Risk Aggregation, Numerical Stability and a Variation of Panjer's Recursion

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Outline of Presentation

- Motivation
- CreditRisk$^+$ and extensions
- Panjer's recursion
- Numerical stability
  and extensions of Panjer's recursion
- Applications and examples
- Quantiles, expected shortfall, risk contributions
- Application to operational risk (time permitting)

Software implementation:

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Motivation: Bernoulli Model for Defaults

- Bernoulli loss indicators

$$N_i = \begin{cases} 1 & \text{if obligor } i \text{ defaults (within one year),} \\ 0 & \text{otherwise.} \end{cases}$$

- Default probability $p_i = \mathbb{P}(N_i = 1)$ for $i = 1, \dots, m$.
- Random number of defaults $N = N_1 + \cdots + N_m$.
- Probability distribution for $n \in \{0, \dots, m\}$

$$\mathbb{P}(N = n) = \sum_{\substack{I \subset \{1,\dots,m\} \\ |I| = n}} \underbrace{\mathbb{P}(N_i = 1_I(i) \text{ for } i = 1, \dots, m)}_{\overset{\text{if ind.}}{=} \left(\prod_{i \in I} p_i\right) \prod_{i \in \{1,\dots,m\} \setminus I} (1 - p_i)}$$

$$m = 1000,\ n = 100 \implies \binom{1000}{100} \approx 6.4 \times 10^{139} \text{ terms}$$

Observations ...

- Already the Bernoulli model with independent loss indicators has far too many terms for the calculation of the portfolio loss distribution in the general case.
- In the general Bernoulli mixture model, individual terms are too complicated to compute numerically.
- Different exposures and recovery rates are not even considered.

... and Conclusions

- Simplifying assumptions are necessary.
- Approximations need to be considered.

Poisson Approximation

- $X_1, \ldots, X_m$ independent default 0-1-indicators
- Intensity $\lambda = \sum_{i=1}^m p_i$ with $p_i = \mathbb{P}(X_i = 1)$
- Number of default events $W = \sum_{i=1}^m X_i$
- Total variation distance

$$d_{\mathsf{TV}}(\mu, \nu) = \sup_{A \subset \mathbb{N}_0} |\mu(A) - \nu(A)|$$

Quality of Poisson approximation (Barbour/Hall, 1984):

$$d_{\mathsf{TV}}\big(\mathcal{L}(W), \mathrm{Poisson}(\lambda)\big) \le \frac{1 - e^{-\lambda}}{\lambda} \sum_{i=1}^m p_i^2$$

For full proof with Stein–Chen method, see e.g. Barbour, Holst and Janson: *Poisson Approximation*, Clarendon Press (1992).

Simple Poisson Model for Defaults

- Number N_i of defaults of obligor $i \in \{1, \ldots, m\}$
- Assume $N_i \sim \mathrm{Poisson}(\lambda_i)$ for all $i \in \{1, \ldots, m\}$ (several defaults of an obligor possible).
- Assume independence of $N_1, \ldots, N_m$.
- Random number of defaults $N = N_1 + \cdots + N_m$.
- $N \sim \mathrm{Poisson}(\lambda)$ with $\lambda = \lambda_1 + \cdots + \lambda_m$, i.e.,

$$\mathbb{P}(N = n) = \frac{\lambda^n}{n!} e^{-\lambda} \quad \text{for all } n \in \mathbb{N}_0.$$

- $m = 20$, $\lambda_i = 0.2 \implies \mathbb{P}(N > 20) \le 2 \times 10^{-9}$.

Introduction to CreditRisk$^+$, Standard Features

- Developed by Credit Suisse First Boston.
- Actuarial model for the aggregation of credit risks.
- Based on the Poisson approximation of individual defaults and the divisibility of the Poisson distribution.
- Allows for deterministic exposures/recovery rates.
- Several independent risk factors for dependence of default frequencies can be considered.
- Probability generating function φ_L of the credit portfolio loss L is available in closed form.
  $\rightarrow$ No Monte Carlo simulation, no stochastic error!

Extensions of CreditRisk$^+$

- Stochastic losses of individual obligors are allowed, distribution may depend on the causing risk factor.
- Risk groups with dependent stochastic losses given default are possible.
- Risk factors for default frequencies may be dependent.
- Risk contributions of obligors can be calculated.
- Even with all the extensions, the probability generating function φ_L of the credit portfolio loss L is available in closed form.
  $\rightarrow$ No Monte Carlo simulation, no stochastic error!
- Distribution of L and risk contributions can be calculated from $\varphi_{\gamma,L}$ with a numerically stable algorithm.

Input Parameters of CreditRisk$^+$ (Extended Version)

- Number of obligors $m \in \mathbb{N}$.
- Basic loss unit $E > 0$.
- Number $K \in \mathbb{N}_0$ of risk factors or non-idiosyncratic, (independent) default causes.
- Relative default variances $\sigma_k^2 > 0$ of risk factors $k \in \{1, \ldots, K\}$.
- Collection G of nonempty subsets of all obligors $\{1, \ldots, m\}$, called risk groups.

Input Parameters of CreditRisk$^+$ (Cont.)

For every group $g \in G$ we need

- the (one year) default probability $p_g \in [0, 1]$,
- the susceptibility $w_{g,0} \in [0, 1]$ to idiosyncratic risk,
- the susceptibilities $w_{g,k} \in [0, 1]$ to risk factors $k \in \{1, \ldots, K\}$,
- the multivariate probability distributions $Q_{g,k} = \{q_{g,k,\mu}\}_{\mu \in \mathbb{N}_0^g}$ on $\mathbb{N}_0^g$ describing the stochastic losses of all the obligors $i \in g$ in multiples of the basic loss unit E in case the risk group g defaults due to risk $k \in \{0, \ldots, K\}$.

Further Assumptions, Notation

- We assume that every obligor $i \in \{1, \ldots, m\}$ belongs to at least one group $g \in G$.
- Let $G_i := \{g \in G \,|\, i \in g\}$ denote the set of all risk groups to which obligor $i \in \{1, \ldots, m\}$ belongs, by assumption $G_i \neq \varnothing$.
- We assume that for each group the susceptibilities (also called weights) exhaustively describe the risk factors. That is, for all $g \in G$,

$$\sum_{k=0}^{K} w_{g,k} = 1.$$

Notation for Default Events of Risk Groups

Number of defaults for every risk group $g \in G$:

- $N_{g,0}$ due to idiosyncratic risk,
- $N_{g,k}$ due to risk $k \in \{1, \ldots, K\}$,
- $N_g := \sum_{k=0}^{K} N_{g,k}$ total.

Notation for Default Events of Individual Obligors

Number of defaults for every obligor $i \in \{1, \ldots, m\}$

- $N_{i,0} := \sum_{g \in G_i} N_{g,0}$ due to idiosyncratic risk,
- $N_{i,k} := \sum_{g \in G_i} N_{g,k}$ due to risk $k \in \{1, \ldots, K\}$,
- $N_i := \sum_{k=0}^{K} N_{i,k} = \sum_{g \in G_i} N_g$ total.

Notation for Stochastic Losses

Loss at default number $n \in \mathbb{N}$ of risk group $g \in G$ due to risk factor $k \in \{1, \dots, K\}$ or idiosyncratic risk $k = 0$

- $L_{g,i,k,n}$ part attributed to obligor $i \in g$
- $L_{g,k,n} := \sum_{i \in g} L_{g,i,k,n}$ loss of entire group

Summation over default numbers, risks and groups:

- $L_{g,k} := \sum_{n=1}^{N_{g,k}} L_{g,k,n}$ total loss of the group for risk k
- $L_g := \sum_{k=0}^{K} L_{g,k}$ total loss of the risk group
- $L = \sum_{g \in G} L_g$ portfolio loss

Loss Attributed to Obligor $i \in \{1, \dots, m\}$

- Due to group $g \in G_i$ and risk $k \in \{0, \dots, K\}$

$$L_{g,i,k} := \sum_{n=1}^{N_{g,k}} L_{g,i,k,n}.$$

- Due to risk $k \in \{0, \dots, K\}$

$$L_{i,k} := \sum_{g \in G_i} L_{g,i,k}.$$

- Total attributed loss

$$L_i := \sum_{k=0}^{K} L_{i,k}.$$

Probabilistic Assumptions for the Extended Version of CreditRisk$^+$

- For every group $g \in G$ and every risk $k \in \{0, \dots, K\}$, the sequence of $\mathbb{N}_0^g$-valued random vectors $(L_{g,i,k,n})_{i \in g}$ with $n \in \mathbb{N}$ is i.i.d. and independent of all other random variables, with distribution

$$\mathbb{P}(L_{g,i,k,1} = \mu_i \text{ for all } i \in g) = q_{g,k,\mu}, \qquad \mu \in \mathbb{N}_0^g.$$

- For each group $g \in G$, the number $N_{g,0}$ of idiosyncatic defaults is Poisson distributed according to the Poisson intensity λ_g and the susceptibility $w_{g,0}$, i.e.,

$$N_{g,0} \sim \text{Poisson}(\lambda_g w_{g,0}) \qquad \text{for every } g \in G.$$

Probabilistic Assumptions (Cont.)

- The group default numbers $\{N_{g,0}\}_{g \in G}$ due to idiosyncratic risk are independent from one another and from all other random variables.
- The risks factors $\Lambda_1, \dots, \Lambda_K$ are independent, each one gamma distributed with $\mathbb{E}[\Lambda_k] = 1$ and $\text{Var}(\Lambda_k) = \sigma_k^2 > 0$, i.e., $\alpha_k = \beta_k = 1/\sigma_k^2$.
- For all groups $g \in G$ and risks $k \in \{1, \dots, K\}$,

$$\mathcal{L}(N_{g,k} | \Lambda_1, \dots, \Lambda_K) \overset{\text{a.s.}}{=} \mathcal{L}(N_{g,k} | \Lambda_k) \overset{\text{a.s.}}{=} \text{Poisson}(\lambda_g w_{g,k} \Lambda_k).$$

- Conditionally on $\Lambda_1, \dots, \Lambda_K$, the risk factor based defaults $\{N_{g,k} \,|\, g \in G, k \in \{1, \dots, K\}\}$ are independent.

Weighted Probability Generating Function

In order to calculate terms needed for the risk contributions we will need what we call weighted probability generating functions.

Definition: For $L : \Omega \to \mathbb{N}_0$ and an integrable random variable $X : \Omega \to \mathbb{R}$, we define the X-weighted probability generating function by

$$\varphi_{L,X}(s) = \mathbb{E}\big[X s^L\big] = \sum_{n=0}^{\infty} \mathbb{E}\big[X 1_{\{L=n\}}\big]\, s^n,$$

which is meaningful at least for all $s \in \mathbb{C}$ with $|s| \le 1$.

Weighted Probability Generating Function (Cont.)

We will need expressions of the form $\mathbb{E}[\Lambda_k 1_{\{L=n\}}]$ for $k \in \{1, \dots, K\}$ and $n \in \mathbb{N}_0$, which can be derived by

$$\varphi^{(n)}_{L,\Lambda_k}(0) = n!\,\mathbb{E}\big[\Lambda_k 1_{\{L=n\}}\big]\,.$$

Unifying approach for the γ-weighted probability generating function of the loss:

Fix $\gamma = (\gamma_1, \dots, \gamma_K) \in [0, \infty)^K$ and define

$$\varphi_{L,\gamma}(s) := \mathbb{E}\big[\Lambda_1^{\gamma_1} \dots \Lambda_K^{\gamma_K} s^L\big]\,, \qquad |s| \le 1,$$

for the risk factors $\Lambda_1, \dots, \Lambda_K$ and the total loss L.
$\gamma = 0$ gives the probability generating function φ_L of L.

The Closed Form of the WPGF

$$\varphi_{L,\gamma}(s) = C_\gamma \exp\Bigg(\bar\lambda_0(\varphi_0(s) - 1) - \sum_{k=1}^{K}\Big(\frac{1}{\sigma_k^2} + \gamma_k\Big)\log\big(1 - \bar\lambda_k \sigma_k^2(\varphi_k(s) - 1)\big)\Bigg),$$

where $C_\gamma := \prod_{k=1}^{K} \mathbb{E}[\Lambda_k^{\gamma_k}] = 1$ if all $\gamma_k \in \{0, 1\}$, with PGF of mixture distributions (conditioned to be positive)

$$\varphi_k(s) := \sum_{g\in G} \frac{\lambda_g w_{g,k}}{\bar\lambda_k} \varphi_{L_{g,k,1}}(s), \quad \bar\lambda_k := \sum_{g\in G} \lambda_g w_{g,k}(1 - q^{\mathsf{s}}_{g,k,0}).$$

Numerical inversion similar to: H. Haaf, O. Reiß, J. Schoenmakers, *Numerically Stable Computation of CreditRisk$^+$*, 2003.

Distributions in the Panjer Class

Definition: A probability distribution $\{q_n\}_{n\in\mathbb{N}_0}$ is said to belong to the Panjer(a, b, k) class with $a, b \in \mathbb{R}$ and $k \in \mathbb{N}_0$ if $q_0 = q_1 = \dots = q_{k-1} = 0$ and

$$q_n = \Big(a + \frac{b}{n}\Big) q_{n-1} \quad \text{for all } n \in \mathbb{N} \text{ with } n \ge k + 1.$$

Important Examples: (all distributions are known)

- Poisson$(\lambda) \in$ Panjer$(0, \lambda, 0)$ with $\lambda > 0$
- NegBin$(\alpha, p) \in$ Panjer$(q, (\alpha - 1)q, 0)$ with $\alpha > 0$ and $p \in (0, 1)$
- Log$(q) \in$ Panjer$(q, -q, 1)$ with $q \in (0, 1)$ and $q_n = -\frac{q^n}{n \log(1-q)}$ for all $n \in \mathbb{N}$

Extended Panjer Recursion

If $\mathcal{L}(N) \in \text{Panjer}(a,b,k)$, independent of the i. i. d. $\mathbb{N}_0$-valued sequence $\{X_n\}_{n\in\mathbb{N}}$, and $a\mathbb{P}(X_1 = 0) \neq 1$, then $S := X_1 + \cdots + X_N$ satisfies

$$\mathbb{P}(S = 0) = \varphi_N(\mathbb{P}(X_1 = 0))$$

with φ_N probability generating function of N, and

$$\mathbb{P}(S = n) = \frac{1}{1 - a\mathbb{P}(X_1 = 0)}\biggl(\mathbb{P}(S_k = n)\mathbb{P}(N = k) + \sum_{j=1}^{n}\Bigl(a + \frac{bj}{n}\Bigr)\mathbb{P}(X_1 = j)\mathbb{P}(S = n - j)\biggr)$$

for all $n \in \mathbb{N}$, where $S_k = X_1 + \cdots + X_k$.

Application of Extended Panjer Recursion

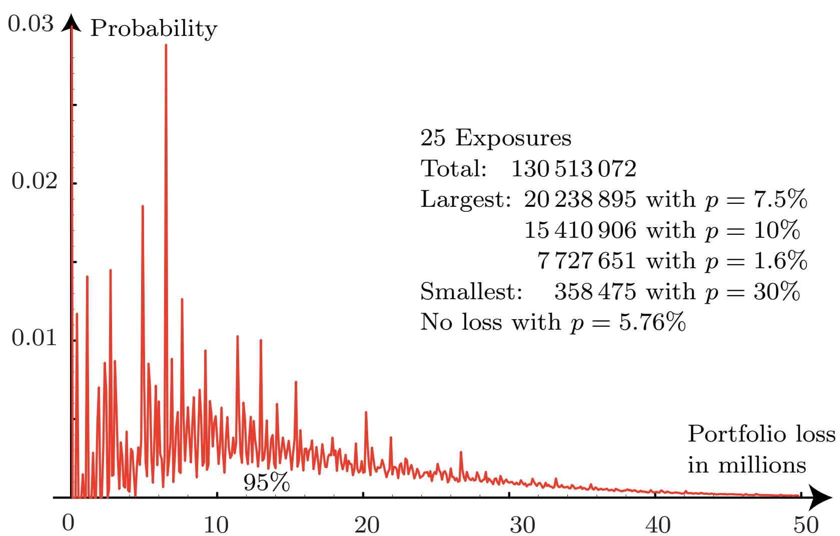
Remark: Recursion scheme is numerically stable for $\text{Poisson}(\lambda)$, $\text{NegBin}(\alpha, p)$, and $\text{Log}(q)$.

Observation: If $N \sim \text{Poisson}(\lambda)$, independent of the i. i. d. sequence $\{X_n\}_{n\in\mathbb{N}}$ with $X_1 \sim \text{Log}(q)$, then

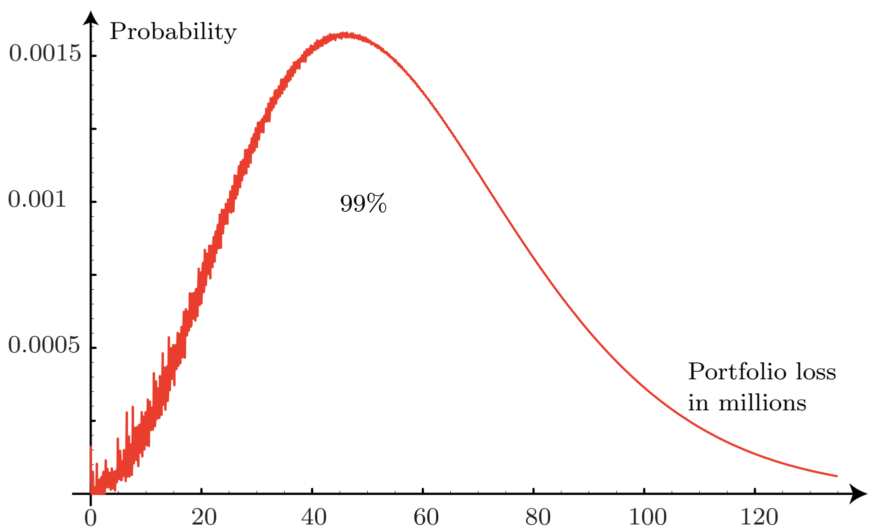
$$S = X_1 + \cdots + X_N \sim \text{NegBin}\Bigl(-\frac{\lambda}{\log(1-q)}, 1 - q\Bigr).$$

Application: Calculate

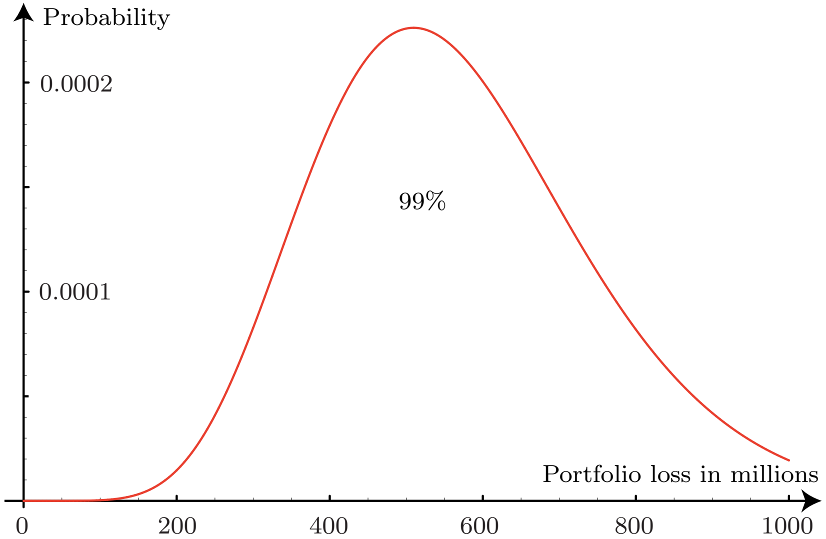
- Mixture distribution φ_k for risks $k \in \{0, \ldots, K\}$.
- Panjer recursion for log. dist. for risks $k \in \{1, \ldots, K\}$.
- Mixture distribution of φ_0 and recursion results.
- Final Panjer recursion for Poisson distribution.



Loss distribution in a credit portfolio of 25 exposures calculated with CreditRisk$^+$ and basic loss unit 100 000.



Loss distribution in a credit portfolio of 100 exposures calculated with CreditRisk$^+$ and basic loss unit 100 000.



Loss distribution in a credit portfolio of 1000 exposures calculated with CreditRisk$^+$ and basic loss unit 100 000.

Extended Logarithmic Distribution

For $k \in \mathbb{N}\setminus\{1\}$ and $q \in (0,1]$ define $q_0 = \cdots = q_{k-1} = 0$,

$$q_n = \frac{\binom{n}{k}^{-1} q^n}{\sum_{l=k}^{\infty} \binom{l}{k}^{-1} q^l} \quad \text{for } n \ge k.$$

ExtLog(k,q) is in Panjer$(q, -kq, k)$.

Extended Negative Binomial Distribution

For $k \in \mathbb{N}$, $\alpha \in (-k, -k+1)$ and $p \in [0,1)$ define $q = 1-p$, $q_0 = \cdots = q_{k-1} = 0$ and

$$q_n = \frac{\binom{\alpha+n-1}{n} q^n}{p^{-\alpha} - \sum_{j=0}^{k-1} \binom{\alpha+j-1}{j} q^j} \quad \text{for } n \ge k.$$

ExtNegBin(α, k, p) is in Panjer$(q, (\alpha-1)q, k)$.

Example for Numerical Instability

Take $N \sim$ ExtNegBin(α, k, p) with $k \in \mathbb{N}$, $\varepsilon, p \in (0,1)$ and $\alpha = -k + \varepsilon$. Consider the loss distribution $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = l) = 1/2$ with $l \ge 3$. Then

$$p_{k+l} = q\frac{k(l-1)+\varepsilon k}{k+l}\Big(\frac{q_k}{2^{k+1}} + \frac{q_{k+l-1}}{k2^{k+l}}\Big) - q\frac{k(l-1)-\varepsilon l}{k+l}\,\frac{q_k}{2^{k+1}}.$$

With $\varepsilon = 1/10\,000$, $k = 1$, $l = 5$, $p = 1/10$:

$$p_6 = 0.1499926 - 0.1499701 = 0.0000225\,.$$

Panjer recursion with five significant digits gives

$$p_6 = 0.0000400\ldots$$

Panjer Recursion Replaced by Weighted Convolution

Fix $l \in \mathbb{N}$, consider $N \sim \{q_n\}_{n\in\mathbb{N}_0}$ and $\tilde N_i \sim \{\tilde q_{i,n}\}_{n\in\mathbb{N}_0}$, define $S = X_1 + \cdots + X_N \sim \{p_n\}_{n\in\mathbb{N}_0}$ and $\tilde S_{(i)} = X_1 + \cdots + X_{\tilde N_i} \sim \{\tilde p_{i,n}\}_{n\in\mathbb{N}_0}$ for $i \in \{1,\ldots,l\}$. Assume there exist $k \in \mathbb{N}_0$ and $a_1,\ldots,a_l, b_1,\ldots,b_l \in \mathbb{R}$ such that $\tilde q_{i,0} = \cdots = \tilde q_{i,k+l-i-1} = 0$ for $i \in \{1,\ldots,l\}$ and

$$q_n = \sum_{i=1}^{l}\Big(a_i + \frac{b_i}{n}\Big)\tilde q_{i,n-i} \quad \text{for } n \ge k+l.$$

Then $p_0 = \varphi_N(\mathbb{P}(X_1 = 0))$ and, for $n \in \mathbb{N}$,

$$p_n = \sum_{j=1}^{k+l-1}\mathbb{P}(S_j = n)q_j + \sum_{i=1}^{l}\sum_{j=0}^{n}\Big(a_i + \frac{b_i j}{in}\Big)\mathbb{P}(S_i = j)\tilde p_{i,n-j}.$$

Combination of Truncated Distributions

Fix $k \in \mathbb{N}_0$, $l \in \mathbb{N}$. For all $i \in \{1,\dots,l\}$ assume that $\alpha_i \ge 0$, $\beta_i \ge -i\alpha_i$ (at least one $\neq$) and that the $\mathbb{N}_0$-valued $\tilde{N}_i$ satisfies $\mathbb{P}(\tilde{N}_i < k+l-i) = 0$. Consider $q_0,\dots,q_{k+l-1} \ge 0$ with $q_0 + \cdots + q_{k+l-1} \le 1$. Define

$$q_n = c \sum_{i=1}^{l} \Big(\alpha_i + \frac{\beta_i}{n}\Big) \mathbb{P}(\tilde{N}_i = n-i) \quad \text{for } n \ge k+l,$$

$$c = \Big(\Big(1 - \sum_{n=0}^{k+l-1} q_n\Big) \Big/ \sum_{i=1}^{l} \Big(\alpha_i + \beta_i\, \mathbb{E}\Big[\frac{1}{i+\tilde{N}_i}\Big]\Big)\Big).$$

Then $\{q_n\}_{n\in\mathbb{N}_0}$ is a probability distribution satisfying the recursion condition with $a_i = c\alpha_i$ and $b_i = c\beta_i$ and the calculation of $\{p_n\}_{n\in\mathbb{N}_0}$ is numerically stable.

Weighted Convolution for ExtLog

Let $k \in \mathbb{N}$ and $q \in (0,1)$. Let $N \sim \text{ExtLog}(k+1,q)$ and $\tilde{N} \sim \text{ExtLog}(k,q)$, where $\text{ExtLog}(1,q)$ means $\text{Log}(q)$. Define $S = X_1 + \cdots + X_N$ and $\tilde{S} = X_1 + \cdots + X_{\tilde{N}}$. Then, with an explicit $b_1 > 0$, the weighted convolution

$$\mathbb{P}(S=n) = \frac{b_1}{n} \sum_{j=1}^{n} j\,\mathbb{P}(X_1 = j)\mathbb{P}(\tilde{S} = n-j), \quad n \in \mathbb{N},$$

is numerically stable. Algorithm:

- Panjer recursion for $\text{Log}(1,q)$
- $k-1$ weighted convolutions: $\text{Log}(1,q) \to \text{ExtLog}(2,q) \to \cdots \to \text{ExtLog}(k-1,q) \to \text{ExtLog}(k,q)$

Stable Algorithm for ExtLog(2,1)

Let $N \sim \text{ExtLog}(2,1)$. For $S = X_1 + \cdots + X_N$ we have

$$\mathbb{P}(S=0) = \mathbb{P}(X_1 = 0) + \mathbb{P}(X_1 \ge 1) \log \mathbb{P}(X_1 \ge 1))$$

with $0 \log 0 := 0$ and, in the case $\mathbb{P}(X_1 \ge 1) > 0$,

$$\mathbb{P}(S=n) = \frac{1}{n} \sum_{j=1}^{n} j\,\mathbb{P}(X_1 = j) r_{n-j}, \quad n \in \mathbb{N},$$

where $r_0 = -\log \mathbb{P}(X_1 \ge 1)$ and, recursively for $n \in \mathbb{N}$,

$$r_n = \frac{1}{\mathbb{P}(X_1 \ge 1)} \Big(\mathbb{P}(X_1 = n) + \frac{1}{n} \sum_{j=1}^{n} j\,\mathbb{P}(X_1 = n-j) r_j\Big)$$

Weighted Convolution for ExtNegBin

Let $k \in \mathbb{N}_0$, $\alpha \in (-k, -k+1)$ and $p \in (0,1)$. Let $N \sim \text{ExtNegBin}(\alpha-1, k+1, p)$ and $\tilde{N} \sim \text{ExtNegBin}(\alpha, k, p)$, where $\text{ExtNegBin}(\alpha, 0, p)$ means $\text{NegBin}(\alpha, p)$. Define $S = X_1 + \cdots + X_N$ and $\tilde{S} = X_1 + \cdots + X_{\tilde{N}}$. Then

$$\mathbb{P}(S=n) = \frac{b_1}{n} \sum_{j=1}^{n} j\,\mathbb{P}(X_1 = j)\mathbb{P}(\tilde{S} = n-j), \quad n \in \mathbb{N},$$

with an explicit $b_1 > 0$ is numerically stable. Algorithm:

- Panjer recursion for $\text{NegBin}(\alpha+k, p)$
- k weighted convolutions:
  $\text{NegBin}(\alpha+k,p) \to \text{ExtNegBin}(\alpha+k-1,1,p) \to \cdots \to \text{ExtNegBin}(\alpha+1,k-1,p) \to \text{ExtNegBin}(\alpha,k,p)$

Stable Algorithm for ExtNegBin($\alpha-1$,1,0)

Let $N \sim$ ExtNegBin$(\alpha-1,1,0)$ with $\alpha \in (0,1)$. For $S = X_1 + \cdots + X_N$ we have

$$\mathbb{P}(S=0) = 1 - \big(\mathbb{P}(X_1 \ge 1)\big)^{1-\alpha}$$

and in the case $\mathbb{P}(X_1 \ge 1) > 0$

$$\mathbb{P}(S=n) = \frac{1-\alpha}{n}\sum_{j=1}^{n} j\,\mathbb{P}(X_1=j)\,r_{n-j}, \quad n \in \mathbb{N},$$

where $r_0 = \big(\mathbb{P}(X_1 \ge 1)\big)^{-\alpha}$ and, recursively for $n \in \mathbb{N}$,

$$r_n = \frac{1}{\mathbb{P}(X_1 \ge 1)}\sum_{j=1}^{n}\frac{n-j+\alpha j}{n}\mathbb{P}(X_1=j)\,r_{n-j}$$

Application: Poisson–Tempered α-Stable Mixtures

Let Y be α-stable on $[0,\infty)$ with Laplace transform

$$\mathbb{E}[\exp(-sY)] = \exp(-\gamma_{\alpha,\sigma}s^{\alpha}), \quad s \ge 0,$$

where $\alpha \in (0,1)$, $\sigma > 0$ and $\gamma_{\alpha,\sigma} := \sigma^{\alpha}/\cos(\frac{\alpha\pi}{2})$.
For $\tau \ge 0$ define tempered stable distribution

$$F_{\alpha,\sigma,\tau}(y) := \mathbb{E}[e^{-\tau Y}1_{\{Y\le y\}}]/\mathbb{E}[e^{-\tau Y}]. \quad y \in \mathbb{R}.$$

Let $\Lambda \sim F_{\alpha,\sigma,\tau}$ and $\mathcal{L}(N|\Lambda) \overset{\text{a.s.}}{=}$ Poisson$(\lambda\Lambda)$ for $\lambda > 0$.
Then $N \overset{\text{d}}{=} N_1 + \cdots + N_M$ with independent

$$M \sim \text{Poisson}(\gamma_{\alpha,\sigma}((\lambda+\tau)^{\alpha} - \tau^{\alpha}))$$

and

$$N_m \sim \text{ExtNegBin}\Big(-\alpha, 1, \frac{\tau}{\lambda+\tau}\Big), \quad m \in \mathbb{N}.$$

Example: Poisson–Lévy Mixture

Special case for $\alpha = 1/2$ and $\tau = 0$.

Lévy distribution: Given $\sigma > 0$, assume that

$$\Lambda = Y \sim f_{\mathsf{L}}(x) = \Big(\frac{\sigma}{2\pi x^3}\Big)^{1/2}\exp\Big(-\frac{\sigma}{2x}\Big), \quad x > 0.$$

Let $\mathcal{L}(N|\Lambda) \overset{\text{a.s.}}{=}$ Poisson$(\lambda\Lambda)$ with $\lambda > 0$.
Then $N \overset{\text{d}}{=} N_1 + \cdots + N_M$ with independent

$$M \sim \text{Poisson}\big(\sqrt{\lambda\sigma/2}\big)$$

and

$$N_m \sim \text{ExtNegBin}(-1/2, 1, 0), \quad m \in \mathbb{N},$$

and our numerically stable recursion is again applicable.

Example: Poisson–Inverse Gaussian Mixture

Fix $\mu, \tilde\sigma > 0$, define $\sigma = \mu^2/\tilde\sigma^2$ and $\tau = 1/(2\tilde\sigma^2)$.

Inverse Gaussian distribution: $\Lambda \sim F_{1/2,\sigma,\tau}$ has density

$$f_{\mathsf{IG}}(x) = \frac{\mu}{\sqrt{2\pi\tilde\sigma^2 x^3}}\exp\Big(-\frac{(x-\mu)^2}{2\tilde\sigma^2 x}\Big), \quad x > 0.$$

Let $\mathcal{L}(N|\Lambda) \overset{\text{a.s.}}{=}$ Poisson$(\lambda\Lambda)$ with $\lambda > 0$.
Then $N \overset{\text{d}}{=} N_1 + \cdots + N_M$ with independent

$$M \sim \text{Poisson}\big(\sqrt{\sigma/2}(\sqrt{\lambda+\tau} - \sqrt{\tau})\big)$$

and

$$N_m \sim \text{ExtNegBin}\Big(-1/2, 1, \frac{\tau}{\lambda+\tau}\Big), \quad m \in \mathbb{N}.$$

Poisson–Reciprocal Inverse Gaussian Mixture

Fix $\mu, \tilde\sigma > 0$, define $\sigma = \mu^2/\tilde\sigma^2$ and $\tau = 1/(2\tilde\sigma^2)$.

Reciprocal inverse Gaussian distribution: Assume

$$\Lambda \sim f_{\mathrm{RIG}}(x) = \frac{1}{\sqrt{2\pi\tilde\sigma^2 x}} \exp\biggl(-\frac{(x-\mu)^2}{2\tilde\sigma^2 x}\biggr), \quad x > 0.$$

Let $\mathcal{L}(N|\Lambda) \stackrel{\text{a.s.}}{=} \mathrm{Poisson}(\lambda\Lambda)$ with $\lambda > 0$.

Then $N \stackrel{\text{d}}{=} N_0 + N_1 + \cdots + N_M$ with independent

$$\begin{aligned} M &\sim \mathrm{Poisson}\bigl(\sqrt{\sigma/2}(\sqrt{\lambda+\tau}-\sqrt{\tau})\bigr),\\ N_0 &\sim \mathrm{NegBin}(1/2, p), \quad p = \tau/(\lambda+\tau),\\ N_m &\sim \mathrm{ExtNegBin}(-1/2, 1, p), \quad m \in \mathbb{N}. \end{aligned}$$

Measuring Risk by Quantiles

Let X be a loss variable and $\delta \in (0,1)$ a level.

Definition: Lower δ-quantile of X

$$q_\delta(X) := \min\{x \in \mathbb{R} \mid \mathbb{P}(X \le x) \ge \delta\}.$$

Remark: Quantiles are used as value-at-risk, they have bad properties concerning diversification.

Properties: $q_\delta(X)$ can jump when

- the level δ varies slightly,
- the loss variable X varies slightly.

Measuring Risk by Expected Shortfall

Let X be a loss variable and $\delta \in (0,1)$ a level.

Definition: The expected shortfall is defined as

$$\mathrm{ES}_\delta[X] := \frac{\mathbb{E}[X 1_{\{X > q_\delta(X)\}}] + q_\delta(X)(\mathbb{P}(X \le q_\delta(X)) - \delta)}{1-\delta}.$$

Remark: If $\mathbb{P}(X \le q_\delta(X)) = \delta$, in particular if the distribution function $\mathbb{R} \ni x \mapsto \mathbb{P}(X \le x)$ of X is also left-continuous at $x = q_\delta(X)$, then

$$\mathrm{ES}_\delta[X] = \mathbb{E}[X \,|\, X > q_\delta(X)].$$

Calculation of Expected Shortfall in CreditRisk$^+$

- Credit portfolio loss L is a discrete random variable, $\rightarrow$ More complicated definition has to be used.
- The lower quantile $q_\delta(L)$ and $\mathbb{P}(L \le q_\delta(L))$ can be calculated using the CreditRisk$^+$ algorithm.
- Furthermore $\mathbb{E}[L 1_{\{L > q_\delta(L)\}}] = \mathbb{E}[L] - \mathbb{E}[L 1_{\{L \le q_\delta(L)\}}]$ with

$$\mathbb{E}[L] = \sum_{g \in G} \sum_{k=0}^{K} \lambda_g w_{g,k} \mathbb{E}[L_{g,k,1}]$$

and

$$\mathbb{E}\bigl[L 1_{\{L \le q_\delta(L)\}}\bigr] = \sum_{l=1}^{q_\delta(L)} l\, \mathbb{P}(L = l).$$

Contributions to Expected Shortfall – Definition

Definition: For a subportfolio loss $X \in \mathcal{L}_1(\mathbb{P})$ within a portfolio loss $Y \in \mathcal{L}_1(\mathbb{P})$ define the expected shortfall contribution at level $\delta \in (0,1)$ of X to Y by

$$\mathrm{ES}_\delta[X,Y] = \frac{E[X1_{\{Y>q_\delta(Y)\}}] + \beta_Y \mathbb{E}[X1_{\{Y=q_\delta(Y)\}}]}{1-\delta}$$

where

$$\beta_Y = \frac{\mathbb{P}(Y \le q_\delta(Y)) - \delta}{\mathbb{P}(Y = q_\delta(Y))}$$

if $\mathbb{P}(Y = q_\delta(Y)) > 0$ and 0 otherwise.

Remark: If $\mathbb{P}(Y \le q_\delta(Y)) = \delta$, then $\beta_Y = 0$ and

$$\mathrm{ES}_\delta[X,Y] = \mathbb{E}[X|Y > q_\delta(Y)].$$

Contributions to Expected Shortfall – Calculation in Extended CreditRisk$^+$

By consistency and linearity of the allocation

$$\mathrm{ES}_\delta[L] = \mathrm{ES}_\delta[L,L] = \sum_{i=1}^{m} \sum_{g\in G_i} \sum_{k=0}^{K} \mathrm{ES}_\delta[L_{g,i,k}, L].$$

Since

$$\mathbb{E}\big[L_{g,i,k}1_{\{L>q_\delta(L)\}}\big] = \underbrace{\mathbb{E}[L_{g,i,k}]}_{=\lambda_g w_{g,k}\,\mathbb{E}[L_{g,i,k,1}]} - \mathbb{E}\big[L_{g,i,k}1_{\{L\le q_\delta(L)\}}\big],$$

we will compute $\mathbb{E}[L_{g,i,k}1_{\{L=l\}}]$ for $l \in \{1,\dots,q_\delta(L)\}$. This can be done using the following lemma.

Lemma on Risk Contributions in CreditRisk$^+$

For every obligor $i \in \{1,\dots,m\}$, every group $g \in G_i$ and total loss $l \in \mathbb{N}_0$,

$$\mathbb{E}[L_{g,i,0}1_{\{L=l\}}] = \lambda_g w_{g,0} \sum_{\nu=1}^{l} \mathbb{E}\big[L_{g,i,0,1}1_{\{L_{g,0,1}=\nu\}}\big]\,\mathbb{P}(L = l-\nu)$$

and, for every risk $k \in \{1,\dots,K\}$,

$$\mathbb{E}[L_{g,i,k}1_{\{L=l\}}] = \lambda_g w_{g,k} \sum_{\nu=1}^{l} \mathbb{E}[L_{g,i,k,1}1_{\{L_{g,k,1}=\nu\}}]\,\mathbb{E}[\Lambda_k 1_{\{L=l-\nu\}}].$$

Possible Approaches to Operational Risk Modelling

The Basel committee defined three approaches towards the quantification of operational risk. The two simple ones define concrete formulae for the risk capital, namely

- Basic indicator approach (BIA).
- Standardized approach (SA).

To reduce supervisory capital needs, an individual

- Advanced measurement approach (AMA)

can be chosen.

Business Lines for Operational Risk

- Eight business lines in the standardized approach:

(1) Corporate finance,	(5) Payment & settlement,
(2) Trading & sales,	(6) Agency services,
(3) Retail banking,	(7) Asset management,
(4) Commercial banking,	(8) Retail brokerage.

- These business lines also serve as categories for an advanced measurement approach.

Seven Loss-Types Distinguished for the Advanved Measurement Approach

- Internal fraud
- External fraud
- Employment practices & workplace safety
- Clients, products & business practice
- Damage to physical assets
- Business disruption & system failures
- Execution, delivery & process management

Application of Extended CreditRisk$^+$ Methodology to Operational Risk: Reinterpretation of the Credit Risk Notation

- Number m of obligors $\rightarrow$ number of business lines ($m = 8$ for the ones given in the Basel commitee's document is an appropriate choice).
- Basic loss unit E stays the same ($E = 10\,000$).
- Number K of non-ideosyncratic risk factors $\rightarrow$ number of loss types ($K = 7$ for above list).
- Numbers $\sigma_k^2 > 0$ denote the relative variance of occurrences of losses of type $k \in \{1, \dots, K\}$.
- G contains the subsets of all business lines which can incur a loss due to the same event.

Notation for Business Lines and Risk Groups

We need for every risk group $g \in G$ of business lines

- the (one year) intensity $\lambda_g \geq 0$ for being hit by an operational loss event,
- the conditional probabilty $w_{g,0} \in [0,1]$ for an idiosyncratic operational loss event not to belong to the types in $\{1, \dots, K\}$, of course $w_{g,0} = 0$ is a possible choice,
- the conditional probabilities $w_{g,k} \in [0,1]$ for an operational loss event to be of type $k \in \{1, \dots, K\}$,

- the multivariate probability distribution $Q_{g,k} = \{q_{g,k,\mu}\}_{\mu \in \mathbb{N}_0^g}$ on $\mathbb{N}_0^g$ describing the severity of the stochastic losses of the business lines $i \in g$ in case an operational loss event of type $k \in \{0, \dots, K\}$ hits the group g of business lines.

Operational Risk Management

With the adoption of the extended CreditRisk$^+$ model for operational risk, a risk manager can

- calculate the distribution of the operational loss,
- calculate risk measures such as value-at-risk and expected shortfall (might be infinity)
- and identify risky business lines and groups by their risk contribution (in case of finite expected shortfall).