numerically stable algorithm for ExtLog$(2, q)$ with $q = 1$

Lemma (Gerhold, S., Warnung, 2010)

Let $N \sim \mathrm{ExtLog}(2,1)$. *For* $S = X_1 + \cdots + X_N$ *we have*

$$\mathbb{P}(S=0) = \mathbb{P}(X_1 = 0) + \mathbb{P}(X_1 \geq 1) \log \mathbb{P}(X_1 \geq 1)$$

with $0 \log 0 := 0$ *and, in the case* $\mathbb{P}(X_1 \geq 1) > 0$,

$$\mathbb{P}(S=n) = \frac{1}{n} \sum_{j=1}^{n} j \, \mathbb{P}(X_1 = j) \, r_{n-j}, \quad n \in \mathbb{N},$$

where $r_0 = -\log \mathbb{P}(X_1 \geq 1)$ *and, recursively for* $n \in \mathbb{N}$,

$$r_n = \frac{1}{\mathbb{P}(X_1 \geq 1)} \left(\mathbb{P}(X_1 = n) + \frac{1}{n} \sum_{j=1}^{n-1} j \, \mathbb{P}(X_1 = n-j) \, r_j \right).$$

Weighted convolution for extended negative binomial dist.

Corollary

Let $k \in \mathbb{N}_0$, $\alpha \in (-k, -k+1)$ *and* $p \in (0,1)$. *Let* $N \sim \mathrm{ExtNegBin}(\alpha - 1, k+1, p)$ *and* $\tilde{N} \sim \mathrm{ExtNegBin}(\alpha, k, p)$, *where* $\mathrm{ExtNegBin}(\alpha, 0, p) := \mathrm{NegBin}(\alpha, p)$. *Let* $S = X_1 + \cdots + X_N$ *and* $\tilde{S} = X_1 + \cdots + X_{\tilde{N}}$. *Then, with an explicit* $b_1 > 0$,

$$\mathbb{P}(S = n) = \frac{b_1}{n} \sum_{j=1}^{n} j\,\mathbb{P}(X_1 = j)\,\mathbb{P}(\tilde{S} = n - j), \quad n \in \mathbb{N}.$$

Algorithm (for $\mathrm{ExtNegBin}(\alpha, k, p)$, numerically stable, $p \neq 0$)

- *Panjer recursion for* $N \sim \mathrm{NegBin}(\alpha + k, p)$
- k *weighted convolutions:*
  $\mathrm{NegBin}(\alpha + k, p) \to \mathrm{ExtNegBin}(\alpha + k - 1, 1, p) \to \cdots \to \mathrm{ExtNegBin}(\alpha + 1, k - 1, p) \to \mathrm{ExtNegBin}(\alpha, k, p)$

Numerically stable algorithm for $\mathrm{ExtNegBin}(\alpha-1,1,0)$

Lemma (Gerhold, S., Warnung, 2010)

Let $N \sim \mathrm{ExtNegBin}(\alpha-1,1,0)$ *with* $\alpha \in (0,1)$. *For* $S = X_1 + \cdots + X_N$ *we have*

$$\mathbb{P}(S=0) = 1 - \big(\mathbb{P}(X_1 \geq 1)\big)^{1-\alpha}$$

and in the non-trivial case $\mathbb{P}(X_1 \geq 1) > 0$

$$\mathbb{P}(S=n) = \frac{1-\alpha}{n}\sum_{j=1}^{n} j\,\mathbb{P}(X_1=j)\,r_{n-j}, \quad n \in \mathbb{N},$$

where $r_0 = \big(\mathbb{P}(X_1 \geq 1)\big)^{-\alpha}$ *and, recursively for* $n \in \mathbb{N}$,

$$r_n = \frac{1}{\mathbb{P}(X_1 \geq 1)}\sum_{j=1}^{n} \frac{n-j+\alpha j}{n}\mathbb{P}(X_1=j)\,r_{n-j}.$$

Application: Poisson–tempered-α-stable mixtures

Definition (τ-tempered α-stable distribution $F_{\alpha,\sigma,\tau}$)

For index $\alpha \in (0,1)$, scale $\sigma > 0$ and tempering $\tau \geq 0$ define

$$F_{\alpha,\sigma,\tau}(y) := \mathbb{E}[e^{-\tau Y} 1_{\{Y \leq y\}}]/\mathbb{E}[e^{-\tau Y}], \quad y \in \mathbb{R}.$$

where Y is α-stable on $[0,\infty)$ with Laplace transform $\mathbb{E}[\exp(-sY)] = \exp(-\gamma_{\alpha,\sigma} s^{\alpha})$ for $s \geq 0$, where $\gamma_{\alpha,\sigma} = \frac{\sigma^{\alpha}}{\cos(\alpha\pi/2)}$.

Theorem (Gerhold, S., Warnung, 2010)

Let $\Lambda \sim F_{\alpha,\sigma,\tau}$ *and* $\mathcal{L}(N|\Lambda) \overset{a.s.}{=} \text{Poisson}(\lambda\Lambda)$ *with* $\lambda > 0$. *Then*

$$N \overset{d}{=} N_1 + \cdots + N_M$$

with independent $M \sim \text{Poisson}(\gamma_{\alpha,\sigma}((\lambda+\tau)^{\alpha} - \tau^{\alpha}))$ *and* $N_m \sim \text{ExtNegBin}\big(-\alpha, 1, \frac{\tau}{\lambda+\tau}\big)$ *for* $m \in \mathbb{N}$.

Application: Poisson–tempered α-stable mixtures (cont.)

Let $\Lambda \sim F_{\alpha,\sigma,\tau}$ and $\mathcal{L}(N|\Lambda) \overset{\text{a.s.}}{=} \text{Poisson}(\lambda\Lambda)$ with $\lambda > 0$. Then the stochastic representation $N \overset{\text{d}}{=} N_1 + \cdots + N_M$ leads to

$$S = \sum_{j=1}^{N} X_j \overset{\text{d}}{=} \sum_{i=1}^{M} \sum_{j=N_1+\cdots+N_{i-1}+1}^{N_1+\cdots+N_i} X_j \overset{\text{d}}{=} \sum_{i=1}^{M} \sum_{j=1}^{N_i} X_{i,j}\,,$$

where $\{X_{i,j}\}_{i,j\in\mathbb{N}}$ are i.i.d. with $X_{i,j} \overset{\text{d}}{=} X_1$.

Algorithm (numerically stable, $\tau \neq 0$)

- *Panjer recursion for* $\tilde{N} \sim \text{NegBin}\big(1-\alpha, \frac{\tau}{\lambda+\tau}\big)$
- *Weighted convolution:* $N_1 \sim \text{ExtNegBin}\big(-\alpha, 1, \frac{\tau}{\lambda+\tau}\big)$
- *Panjer recursion for* $M \sim \text{Poisson}(\gamma_{\alpha,\sigma}((\lambda+\tau)^\alpha - \tau^\alpha))$

If $\tau = 0$, use the special algorithm for $N_1 \sim \text{ExtNegBin}(-\alpha, 1, 0)$.

Examples for τ-tempered $\frac{1}{2}$-stable distributions

Definition (Lévy distribution with scale parameter $\sigma > 0$)

A density of $F_{1/2,\sigma,0}$ is

$$f_{\text{Lévy},\sigma}(x) = \Big(\frac{\sigma}{2\pi x^3}\Big)^{1/2} \exp\Big(-\frac{\sigma}{2x}\Big), \quad x > 0.$$

Definition (inverse Gaussian distribution, parameters $\mu, \tilde{\sigma} > 0$)

Define $\sigma = \mu^2/\tilde{\sigma}^2$ and $\tau = 1/(2\tilde{\sigma}^2)$. A density of $F_{1/2,\sigma,\tau}$ is

$$f_{\text{IG},\mu,\tilde{\sigma}}(x) = \frac{\mu}{\sqrt{2\pi\tilde{\sigma}^2 x^3}} \exp\bigg(-\frac{(x-\mu)^2}{2\tilde{\sigma}^2 x}\bigg), \quad x > 0.$$

Additional examples of probability distributions for the Poisson mixture we can handle

- Generalized τ-tempered α-stable distributions (one additional parameter $m \in \mathbb{N}_0$)
- Inverse gamma distribution (with half-integer shape parameter)
- Generalized inverse Gaussian distribution (with additional half-integer parameter $m + \frac{1}{2}$)

With an additional convolution:

- Reciprocal generalized inverse Gaussian distribution (with additional half-integer parameter $m + \frac{1}{2}$)

Definition of a generalized gamma convolution

These distributions arise as the weak limit of sums of independent gamma distributed random variables.

Definition (cf. Bondesson 1992, Lecture Notes in Statistics)

An (a, U)-generalized gamma convolution (g.g.c.) is a probability distribution F on $\mathbb{R}_+ = [0, \infty)$ with moment generating function

$$M(s) = \int_0^\infty e^{sx} F(dx) = \exp\biggl(as + \int_{(0,\infty)} \ln\Bigl(\frac{t}{t-s}\Bigr)\, U(dt)\biggr),$$

for $s \leq 0$, where $a \geq 0$ and U is a locally finite non-negative measure on $(0, \infty)$ satisfying

$$\int_{(0,1]} |\ln t|\, U(dt) < \infty, \qquad \int_{(1,\infty)} \frac{1}{t}\, U(dt) < \infty.$$

Examples of generalized gamma convolutions

- Consider a finite sum $Y = a + \sum_{j=1}^{n} Y_j$ with $a \geq 0$ of independent random variables with $Y_j \sim \text{Gamma}(\alpha_j, \beta_j)$, $j \in \{1, \ldots, n\}$. Then

$$\mathbb{E}\left[e^{sY}\right] = e^{as} \prod_{j=1}^{n} \left(\frac{\beta_j}{\beta_j - s}\right)^{\alpha_j} = \exp\left(as + \sum_{j=1}^{n} \alpha_j \ln\left(\frac{\beta_j}{\beta_j - s}\right)\right)$$

  for $s < \min\{\beta_1, \ldots, \beta_n\}$, hence $U = \sum_{j=1}^{n} \alpha_j \delta_{\beta_j}$.
- Pareto distribution, τ-tempered α-stable distribution, lognormal distribution, inverse Gaussian distribution, etc.

A closure theorem

This theorem is useful for the construction of an approximation.

Theorem (cf. Bondesson 1992, Thorin 1977)

Let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of (a_n, U_n)-generalized gamma convolutions and F a probability distribution. Then $\{F_n\}_{n\in\mathbb{N}}$ converges weakly to F as $n \to \infty$ and F is an (a, U)-generalized gamma convolution if and only if

1. $U_n \to U$ *vaguely on* $(0, \infty)$ *as* $n \to \infty$,
2. $a = \lim_{A\to\infty} \lim_{n\to\infty} \left(a_n + \int_{(A,\infty)} \frac{1}{t}\, U_n(dt) \right)$,
3. $\lim_{\varepsilon\searrow 0} \limsup_{n\to\infty} \int_{(0,\varepsilon)} |\ln t|\, U_n(dt) = 0$.

Approximation by finite gamma convolutions

Proposition

For every (a, U)-generalized gamma convolution F with $\Lambda \sim F$ there exists a weakly convergent sequence $\{\Lambda_n\}_{n\in\mathbb{N}} \sim \{F_n\}_{n\in\mathbb{N}}$ of (a, U_n)-generalized gamma convolutions with

$$U_n = \sum_{i=1}^{n} \alpha_i^{(n)} \delta_{\beta_i^{(n)}} \,,$$

which converges vaguely to U as $n \to \infty$. Then $F_n = \delta_a \star \mathrm{Gamma}\big(\alpha_1^{(n)}, \beta_1^{(n)}\big) \star \cdots \star \mathrm{Gamma}\big(\alpha_n^{(n)}, \beta_n^{(n)}\big)$.

Representation and convergence of Poisson mixtures

Lemma (Rudolph & S.)

Fix $\lambda > 0$. Let Λ and Λ_n as in the Proposition. For each $n \in \mathbb{N}$ let N_n be a random variable such that $\mathcal{L}(N_n|\Lambda_n) \overset{a.s.}{=} \mathrm{Poisson}(\lambda\Lambda_n)$ and

$$M_n := P + \sum_{j=1}^{n} R_j^{(n)},$$

where $P \sim \mathrm{Poisson}(a)$ and

$$R_j^{(n)} \sim \mathrm{NegBin}\Bigl(\alpha_j^{(n)}, \tfrac{1}{1+\lambda/\beta_j^{(n)}}\Bigr), \quad j \in \{1, \ldots, n\},$$

are independent. Then $M_n \overset{d}{=} N_n$ for all $n \in \mathbb{N}$ and $\{N_n\}_{n\in\mathbb{N}}$ converges weakly to some random variable N satisfying

$$\mathcal{L}(N|\Lambda) \overset{a.s.}{=} \mathrm{Poisson}(\lambda\Lambda)\,.$$

Total variation distance for Poisson–g.g.c. mixtures (I)

Assumption

1. *$\lambda > 0$ and Λ denotes an (a, U)-g.g.c. and Ψ a (b, V)-g.g.c.,*
2. $\int_{(0,T]} \frac{U(dt)}{t+\lambda} \geq \int_{(0,T]} \frac{V(dt)}{t+\lambda}$ *for all $T > 0$,*
3. *the random variables N and M satisfy*

$$\mathcal{L}(N|\Lambda) \overset{a.s.}{=} \text{Poisson}(\lambda\Lambda) \quad \textit{and} \quad \mathcal{L}(M|\Psi) \overset{a.s.}{=} \text{Poisson}(\lambda\Psi)\,,$$

4. *$\{X_i\}_{i\in\mathbb{N}}$ is a sequence of i.i.d. non-negative random variables independent of N and M,*
5. $S := \sum_{i=1}^{N} X_i$ *and* $T := \sum_{i=1}^{M} X_i$.

Remark

Assumption 2 is satisfied if $U((0,T]) \geq V((0,T])$ for all $T > 0$.

Total variation distance for Poisson–g.g.c. mixtures (II)

Theorem (Rudolph & S.)

Let the assumption be satisfied. Then the total variation distance is

$$d_{TV}(\mathcal{L}(S), \mathcal{L}(T)) \leq \frac{3}{2}|\mu' - \nu'| + \frac{\mu - \nu}{2} + \frac{\lambda|a\nu' - b\mu'|}{2\nu'},$$

where

$$\mu = \int_{(0,\infty)} \ln\left(\frac{t+\lambda}{t}\right) U(dt), \quad \nu = \int_{(0,\infty)} \ln\left(\frac{t+\lambda}{t}\right) V(dt),$$

$\mu' = \mu + a\lambda$ *and* $\nu' = \nu + b\lambda$.

Remark

If $a = b = 0$, then $d_{\mathrm{TV}}(\mathcal{L}(S), \mathcal{L}(T)) \leq 2(\mu - \nu)$.

Sketch of the proof

- $d_{\mathrm{TV}}(\mathcal{L}(S), \mathcal{L}(T)) \leq d_{\mathrm{TV}}(\mathcal{L}(M), \mathcal{L}(N))$, see Lemma 3.1 in Chaudhuri and Vellaisamy (1996).
- Representation as compound Poisson distributions: Let $M \sim \mathrm{CPoi}(\nu', H)$ and $N \sim \mathrm{CPoi}(\mu', F)$.
- Applying Corollary 3.2 in Chaudhuri and Vellaisamy (1996) gives

$$d_{\mathrm{TV}}(\mathcal{L}(M), \mathcal{L}(N)) \leq \min\{|\sqrt{\mu'} - \sqrt{\nu'}|, |\mu' - \nu'|\} + \min\{\mu', \nu'\} d_{\mathrm{TV}}(F, H).$$

- Determine the distributions F and H using the moment generating function of Λ and Ψ, respectively.

Approximation quality by finite gamma convolutions

Lemma (Rudolph & S.)

- *For* $\lambda > 0$, $n \in \mathbb{N}$ *and* U *denotes the measure of a g.g.c.*
- $0 = \beta_0 < \beta_1 < \cdots < \beta_n$ *with atoms* $U(\{\beta_1\}), \ldots, U(\{\beta_n\})$
- $\alpha_i = \int_{(\beta_{i-1},\beta_i)} \frac{\beta_i+\lambda}{t+\lambda} U(dt)$ *for* $i \in \{1, \ldots, n\}$
- $V := \sum_{i=1}^{n} \bigl(\alpha_i + U(\{\beta_i\})\bigr)\delta_{\beta_i}$,
- $\nu := \int_0^\infty \ln\bigl(\frac{t+\lambda}{t}\bigr)\, V(dt)$,

the previous theorem is applicable and

$$0 \le \mu - \nu = \sum_{i=1}^{n} \biggl(\int_{(\beta_{i-1},\beta_i)} \ln\Bigl(\frac{t+\lambda}{t}\Bigr)\, U(dt) - \ln\Bigl(\frac{\beta_i+\lambda}{\beta_i}\Bigr)\alpha_i \biggr) + \int_{(\beta_n,\infty)} \ln\Bigl(\frac{t+\lambda}{t}\Bigr)\, U(dt).$$

Algorithm for computing the approximation (I)

For $\varepsilon > 0$ choose $\beta_1, \dots, \beta_n$ such that $\mu - \nu \le \varepsilon$ and use the algorithm given in Gerhold, S. and Warnung (2010):

- $M_n = P + \sum_{i=1}^n R_i$, where $R_i \sim \mathrm{NegBin}\Big(\alpha_i, \frac{1}{1+\lambda/\beta_i}\Big)$
- $R_i = \sum_{j=1}^{P_i} L_{i,j}$ where $P_i \sim \mathrm{Poisson}(\alpha_i \ln(1 + \lambda/\beta_i))$ and $L_{i,j} \sim \mathrm{Log}\Big(\frac{\lambda/\beta_i}{1+\lambda/\beta_i}\Big)$ are Panjer class distributions.
- $S_n = \sum_{i=1}^{M_n} X_i \overset{\mathrm{d}}{=} \sum_{i=1}^{P} X_i + \sum_{i=1}^n \sum_{k=1}^{R_i} X_{i,k}$, and $X_i \overset{\mathrm{d}}{=} X_{i,k}$.
- Let $S_{i,j} = \sum_{k=L_{i,1}+\cdots+L_{i,j-1}+1}^{L_{i,1}+\cdots+L_{i,j}} X_{i,k} \overset{\mathrm{d}}{=} \sum_{k=1}^{L_{i,1}} X_{i,k}$.
- Compute $\mathcal{L}(S_{i,1})$ for $i \in \{1, \dots, n\}$ by numerically stable Panjer recursions.

Algorithm for computing the approximation (II)

The probability generating function of

$$S_n = \sum_{i=1}^{P} X_i + \sum_{i=1}^{n} \sum_{j=1}^{P_i} S_{i,j}$$

is

$$\begin{aligned} G_{S_n}(z) &= \exp\bigl(a(G_{X_1}(z) - 1)\bigr) \prod_{i=1}^{n} \exp\Bigl(\alpha_i \ln(1 + \lambda/\beta_i)\,(G_{S_{i,1}}(z) - 1)\Bigr) \\ &= \exp\bigl((a + \nu)(G(z) - 1)\bigr), \qquad |z| \le 1, \end{aligned}$$

with $\nu = \sum_{i=1}^{n} \alpha_i \ln(1 + \lambda/\beta_i)$ and convex combination

$$G(z) := \frac{a}{a + \nu} G_{X_1}(z) + \sum_{i=1}^{n} \frac{\alpha_i \ln(1 + \lambda/\beta_i)}{a + \nu} G_{S_{i,1}}(z).$$

$G_{S_{i,1}}$ for $i \in \{1, \ldots, n\}$ were computed previously. Conduct another numerically stable Panjer recursion for $\mathrm{CPoi}(a + \nu, F_G)$.

Comparison of distributions obtained by FFT with exponential tilting and iterated Panjer recursion

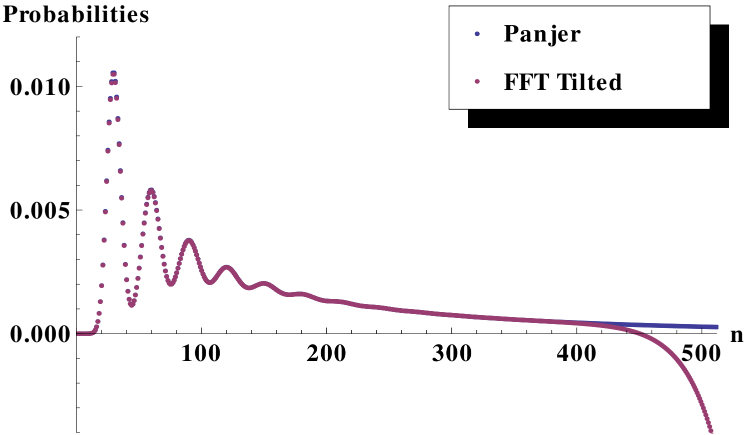

Figure : Approximations of $S = X_1 + \cdots + X_N$, where $X_1 \sim \text{Poisson}(30)$ and $\mathcal{L}(N|\Lambda) \stackrel{\text{a.s.}}{=} \text{Poisson}(20\Lambda)$ with $\Lambda \sim \text{Pareto}(0.5, 2.5)$ and $\varepsilon = 0.01$.

Comparison of FFT and iterated Panjer recursion

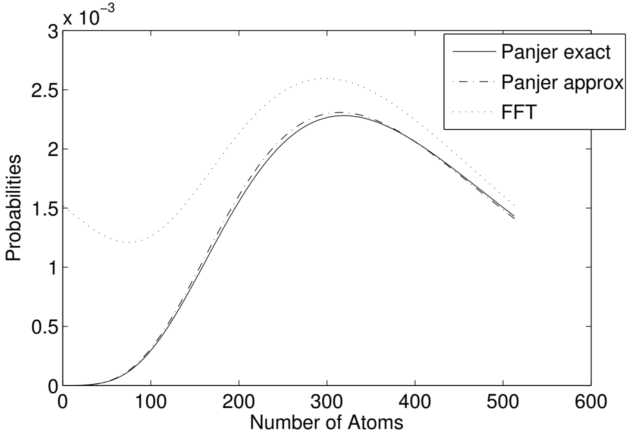

Figure : Approximations of the distribution of $S = X_1 + \cdots + X_N$, where $X_1 \sim \text{Poisson}(10)$ and $\mathcal{L}(N|\Lambda) \overset{\text{a.s.}}{=} \text{Poisson}(30\Lambda)$ with $\Lambda \sim F_{0.4,50,2}$ and $\varepsilon = 0.1$. Here $F_{\alpha,\sigma,\tau}$ denotes a τ-tempered α-stable distribution.

Several dependent collective risk models

New task: Calculate (in a fast and numerically stable way if possible) the distribution of the sum of random sums

$$S = \sum_{j=1}^{N_1} X_{1,j} + \cdots + \sum_{j=1}^{N_m} X_{m,j}$$

where

- the losses $\{X_{i,j}\}_{j\in\mathbb{N}}$ with $i \in \{1,\ldots,m\}$ are independent sequences of $\mathbb{N}_0$-valued i.i.d. random variables,
- $(N_1,\ldots,N_m)$ is an $\mathbb{N}_0^m$-valued random vector, independent of all individual losses but with possibly dependent components.

Remark: Independence is lost, no convolutions of random sums.

Question: Which multivariate distributions for the claim numbers $(N_1,\ldots,N_m)$ can we handle with recursive methods?

Random scenarios of linear dependence

- Let $A_1, \dots, A_k \in [0,\infty)^{m\times n}$ be matrices with non-negative entries describing dependence scenarios.
- Let J be a random variable selecting one of the k scenarios.
- Let $R_1, \dots, R_n$ be independent and non-negative random variables (risk factors), independent of J.
- Define random Poisson intensities by

$$(\Lambda_1, \dots, \Lambda_m)^\top = A_J (R_1, \dots, R_n)^\top.$$

- Let $N = (N_1, \dots, N_m)$ be a random vector with conditionally independent components given $\Lambda_1, \dots, \Lambda_m$ such that

$$\mathcal{L}(N_i \,|\, \Lambda_1, \dots, \Lambda_m) \stackrel{\text{a.s.}}{=} \mathcal{L}(N_i \,|\, \Lambda_i) \stackrel{\text{a.s.}}{=} \text{Poisson}(\lambda_i \Lambda_i)\,,$$

where $\lambda_i \geq 0$ for each $i \in \{1, \dots, m\}$.

Equivalent stochastic representation of claim numbers

Theorem (Rudolph & S.)

Let $\{E^j_{h,l}\}_{h\in\mathbb{N}}$ *for each scenario* $j \in \{1,\dots,k\}$ *and risk* $l \in \{1,\dots,n\}$ *be independent sequences of i.i.d. random vectors, independent of all other random variables, where* $E^j_{1,l} \sim \mathrm{Multinomial}\big(1; \lambda_i a^j_{i,l}/\lambda_{j,l}, i = 1,\dots,m\big)$ *with* $\lambda_{j,l} := \sum_{d=1}^m \lambda_d a^j_{d,l} > 0$. *Define the* $\mathbb{N}_0^m$*-valued random vector*

$$M = \sum_{j=1}^{k} 1_{\{J=j\}} \sum_{l=1}^{n} \sum_{h=1}^{\bar{M}_{j,l}} E^j_{h,l},$$

where $\mathcal{L}\big(\bar{M}_{j,l} \,|\, J, R_1,\dots,R_n\big) \overset{a.s.}{=} \mathcal{L}\big(\bar{M}_{j,l} \,|\, R_l\big) \overset{a.s.}{=} \mathrm{Poisson}(\lambda_{j,l} R_l)$ *for* $j \in \{1,\dots,k\}$ *and* $l \in \{1,\dots,n\}$ *are conditionally independent given* $J, R_1,\dots,R_n$. *Then* M *and* N *have the same distribution.*

Application to recursive algorithms

- Rewrite the random sum using the equivalent representation to restore independence via scenarios and risk factors:

$$S = \sum_{i=1}^{m} \sum_{j=1}^{N_i} X_{i,j} \stackrel{\mathrm{d}}{=} \sum_{j=1}^{k} 1_{\{J=j\}} \sum_{l=1}^{n} \sum_{h=1}^{\bar{M}_{j,l}} X_{h,j,l},$$

  where $\{X_{h,j,l}\}_{h\in\mathbb{N}}$ are independent sequences of i.i.d. random variables such that $X_{1,j,l} \stackrel{\mathrm{d}}{=} \sum_{i=1}^{m} E_{1,l}^{j,i} X_{i,1}$
- For the random sums, if $\mathcal{L}(\bar{M}_{j,l})$ permits, recursive methods are applicable,[1] see the previous sections.
- If every $\mathcal{L}(\bar{M}_{j,l})$ is a compond Poisson distribution, then the $n-1$ convolutions might be replaced by a convex combination and one Panjer recursion.

[1] FFT and FFT with tilting can also be applied.

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