Modelling and Estimation of Stochastic Dependence

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Research areas for stochastic dependence (1)

- Modelling and estimation of dependent credit rating transitions
  $\rightarrow$ Ph. D. thesis of Verena Goldammer (2010)
- Market and credit risk aggregation: a bottom-up approach
  $\rightarrow$ Ph. D. thesis of Robert Schöftner (2010)
- Adapted dependence
  $\rightarrow$ Ph. D. project of Karin Hirhager
  - Relaxing the independence of biometric and financial market risks when estimating the risk of unit-linked life insurance contracts
  - Modelling consumer behaviour dependent on financial market development (related to American option)
    $\rightarrow$ Variable annuities

Research areas for stochastic dependence (2)

- Generalization of Panjer's recursion for dependent claim numbers (collective model, CreditRisk^{+})
  $\rightarrow$ Ph. D. project of Cordelia Rudolph
- Joint term-structure models for credit spreads and risk-free interest rates
  $\rightarrow$ Ph. D. project of Sühan Altay
  We aim for
  - Non-negative interest rates and credit spreads,
  - Negative covariation between them,
  - Zero-coupon bond prices easy to calculate.
- Asymptotic variance of estimators of dependence (linear correlation, Kendall's tau)
  $\rightarrow$ mainly the Ph. D. thesis of Barbara Dengler (2010)

Outline

Linear correlation coefficient

Definition

The linear correlation coefficient for a random vector (X, Y) with non-zero finite variances is defined as

$$\varrho = \frac{\mathbb{C}\mathrm{ov}[X, Y]}{\sqrt{\mathbb{V}\mathrm{ar}[X]}\sqrt{\mathbb{V}\mathrm{ar}[Y]}}.$$

Estimator

The standard estimator for a sample $(X_1, Y_1), \ldots, (X_n, Y_n)$ is

$$\hat{\varrho}_n = \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Y_i - \bar{Y}_n)}{\sqrt{\sum_{i=1}^n (X_i - \bar{X}_n)^2}\sqrt{\sum_{i=1}^n (Y_i - \bar{Y}_n)^2}}$$

where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ and $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^n Y_i$.

Asymptotic behaviour of the standard estimator

Theorem (Asymptotic normality, e.g. Witting/Müller-Funk '95, p. 108)

For an i. i. d. sequence of non-degenerate real-valued random variables (X_j, Y_j), $j \in \mathbb{N}$, with $\mathbb{E}[X^4] < \infty$ and $\mathbb{E}[Y^4] < \infty$, the standard estimators $\hat{\varrho}_n$, normalized with $\sqrt{n}$, are asymptotically normal,

$$\sqrt{n}\,(\hat{\varrho}_n - \varrho) \xrightarrow{\text{d}} \mathcal{N}(0, \sigma_\varrho^2), \quad n \to \infty .$$

The asymptotic variance is

$$\sigma_\varrho^2 = \Bigl(1 + \frac{\varrho^2}{2}\Bigr)\frac{\sigma_{22}}{\sigma_{20}\,\sigma_{02}} + \frac{\varrho^2}{4}\Bigl(\frac{\sigma_{40}}{\sigma_{20}^2} + \frac{\sigma_{04}}{\sigma_{02}^2} - \frac{4\,\sigma_{31}}{\sigma_{11}\,\sigma_{20}} - \frac{4\,\sigma_{13}}{\sigma_{11}\,\sigma_{02}}\Bigr),$$

where $\sigma_{kl} := \mathbb{E}[(X - \mu_X)^k (Y - \mu_Y)^l]$, $\mu_X := \mathbb{E}[X]$, $\mu_Y := \mathbb{E}[Y]$.

Kendall's tau

Definition

Kendall's tau for a random vector (X, Y) is defined as

$$\tau = \mathbb{P}[\underbrace{(X - \widetilde{X})(Y - \widetilde{Y}) > 0}_{\text{concordance}}] - \mathbb{P}[\underbrace{(X - \widetilde{X})(Y - \widetilde{Y}) < 0}_{\text{discordance}}]$$
$$= \mathbb{E}[\,\mathrm{sgn}(X - \widetilde{X})\,\mathrm{sgn}(Y - \widetilde{Y})\,],$$

where $(\widetilde{X}, \widetilde{Y})$ is an independent copy of (X, Y).

Estimator (Representation as U-statistic)

The tau-estimator for a sample $(X_1, Y_1), \ldots, (X_n, Y_n)$ is

$$\hat{\tau}_n = \binom{n}{2}^{-1} \sum_{1 \le i < j \le n} \mathrm{sgn}(X_i - X_j)\,\mathrm{sgn}(Y_i - Y_j)\,.$$

U-statistics

Definition

Fix $m \in \mathbb{N}$. For $n \geq m$ let $Z_1, \ldots, Z_n$ be random variables taking values in the measurable space $(\mathcal{Z}, \mathfrak{Z})$ and let $\kappa : \mathcal{Z}^m \to \mathbb{R}$ be a symmetric measurable function. The U-statistic $\hat{U}_n(\kappa)$ belonging to the kernel κ of degree m is defined as

$$\hat{U}_n(\kappa) := \binom{n}{m}^{-1} \sum_{1 \leq i_1 < \cdots < i_m \leq n} \kappa(Z_{i_1}, \ldots, Z_{i_m}) \,.$$

The tau-estimator is a U-statistic with kernel κ_τ of degree 2:

$$\kappa_\tau : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \,,$$
$$\kappa_\tau\big((x, y), (x', y')\big) = \operatorname{sgn}(x - x') \operatorname{sgn}(y - y') \,.$$

Properties of the tau-estimator

If the observations are i. i. d., then $\hat{\tau}_n$ is an unbiased estimate of τ.

Theorem (Asymptotic normality, e.g. Borovskikh '96)

For an i. i. d. sequence of $\mathbb{R}^2$-valued random vectors, the tau-estimators $\hat{\tau}_n$, normalized with $\sqrt{n}$, are asymptotically normal,

$$\sqrt{n}\,(\hat{\tau}_n - \tau) \xrightarrow{\mathrm{d}} \mathcal{N}\big(0, \sigma_\tau^2\big), \quad n \to \infty\,.$$

The asymptotic variance is

$$\sigma_\tau^2 = 4\,\mathbb{V}\mathrm{ar}\big[\mathbb{E}[\operatorname{sgn}(X - \widetilde{X})\operatorname{sgn}(Y - \widetilde{Y})\,|\,X, Y\,]\big]\,,$$

where $(\widetilde{X}, \widetilde{Y})$ is an independent copy of (X, Y).

Applications of asymptotic variance

- Asymptotic normality leads to asymptotic confidence intervals of the form

$$\left[\hat{\tau}_n - \frac{\sigma_\tau}{\sqrt{n}}\, u_{\frac{1+\alpha}{2}},\ \hat{\tau}_n + \frac{\sigma_\tau}{\sqrt{n}}\, u_{\frac{1+\alpha}{2}}\right]$$

  for given confidence level $\alpha \in (0, 1)$, where $u_{\frac{1+\alpha}{2}}$ is the corresponding quantile of the standard normal distribution.

- This allows in particular to test for dependence.
- Estimators can be evaluated by their asymptotic variance and different ways of estimation can be compared, e.g. for elliptical distributions.

Definition of a copula and Sklar's theorem

Definition

A two-dimensional copula C is a distribution function on $[0,1]^2$ with uniform marginal distributions.

Let (X, Y) be an $\mathbb{R}^2$-valued random vector with marginal distribution functions F and G. Then, by Sklar's theorem, there exists a copula C such that

$$\mathbb{P}[X \le x, Y \le y] = C\big(F(x), G(y)\big), \quad x, y \in \mathbb{R}.$$

If the marginal distribution functions F and G are continuous, then Sklar's theorem also gives uniqueness of the copula C.

Kendall's tau and asymptotic variance for copulas

Assume that X and Y have continuous distribution functions. Then

$$U := F(X) \quad \text{and} \quad V := G(Y)$$

are uniformly distributed on $[0, 1]$ and Kendall's tau becomes

$$\tau = 4\,\mathbb{E}[\,C(U, V)] - 1\,.$$

Theorem (Dengler/Schmock)

The asymptotic variance for the tau-estimators is

$$\sigma_\tau^2 = 16\,\mathbb{V}\mathrm{ar}[2\,C(U, V) - U - V]\,.$$

Note: Both quantities depend only on the copula C.

Examples of copulas for calculating the asymptotic variance for the tau-estimators

- Archimedean copulas
  - Product (independence) copula
  - Clayton copula
  - Ali–Mikhail–Haq copula
- Non-Archimedean copulas
  - Farlie–Gumbel–Morgenstern copula
  - Marshall–Olkin copula

Archimedean copulas

- An Archimedean copula is defined by a generator, i.e., by a continuous, strictly decreasing and convex function $\varphi : [0,1] \to [0,\infty]$ with $\varphi(1) = 0$.
- The pseudo-inverse $\varphi^{[-1]}$ of φ is given by

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t) & \text{for } t \in [0, \varphi(0)]\,, \\ 0 & \text{for } t \in (\varphi(0), \infty]\,. \end{cases}$$

- The copula is defined as

$$C(u,v) = \varphi^{[-1]}(\varphi(u) + \varphi(v))\,, \quad u, v \in [0,1]\,.$$

- If $\varphi(0) = \infty$, then the generator φ and its copula C are called strict.

Product copula

$$C^{\perp} : [0,1]^2 \to [0,1]$$
$$C^{\perp}(u,v) = u\,v$$

- Copula for two independent random variables, $\tau^{\perp} = 0$.
- The product copula is a strict Archimedean copula with generator $\varphi(t) = -\log t$ for $t \in [0,1]$.
- Asymptotic variance of the tau-estimator:

$$(\sigma_{\tau}^{\perp})^2 = \frac{4}{9}$$

Clayton copula with parameter $\theta \in (0, \infty)$

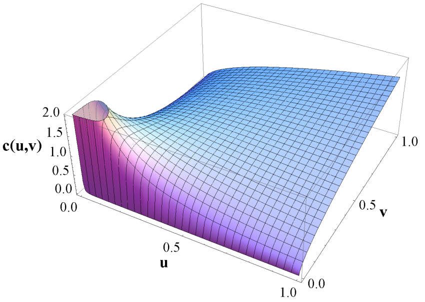
$$C^{\mathrm{Cl},\theta}(u,v) = \begin{cases} \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta} & \text{for } u, v \in (0,1]\,, \\ 0 & \text{otherwise} \end{cases}$$

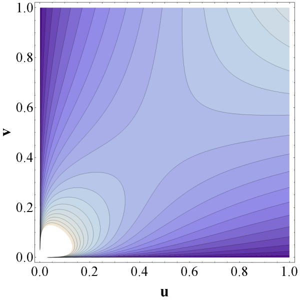
- The Clayton copula is a strict Archimedean copula with generator $\varphi(t) = \frac{1}{\theta}\left(t^{-\theta} - 1\right)$ for $t \in [0,1]$.
- Kendall's tau is $\tau^{\mathrm{Cl},\theta} = \frac{\theta}{\theta+2} \in (0,1)$.
- Asymptotic variance of the tau-estimator for $\theta \in \{1,2\}$:

$$\left(\sigma_\tau^{\mathrm{Cl},1}\right)^2 = \frac{16}{9}\left(6\pi^2 - 59\right) \approx 0.387$$

$$\left(\sigma_\tau^{\mathrm{Cl},2}\right)^2 = \frac{337}{15} - 32\log(2) \approx 0.286$$

Clayton copula, density and results





$$\tau = \frac{2}{9}, \quad \theta = \frac{2\tau}{1-\tau} = \frac{4}{7}, \quad \left(\sigma_\tau^{\mathrm{Cl},\theta}\right)^2 \approx 0.430$$

Note: An estimate for τ gives an estimate for the parameter θ.

Ali–Mikhail–Haq copula with parameter $\theta \in [-1, 1)$

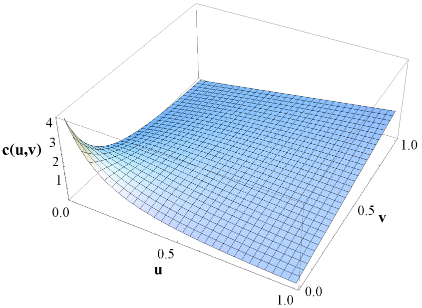
$$C^{\mathrm{AMH},\theta}(u,v) = \frac{u\,v}{1-\theta\,(1-u)\,(1-v)}\,, \quad u,v \in [0,1]$$

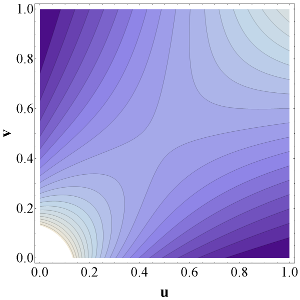
- The AMH copula is a strict Archimedean copula with generator $\varphi(t) = \log\bigl(\frac{1-\theta\,(1-t)}{t}\bigr)$ for $t \in [0,1]$.
- Product copula corresponds to $\theta = 0$.
- Results for $\theta \neq 0$ (with Li_2 denoting the dilogarithm):

$$\tau^{\mathrm{AMH},\theta} = \frac{3\theta-2}{3\theta} - 2\frac{(1-\theta)^2}{3\theta^2}\log(1-\theta)$$

$$\bigl(\sigma_\tau^{\mathrm{AMH},\theta}\bigr)^2 = -\frac{100}{9} - 8\frac{4-(\theta^2+9\theta+2)\,\tau^{\mathrm{AMH},\theta}}{\theta(1-\theta)} + 4\bigl(\tau^{\mathrm{AMH},\theta}\bigr)^2 + 32\frac{\theta+1}{\theta^2}\,\mathrm{Li}_2(\theta)$$

Ali–Mikhail–Haq copula, density and results





$$\tau = \frac{2}{9}, \quad \theta \approx 0.77152, \quad \left(\sigma_\tau^{\mathrm{AMH},\theta}\right)^2 \approx 0.399$$

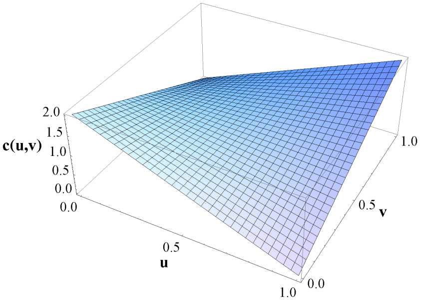
Farlie–Gumbel–Morgenstern copula with $\theta \in [-1, 1]$

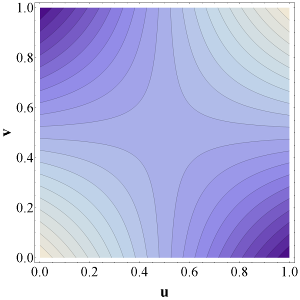
$$C^{\mathrm{FGM},\theta}(u, v) = u\,v + \theta\,u\,v\,(1 - u)\,(1 - v)\,, \quad u, v \in [0, 1]$$

- Kendall's tau is $\tau^{\mathrm{FGM},\theta} = \frac{2\theta}{9} \in [-\frac{2}{9}, \frac{2}{9}]$.
- Asymptotic variance of the tau-estimator:

$$\left(\sigma_\tau^{\mathrm{FGM},\theta}\right)^2 = \frac{4}{9} - \frac{46}{25}\left(\tau^{\mathrm{FGM},\theta}\right)^2$$

Farlie–Gumbel–Morgenstern copula, density and results





$$\tau = \frac{2}{9}, \quad \theta = \frac{9}{2}\tau = 1, \quad \left(\sigma_\tau^{\mathrm{FGM},\theta}\right)^2 = \frac{716}{2025} \approx 0.354$$

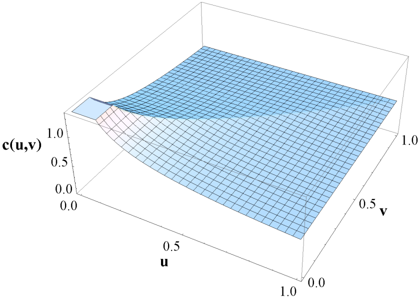
Marshall–Olkin copula with parameters $\alpha, \beta \in (0,1)$

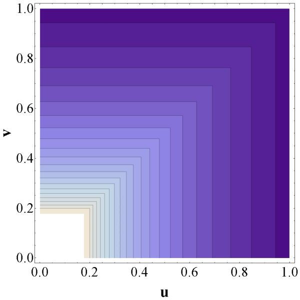
$$C^{\mathrm{MO}}_{\alpha,\beta}(u,v) = \min\{u^{1-\alpha}\, v, u\, v^{1-\beta}\}\,, \quad u,v \in [0,1]$$

- Kendall's tau is $\tau^{\mathrm{MO}}_{\alpha,\beta} = \frac{\alpha\beta}{\alpha+\beta-\alpha\beta} \in (0,1)$.
- Asymptotic variance of the tau-estimator:

$$\left(\sigma_\tau^{\mathrm{MO},\alpha,\beta}\right)^2 = \frac{64\,(\alpha+\beta+\alpha\beta)}{9\,(\alpha+\beta-\alpha\beta)} - \frac{32\,(2\alpha+3\beta+\alpha\beta)}{3\,(2\alpha+3\beta-2\alpha\beta)} - \frac{32\,(3\alpha+2\beta+\alpha\beta)}{3\,(3\alpha+2\beta-2\alpha\beta)} + \frac{16\,(\alpha+\beta)}{(2\alpha+2\beta-\alpha\beta)} + \frac{8\,\alpha\beta}{\alpha+\beta-\alpha\beta} - \frac{4\,\alpha^2\beta^2}{(\alpha+\beta-\alpha\beta)^2} + \frac{20}{3}$$

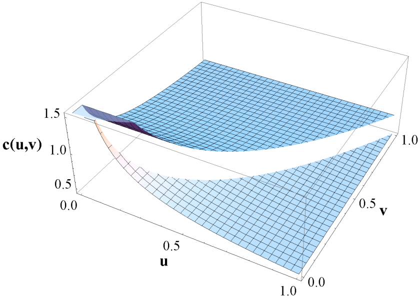
Marshall–Olkin copula, density and results (1)

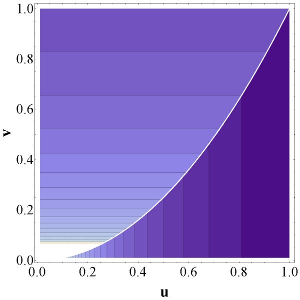




$$\tau = \frac{2}{9}, \quad \alpha = \beta = \frac{4}{11}, \quad \left(\sigma_\tau^{\mathrm{MO},\alpha,\beta}\right)^2 \approx 0.538$$

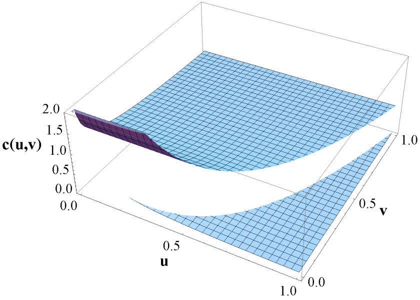
Marshall–Olkin copula, density and results (2)

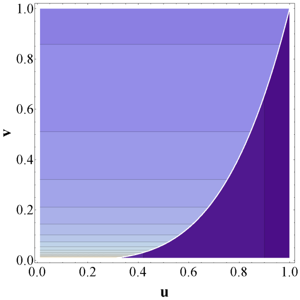




$$\tau = \frac{2}{9}, \quad \alpha = \frac{6}{11}, \quad \beta = \frac{\alpha}{2} = \frac{3}{11}, \quad \left(\sigma_\tau^{\text{MO},\alpha,\beta}\right)^2 \approx 0.505$$

Marshall–Olkin copula, density and results (3)





$$\tau = \frac{2}{9}, \quad \alpha = \frac{10}{11}, \quad \beta = \frac{\alpha}{4} = \frac{5}{22}, \quad \left(\sigma_\tau^{\mathrm{MO},\alpha,\beta}\right)^2 \approx 0.429$$

Spherical distributions

Definition

$X = (X_1, \dots, X_d)^\top$ is spherically distributed if it has the stochastic representation

$$X \overset{\mathrm{d}}{=} RS\,,$$

where

1. S is uniformly distributed on the $(d-1)$-dimensional unit sphere $\mathcal{S}^{d-1} = \{s \in \mathbb{R}^d : s^\top s = 1\}$, and
2. $R \geq 0$ is a radial random variable, independent of S.

Note: A spherical distribution is invariant under orthogonal transformations.

Elliptical distributions

Definition

$X = (X_1, \dots, X_d)^\top$ is elliptically distributed with location vector μ and dispersion matrix Σ, if there exist $k \in \mathbb{N}$, a matrix $A \in \mathbb{R}^{d \times k}$ with $AA^\top = \Sigma$, and random variables R, S satisfying

$$X \stackrel{\mathrm{d}}{=} \mu + RAS\,,$$

where

1. S is uniformly distributed on the unit sphere $\mathcal{S}^{k-1} = \{s \in \mathbb{R}^k : s^\top s = 1\}$, and
2. $R \geq 0$ is a radial random variable, independent of S.

Note: An elliptical distribution is an affine transformation of a spherical distribution.

Linear correlation and standard estimator for non-degenerate elliptical distributions

The (generalized) linear correlation coefficient is defined by

$$\varrho = \frac{\Sigma_{12}}{\sqrt{\Sigma_{11}\,\Sigma_{22}}}\,.$$

Theorem (Dengler/Schmock)

For elliptical distributions the asymptotic variance of the standard estimator simplifies to

$$\sigma_\varrho^2 = \frac{\mathbb{E}[R^4]}{2\,\mathbb{E}[R^2]^2}\,\left(\varrho^2 - 1\right)^2,$$

provided the radial variable R satisfies $0 < \mathbb{E}[R^4] < \infty$.

Connection between the linear correlation coefficient and Kendall's tau for elliptical distributions

Theorem (Lindskog/McNeil/Schmock, 2003)

Let $(X,Y)^\top$ be elliptically distributed with non-degenerate components. Define

$$a_X = \sum_{x\in\mathbb{R}} \bigl(\mathbb{P}[X = x]\bigr)^2,$$

where the sum extends over all atoms of the distribution of X. Then

$$\tau = \frac{2(1-a_X)}{\pi} \arcsin \varrho\,.$$

Transformation of Kendall's tau into an alternative linear correlation estimator

- Define the transformed tau-estimator by

$$\hat{\varrho}_{\tau,n} = \sin\left(\frac{\pi}{2(1-a_X)}\,\hat{\tau}_n\right).$$

If the random variables are non-degenerate, then $\hat{\varrho}_{\tau,n}$ is an estimator for the (generalized) linear correlation ϱ.

- The asymptotic distribution remains normal,

$$\sqrt{n}\,(\hat{\varrho}_{\tau,n} - \varrho) \xrightarrow{\text{d}} \mathcal{N}\big(0, \sigma^2_{\varrho(\tau)}\big), \quad n \to \infty,$$

with

$$\sigma^2_{\varrho(\tau)} = \frac{\pi^2}{4(1-a_X)^2}\,\sigma^2_\tau\,(1-\varrho^2).$$

(e.g. Lehmann/Casella '98, p. 58)

Asymptotic variance for spherical distributions

- Formula for the asymptotic variance of the tau-estimator:

$$\sigma_\tau^2 = 4\,\mathbb{V}\mathrm{ar}\big[\mathbb{E}[\operatorname{sgn}(X-\widetilde{X})\operatorname{sgn}(Y-\widetilde{Y})\,|\,X,Y]\big]\,,$$

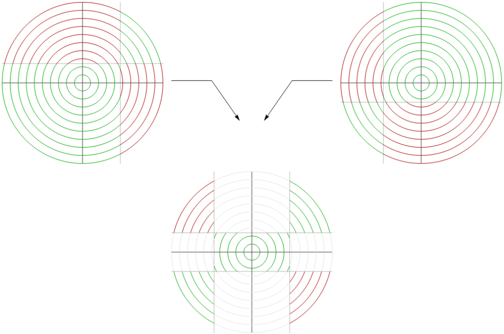
where $(\widetilde{X},\widetilde{Y})$ is an independent copy of (X,Y).

- For two random variables (X,Y) with joint spherical density f, this formula can be simplified to $(\tau=0)$

$$\sigma_\tau^2 = 4\int_{\mathbb{R}^2}\left(4\int_0^{|y|}\int_0^{|x|} f(u,v)\,du\,dv\right)^2 f(x,y)\,d(x,y)\,.$$

Formula for the asymptotic variance for spherical distributions (idea of proof)

$$\sigma_\tau^2 = 4\,\mathbb{E}\big[\,\mathbb{E}[\,\mathrm{sgn}(X-\widetilde{X})\ \mathrm{sgn}(Y-\widetilde{Y})\,|\,X,Y\,]^2\,\big]$$



$$\sigma_\tau^2 = 4\int_{\mathbb{R}^2}\Big(4\int_0^{|y|}\int_0^{|x|} f(u,v)\,du\,dv\Big)^2 f(x,y)\,d(x,y)$$

Normal variance mixture distributions

Definition

$X = (X_1, \ldots, X_d)^\top$ has a normal variance mixture distribution with location vector μ and dispersion matrix Σ, if there exist $k \in \mathbb{N}$, a matrix $A \in \mathbb{R}^{d \times k}$ with $AA^\top = \Sigma$, and random variables W, Z satisfying

$$X \stackrel{\mathrm{d}}{=} \mu + \sqrt{W} AZ,$$

with

1. Z a k-dimensional standard normally distributed random vector, and
2. $W \geq 0$, a radial random variable, independent of Z.

Asymptotic variance of the tau-estimator for standard normal variance mixture distributions

Theorem (Dengler/Schmock)

For a two-dimensional standard normal variance mixture distribution with mixing distribution function G satisfying $G(0) = 0$, the asymptotic variance of the tau-estimator simplifies to

$$\sigma_\tau^2 = \frac{16}{\pi^2} \iiint_{(0,\infty)^3} \arctan^2\Bigl(\frac{\sqrt{\upsilon\xi}}{\sqrt{\zeta}\,\sqrt{\upsilon+\xi+\zeta}}\Bigr)\, dG(\upsilon)\, dG(\xi)\, dG(\zeta)\,.$$

Standard normal distribution

The asymptotic variance of the standard estimator is slightly better than the asymptotic variance of the transformed tau-estimator:

$$\sigma_{\varrho}^2 = 1 \qquad \text{versus} \qquad \sigma_{\varrho(\tau)}^2 = \frac{\pi^2}{4}\sigma_\tau^2 = \frac{\pi^2}{9} \approx 1.097\,,$$

because $(\sigma_\tau^\perp)^2 = 4/9$ for the product copula and also

$$\sigma_\tau^2 = \frac{16}{\pi^2}\arctan^2\frac{1}{\sqrt{3}} = \frac{4}{9}$$

by the previous theorem applied to $G = 1_{[1,\infty)}$.

Student's t-distribution

Definition

A d-dim. t-distribution with location μ, dispersion matrix Σ, and $\nu > 0$ degrees of freedom is defined as the corresponding normal variance mixture distribution, where the mixing random variable W has the inverse Gamma distribution $\mathrm{Ig}(\frac{\nu}{2}, \frac{\nu}{2})$.

For the 2-dim. case with non-degenerate marginal distributions:

- Asymptotic variance of the standard estimator ($\nu > 4$):

$$\sigma_\varrho^2 = \left(1 + \frac{2}{\nu - 4}\right) \left(1 - \varrho^2\right)^2 .$$

- Asymptotic variance of the tau-estimator if $\varrho = 0$ ($\nu > 0$):

$$\sigma_\tau^2 = \frac{32\,\Gamma(\frac{3\nu}{2})}{\pi^2\,\Gamma^3(\frac{\nu}{2})} \int_0^\infty u^{\nu-1} \arctan^2 u \int_0^1 t^{\nu-1} \frac{(1-t)^{\nu-1}}{(u^2+t)^\nu}\, dt\, du .$$

Asymptotic variance for the uncorrelated t-distribution

Theorem (Dengler/Schmock)

For a two-dimensional uncorrelated t-distribution with $\nu \in \mathbb{N}$ degrees of freedom, the asymptotic variance of the tau-estimator has the following representation:

(i) If ν is odd, then

$$\sigma_\tau^2 = \frac{16}{\pi^2}\log^2(2) + \frac{32\,\Gamma(\frac{3\nu}{2})}{\pi\,\Gamma^3(\frac{\nu}{2})}\sum_{k=0}^{\nu-1}\frac{(-1)^{\frac{\nu-1}{2}+k}}{\nu+2k}\binom{\nu-1}{k}\binom{\nu+k-1}{k}$$
$$\times\sum_{h=1}^{\frac{\nu-1}{2}+k}\frac{1}{h}\biggl(\log(2)+\sum_{l=1}^{2h}\frac{(-1)^l}{l}\biggr);$$

(ii) If ν is even, then

$$\sigma_\tau^2 = \frac{32\,\Gamma(\frac{3\nu}{2})}{\pi^2\,\Gamma^3(\frac{\nu}{2})}\sum_{k=0}^{\nu-1}\frac{(-1)^{\frac{\nu}{2}+k-1}}{\nu+2k}\binom{\nu-1}{k}\binom{\nu+k-1}{k}$$
$$\times\sum_{l=\nu/2}^{\nu/2+k-1}\biggl(\frac{\pi^2}{4(l+1)}-\frac{1}{2l+1}\Bigl(\frac{\pi^2}{3}+\sum_{n=1}^{l}\frac{1}{n^2}\Bigr)\biggr).$$

Asymptotic variance of the transformed tau-estimators for the uncorrelated t-distribution with even ν

ν	$\sigma^2_{\varrho(\tau)} = \pi^2\sigma^2_\tau/4$
2	$\frac{8}{3} - \frac{1}{9}\pi^2$
4	$-\frac{1\,000}{27} + \frac{35}{9}\pi^2$
6	$\frac{401\,312}{675} - \frac{541}{9}\pi^2$
8	$-\frac{42\,307\,408}{3675} + \frac{10\,499}{9}\pi^2$
10	$\frac{71\,980\,077\,752}{297\,675} - \frac{220\,501}{9}\pi^2$

Asymptotic variance of the transformed tau-estimators for the uncorrelated t-distribution with odd ν

ν	$\sigma^2_{\varrho(\tau)} = \pi^2\sigma^2_\tau/4$
1	$4\log^2(2)$
3	$30 - 44\log(2) + 4\log^2(2)$
5	$-\frac{20\,221}{54} + \frac{1\,618}{3}\log(2) + 4\log^2(2)$
7	$\frac{342\,071}{50} - \frac{148\,066}{15}\log(2) + 4\log^2(2)$
9	$-\frac{1\,358\,296\,703}{9\,800} + \frac{20\,995\,691}{105}\log(2) + 4\log^2(2)$

Bounds and limits for the asymptotic variance σ_τ^2 of the tau-estimators

Theorem (Dengler/Schmock)

1. General upper bound: $\sigma_\tau^2 \le 4(1-\tau^2)$.
2. For axially symmetric distributions: $\sigma_\tau^2 \le 4/3$.
3. For uncorrelated t-distributions:

$$\lim_{\nu\to\infty} \sigma_\tau^2 = \frac{4}{9} \quad \text{and} \quad \lim_{\nu\searrow 0} \sigma_\tau^2 = \frac{4}{3},$$

hence

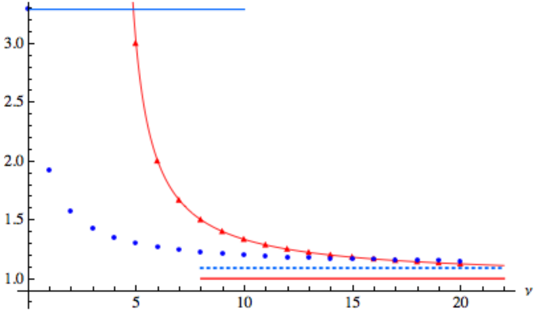
$$\sigma_{\varrho(\tau)}^2 = \frac{\pi^2}{4}\sigma_\tau^2 \to \frac{\pi^2}{3} \approx 3.290 \quad \text{as } \nu \searrow 0.$$

The upper bound in (2) is attained by (RU, RV) with independent, symmetric $\{-1,+1\}$-valued U and V, and $R \ge 0$ with density.

Comparison of the estimators for uncorrelated t-distributions with different degrees ν of freedom

ν	$\nu \downarrow 0$	1	2	3	4	5	6	7	8	9
σ_ϱ^2	n. a.	n. a.	n. a.	n. a.	n. a.	3	2	1.667	1.500	1.400
$\sigma_{\varrho(\tau)}^2$	3.290	1.922	1.570	1.423	1.345	1.296	1.263	1.240	1.222	1.208

ν	10	11	12	13	14	15	16	17	...	∞
σ_ϱ^2	1.333	1.286	1.250	1.222	1.200	1.182	1.167	1.154	...	1
$\sigma_{\varrho(\tau)}^2$	1.197	1.188	1.180	1.174	1.168	1.164	1.159	1.156	...	1.097



Results for the uncorrelated t-distribution

- For heavy-tailed t-distributions ($\nu \leq 4$), the transformed estimator is asymptotically normal with finite asymptotic variance whereas the standard estimator can not be asymptotically normal with finite variances.
- For $\nu \in \{5, 6, \ldots, 16\}$ the transformed estimator has a smaller asymptotic variance than the standard estimator and is in this sense better. Especially for small ν the difference is remarkable.
- The two estimating methods are approximately equivalent for $\nu \approx 17$, where the corresponding t-distribution is already quite similar to the normal distribution.

Asymptotic variance for the t-distribution (1)

Main steps to solve the integrals for even ν:

- Reduce $u^{\nu-1}$ to u by writing

$$u^{\nu-1} = u\,(t+u^2-t)^{\frac{\nu}{2}-1} = u \sum_{j=0}^{\frac{\nu}{2}-1} \binom{\frac{\nu}{2}-1}{j} (t+u^2)^j\,(-t)^{\frac{\nu}{2}-j-1}$$

  and dividing by $(t+u^2)^\nu$ as far as possible.
- Reduce the remaining $(t+u^2)^{\nu-j}$ to $(t+u^2)^2$ by $\nu-j-2$ integrations by parts:

$$\int_0^1 \frac{t^{\frac{3\nu}{2}-j-2}\,(1-t)^{\nu-1}}{(t+u^2)^{\nu-j}}\,dt$$

$$= \sum_{k=0}^{\nu-1} \frac{(-1)^k}{\frac{\nu}{2}+k} \binom{\nu-1}{k} \binom{\frac{3\nu}{2}-j+k-2}{\nu-j-1} \int_0^1 \frac{t^{\frac{\nu}{2}+k}}{(t+u^2)^2}\,dt$$

Asymptotic variance for the t-distribution (2)

- Reduce the $\arctan^2$ by

$$\int_0^\infty \frac{u \arctan^2 u}{(t+u^2)^2}\,du = \int_0^\infty \frac{\arctan u}{(1+u^2)\,(t+u^2)}\,du$$

- To solve the remaining integrals use

$$\frac{t^k - 1}{(1+u^2)\,(t+u^2)} = \Big(\frac{1}{1+u^2} - \frac{1}{t+u^2}\Big)\sum_{l=0}^{k-1} t^l$$

Asymptotic variance for the t-distribution (3)

Main steps to solve the integrals for odd $\nu \geq 3$:

- First steps are similar to the case of even ν.
- With $l \in \mathbb{N}$, reduce the $\arctan^2$ by

$$\int_0^1 t^l \int_0^\infty \frac{u^2 \arctan^2 u}{(t+u^2)^2}\,du\,dt = \frac{\pi^3}{24\,(2l+1)} + \frac{2l}{2l+1} \int_0^1 t^l \int_0^\infty \frac{u \arctan u}{(1+u^2)\,(t+u^2)}\,du\,dt\,.$$

- Show that

$$\int_0^\infty \frac{u \arctan u}{1+u^2} \log\left(1+\frac{1}{u^2}\right) du = \frac{\pi}{2}\left(\frac{\pi^2}{12} - \log^2(2)\right). \quad (1)$$

Some literature

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