Introduction to Risk Measures and Capital Allocation Principles

Introductory Crash Course
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Purpose of Risk Measures

- Concentrate the "relevant" information about the future worth of a risky position into a single number.
- Determine the amount of cash (or units of a reference instrument) needed to make a risky position acceptable for the period.

Remarks

- Connection to premium calculation principles
- Risk measures should have economically meaningful properties, in particular w.r.t. aggregation of risks.
- Focus on the loss part (one-sided measures); variance and standard deviation punish free lottery tickets.

Quantiles and Value-at-Risk

$X\colon \Omega \to \mathbb{R}$ random discounted one-period profit & loss

- Upper α-quantile of X with $\alpha \in (0,1)$

$$q^\alpha(X) = \inf\{x \in \mathbb{R} \mid \mathbb{P}(X \le x) > \alpha\}$$

- Lower α-quantile of X

$$q_\alpha(X) = \inf\{x \in \mathbb{R} \mid \mathbb{P}(X \le x) \ge \alpha\}$$

$q_\alpha(X) = q^\alpha(X) \iff \mathbb{P}(X \le x) = \alpha$ for at most one x

Value-at-Risk of X at level α

$$\mathrm{VaR}_\alpha(X) = -q^\alpha(X) = q_{1-\alpha}(-X)$$

Smallest value when added to X avoids negative results with probability at least $1-\alpha$.

Advantages and Deficiencies of Value-at-Risk

- Robust quantity like the median,
  doesn't depend of the far-out tails.
- "Easy" to calculate and to backtest.
- Applicable for all real-valued random variables.

Deficiencies:

- VaR ignores severity of unfavourable events.
- **VaR can punish diversification!** Example:
  (a) 100 Euro loan with default probability $p = 0.8\%$
  $\implies \mathrm{VaR}_{1\%}(X) = 0$
  (b) Two independent 50 Euro loans with $p = 0.8\%$
  $\implies \mathbb{P}(\text{at least one default}) = 2p - p^2 \ge 1.59\%$
  Therefore $\mathrm{VaR}_{1\%}(X) = 50$

Definition of a Coherent Risk Measure (ADEH 1999)

A map $\varrho \colon L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$, defined on the set of ($\mathbb{P}$-equivalence classes of) $\mathbb{P}$-almost surely bounded random variables on $(\Omega, \mathcal{F})$, is called coherent risk measure if it satisfies

(a) Monotonicity: $X \geq 0 \implies \varrho(X) \leq 0$,

(b) Positive homogeneity: $\varrho(\lambda X) = \lambda \varrho(X)$ for all $\lambda \geq 0$,

(c) Translation invariance:
$\varrho(X + c) = \varrho(X) - c$ for all $c \in \mathbb{R}$,

(d) Subadditivity: $\varrho(X + Y) \leq \varrho(X) + \varrho(Y)$.

VaR satisfies conditions (a)–(c), but not (d).

Examples of Coherent Risk Measures

Let $\mathcal{P}$ be a set of probability measures $\mathbb{Q}$ on $(\Omega, \mathcal{F})$, absolutely continuous w.r.t. $\mathbb{P}$ (think of scenarios). Then a coherent risk measure $\varrho_{\mathcal{P}} \colon L^\infty \to \mathbb{R}$ is given by

$$\varrho_{\mathcal{P}}(X) = \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}[-X], \quad X \in L^\infty.$$

- If $\mathcal{P} = \{\mathbb{P}\}$, then $\varrho_{\mathcal{P}}(X) = \mathbb{E}_{\mathbb{P}}[-X]$ (too tolerant).
- If $\mathcal{P} = \{\mathbb{Q} \mid \mathbb{Q} \ll \mathbb{P}\}$, then $\varrho_{\mathcal{P}}(X) = \operatorname{ess\,sup}_{\mathbb{P}}(-X)$ (too restrictive).
- For $\alpha \in (0,1)$ define $\mathcal{P}_\alpha = \left\{ \mathbb{Q} \,\middle|\, \mathbb{Q} \ll \mathbb{P},\ \frac{d\mathbb{Q}}{d\mathbb{P}} \leq \frac{1}{\alpha} \right\}$.

Tail Mean and Expected Shortfall

For measurable $X : \Omega \to \mathbb{R}$ with $\mathbb{E}[X^-] < \infty$ define the tail mean of X at level $\alpha \in (0,1)$ by

$$\mathrm{TM}_\alpha(X) = \frac{1}{\alpha}\,\mathbb{E}\big[X 1_{\{X<q^\alpha(X)\}}\big] + q^\alpha(X)\frac{\alpha - \mathbb{P}(X < q^\alpha(X))}{\alpha}.$$

If $\mathbb{P}(X \le q^\alpha(X)) = \alpha$, then

$$\mathrm{TM}_\alpha(X) = \mathbb{E}[X \,|\, X \le q^\alpha(X)].$$

Define the expected shortfall of X at level $\alpha \in (0,1)$ by

$$\mathrm{ES}_\alpha(X) = -\,\mathrm{TM}_\alpha(X) \ge -q^\alpha(X) = \mathrm{VaR}_\alpha(X).$$

Theorem: $\varrho_{\mathcal{P}_\alpha}(X) = \mathrm{ES}_\alpha(X)$

Characterization of Coherent Risk Measures

Definition: A coherent risk measure ϱ is said to satisfy the Fatou property, if for every X and every sequence $\{X_n\}_{n\in\mathbb{N}}$ in L^∞ with $\|X_n\|_\infty \le 1$

$$X_n \xrightarrow{\mathbb{P}} X \implies \varrho(X) \le \liminf_{n\to\infty} \varrho(X_n).$$

Theorem: A coherent risk measure $\varrho : L^\infty \to \mathbb{R}$ satisfies the Fatou property is and only if there exists an $L^1(\mathbb{P})$-closed, convex set of probability measures $\mathcal{P}$ with $\mathbb{Q} \ll \mathbb{P}$ for all $\mathbb{Q} \in \mathcal{P}$ such that

$$\varrho(X) = \varrho_{\mathcal{P}}(X) = \sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}[-X], \quad X \in L^\infty.$$

Extension of a Coherent Risk Measure

Let $\varrho_{\mathcal{P}}\colon L^\infty \to \mathbb{R}$ be a coherent risk measure as in the previous theorem. Suppose the exists $\delta > 0$ such that

$$A \in \mathcal{F} \text{ satisfies } \mathbb{P}(A) \le \delta \implies \text{there exists } \mathbb{Q} \in \mathcal{P} \text{ with } \mathbb{Q}(A) = 0.$$

Then

$$\varrho(X) = \lim_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}[-(X \wedge n)]$$

defines an extension $\varrho\colon L^0 \to \mathbb{R} \cup \{\infty\}$ of $\varrho_{\mathcal{P}}$ to the space $L^0(\Omega, \mathcal{F}, \mathbb{P})$ of all random variables preserving monotonicity, positive homogeneity, translation invariance and subadditivity.

Convex Risk Measures (Föllmer/Schied 2000)

Risk for large positions might increase more than linear (due to additional liquidity risk, for example), hence positive homogeneity might need to be relaxed.

Definition: A map $\varrho\colon L^\infty \to \mathbb{R}$ is called convex risk measure if it satisfies

(a) Monotonicity: $X \ge Y \implies \varrho(X) \le \varrho(Y)$,

(b) Convexity: For all $\lambda \in [0,1]$

$$\varrho\bigl(\lambda X + (1-\lambda)Y\bigr) \le \lambda\varrho(X) + (1-\lambda)\varrho(Y),$$

(c) Translation invariance: For all $c \in \mathbb{R}$

$$\varrho(X + c) = \varrho(X) - c.$$

Convex Risk Measure Defined by a Loss Function

- Loss funct. $l\colon \mathbb{R} \to \mathbb{R}$ increasing, convex, non-constant
- Threshold x_0 in the range of l

Then a convex risk measure $\varrho_l\colon L^\infty \to \mathbb{R}$ is defined by

$$\varrho_l(X) = \inf\bigl\{ c \in \mathbb{R} \bigm| \mathbb{E}[l(-(X+c))] \le x_0 \bigr\}.$$

Examples:

- $l(x) = \exp(\lambda x)$ with $\lambda > 0$ and $x_0 = 1$
  $\implies \varrho_l(X) = \frac{1}{\lambda} \log \mathbb{E}[\exp(-\lambda X)]$
- $l(x) = \max\{x, 0\}$ and $x_0 \ge 0$
  $\implies \varrho_l(X)$ is minimal retention level such that the expected excess of loss is bounded by x_0.
- $l(x) = \frac{1}{p}(\max\{x,0\})^p$ with $p \ge 1$

Characterization of Convex Risk Measures

Definition: A convex risk measure ϱ is said to satisfy the Fatou property, if for every X and every sequence $\{X_n\}_{n\in\mathbb{N}}$ in L^∞ with $\|X_n\|_\infty \le 1$

$$X_n \xrightarrow{\mathbb{P}} X \implies \varrho(X) \le \liminf_{n\to\infty} \varrho(X_n)$$

Theorem: A convex risk measure $\varrho\colon L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ satisfies the Fatou property if and only if there exists a "penalty function" $\alpha\colon \mathcal{P} \to \mathbb{R} \cup \{\infty\}$ such that

$$\varrho(X) = \sup_{\mathbb{Q}\in\mathcal{P}} \bigl(\mathbb{E}_\mathbb{Q}[-X] - \alpha(\mathbb{Q})\bigr), \quad X \in L^\infty,$$

where $\mathcal{P}$ is the set of all probability measures $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ which are absolutely continuous w.r.t. $\mathbb{P}$.

The Allocation Problem for Risk Capital

Given risk bearing capital $C > 0$ for a financial institution, how to allocate it to business units for

- Fair distribution of the diversification benefit,
- Consideration of dependencies (ALM),
- Measurement of risk contributions (risk management),
- Performance measurement (for steering the company),
- Determination of bonuses for the management?

Applications on the Portfolio Level

- Security loadings for individual insurance contracts
- Credit spreads for loans and defaultable bonds

A Wish List for the Allocation of Risk Capital

- Coherent risk measure $\varrho\colon L^\infty \to \mathbb{R}$
- Profit & losses $X_1, \ldots, X_m$ of m business units, adding up to the company result $X = X_1 + \cdots + X_m$
- Total risk capital C, capital C_i assigned to unit i

Useful Properties (cf. game theory)

1. Risk sensitivity: $C = \varrho(X)$
2. Additivity: $C = C_1 + \cdots + C_m$
3. No subgroup of units is better off on its own: $\sum_{i \in I} C_i \le \varrho(\sum_{i \in I} X_i)$ for all $I \subset \{1, \ldots, m\}$.
4. If business units can be divided into parts: $\sum_{i=1}^m \alpha_i C_i \le \varrho(\sum_{i=1}^m \alpha_i X_i)$ for all $\alpha_i \in [0, 1]$.

Axiomatic Approach to Risk Capital Allocation

Let $V \subset L^0$ be a linear subspace, e. g. $V = L^\infty$.

Def.: $\Lambda\colon V \times V \to \mathbb{R}$ is called a risk capital allocation principle w.r.t. the coherent risk measure $\varrho\colon V \to \mathbb{R}$, if it satisfies for all $X, Y, Z \in V$ and $\alpha, \beta \in \mathbb{R}$

(a) Risk sensitivity: $\Lambda(X, X) = \varrho(X)$,

(b) Linearity: $\Lambda(\alpha X + \beta Y, Z) = \alpha\Lambda(X, Z) + \beta\Lambda(Y, Z)$,

(c) Diversification: $\Lambda(X, Y) \le \Lambda(X, X)$.

Exercise: Such a Λ has the four useful properties.

Λ is called continuous at $Y \in V$, if for all $X \in V$

$$\lim_{\varepsilon\to 0}\Lambda(X, Y + \varepsilon X) = \Lambda(X, Y).$$

Results About Risk Capital Allocations

Existence: For every $Y \in V$ there exists $h_Y \in V^*$ with $h_Y(Y) = \varrho(Y)$ and $h_Y \le \varrho$ on V. Furthermore,

$$\Lambda(X, Y) = h_Y(X), \qquad X, Y \in V,$$

defines a risk capital allocation principle w.r.t. the coherent risk measure ϱ.

Uniqueness: If the capital allocation principle Λ is continuous at $Y \in V$, then for all $X \in V$

$$\Lambda(X, Y) = \lim_{\varepsilon\to 0}\frac{\varrho(Y + \varepsilon X) - \varrho(Y)}{\varepsilon}$$

(directional derivative of the underlying risk measure).

Capital Allocation by Expected Shortfall (Schmock 1998)

Consider $\alpha \in (0,1)$ and $X, Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$.
A capital allocation principle for $\varrho_{\mathcal{P}_\alpha} = \mathrm{ES}_\alpha$ is

$$\Lambda_\alpha^{\mathrm{ES}}(X,Y) = -\frac{1}{\alpha}\,\mathbb{E}[X 1_{\{Y < q^\alpha(Y)\}}] - \frac{\alpha - \mathbb{P}(Y < q^\alpha(Y))}{\alpha}\,\mathbb{E}[X \,|\, Y = q^\alpha(Y)].$$

If $\mathbb{P}(Y \le q^\alpha(Y)) = \alpha$, then

$$\Lambda_\alpha^{\mathrm{ES}}(X,Y) = -\,\mathbb{E}[X \,|\, Y \le q^\alpha(Y)].$$

If $\mathbb{P}(Y = q^\alpha(Y)) = 0$, then $\Lambda_\alpha^{\mathrm{ES}}$ is continuous at Y.

Further Topics

- Risk measures and acceptance sets
- Risk measures and utility functions
- Value of information and risk measures
- Convex risk measures and convex trading constraints
- Allocation of risk capital and game theory
- Multi-period risk measurement

Some Literature Related to Risk Measures

- C. Acerbi, D. Tasche: *On the coherence of expected short-fall,* Journal of Banking and Finance 26 (2002), 1487–1503.
- Ph. Artzner, F. Delbaen, J.-M. Eber, D. Heath: *Coherent measures of risk,* Math. Finance 9 (1999), no. 3, 203–228.
- F. Delbaen: *Coherent measures of risk on general probability spaces.* In: Advances in Finance and Stochastics, K. Sandmann and P. Schönbucher (Eds.), Springer (2000), 1–37.
- F. Delbaen: *Coherent risk measures,* Lecture notes, Scuola Normale di Pisa (2000).
- H. Föllmer, A. Schied: *Convex measures of risk and trading constraints,* Finance and Stochastics 6 (2000), 429–447.
- M. Kalkbrenner: *An axiomatic approach to capital allocation,* preprint.