CASE STUDY III

EVALUATING MODEL RISK WITHIN THE BLACK–SCHOLES FRAMEWORK

Limiting Model Risk by Short-Selling Constraints

Outline for case study III

- Samuelson (Black–Scholes) model
- Exotic options, unlimited short positions
- Mitigation of model risk
  by short-selling constraints
- Resulting market incompleteness,
  upper hedging price
- Incorporation of constraint into option price
- Option price as stochastic control problem
- Explicit valuation for several examples

References for case study III

- U. Schmock, S. E. Shreve, U. Wystup:
  Valuation of Exotic Options under Shortselling Constraints
  Finance and Stochastics, Vol. 6 (2002) 143–172.

- U. Schmock, S. E. Shreve, U. Wystup:
  Dealing with Dangerous Digitals
  In: J. Hakala and U. Wystup (eds.):
  Foreign Exchange Risk: Models, Instruments and Strategies
  Risk Books, Risk Waters Group (2002) 327–348.

Samuelson model

Geometric Brownian motion for the exchange rate:

$$dS_t = (r_d - r_f)S_t\,dt + \sigma S_t\,dW_t, \quad S_0 > 0$$

S_t	price of one unit of foreign currency in domestic currency at time $t \in [0, T]$
$r_d \in \mathbb{R}$	risk-free domestic interest rate
$r_f \in \mathbb{R}$	risk-free foreign interest rate
$\sigma > 0$	volatility
$(W_t)_{t\in[0,T]}$	Brownian motion (Wiener process) under the risk-neutral measure $\mathbb{P}$
$r \triangleq r_d - r_f$	mean rate of return of the exchange rate

Samuelson model (cont.)

Equity model

S_t	stock price at time t
$r_d \in \mathbb{R}$	risk-free domestic interest rate
$r_f \in \mathbb{R}$	continuously paid dividend rate

Solution of the SDE

$$S_t = S_0 \exp\Big(rt + \sigma W_t - \frac{\sigma^2}{2}t\Big), \quad t \in [0, T]$$

Canonical probability space

$\Omega = C([0,T], \mathbb{R}) \ni \omega \mapsto W_t(\omega) = \omega(t)$
σ-algebra $\mathcal{F}_t$ is the $\mathbb{P}$-completion of $\sigma(W_s; s \in [0,t])$,
i. e., $\{\mathcal{F}_t\}_{t\in[0,T]}$ is a Brownian filtration.

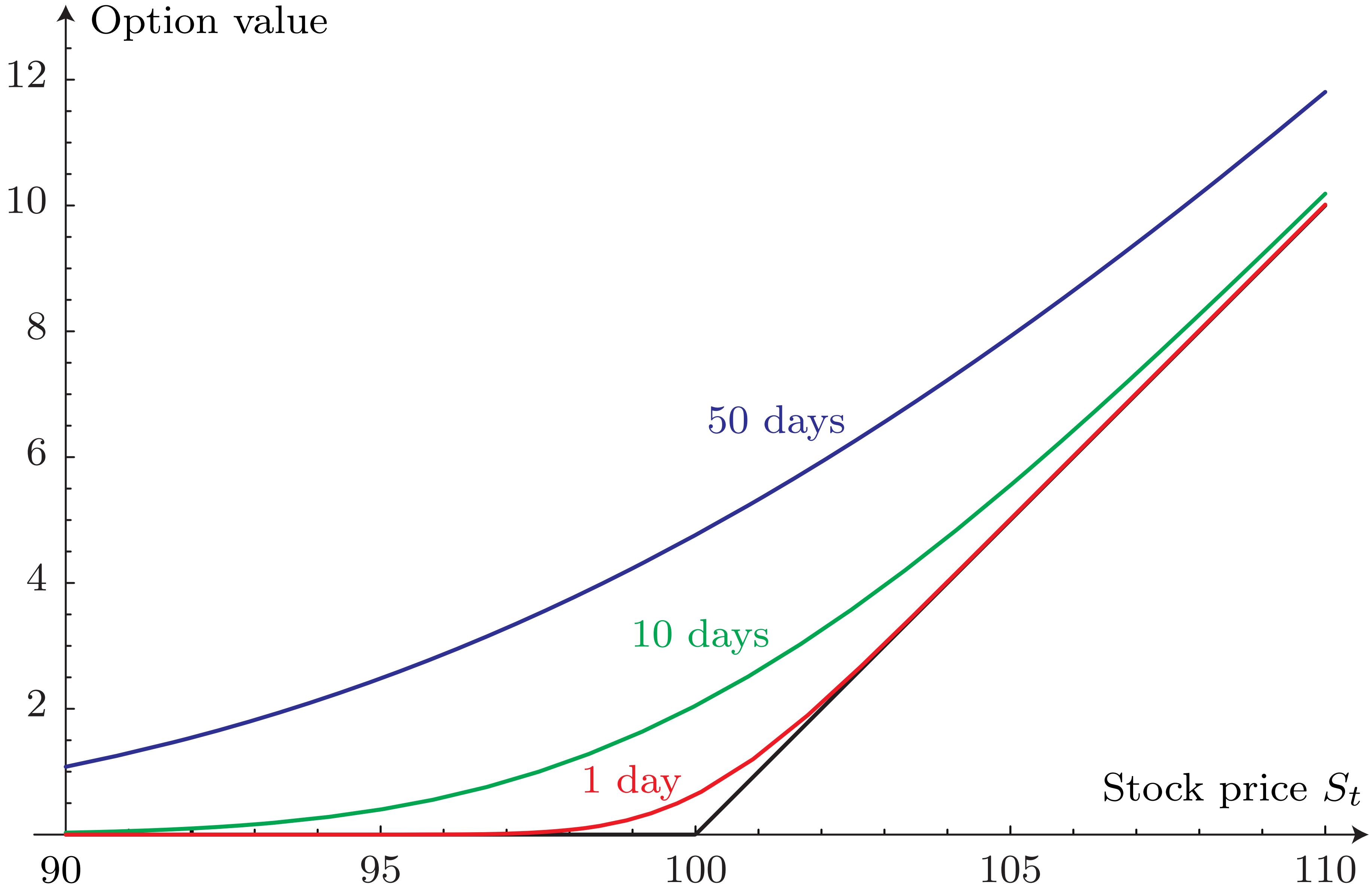
Hedging a European call option with strike K

- Pay-off at maturity T is $(S_T - K)^+$ where S_T is the price of the underlying at time T and $K > 0$ is the strike price.
- To hedge the call, buy a fraction of the underlying (delta-hedging).

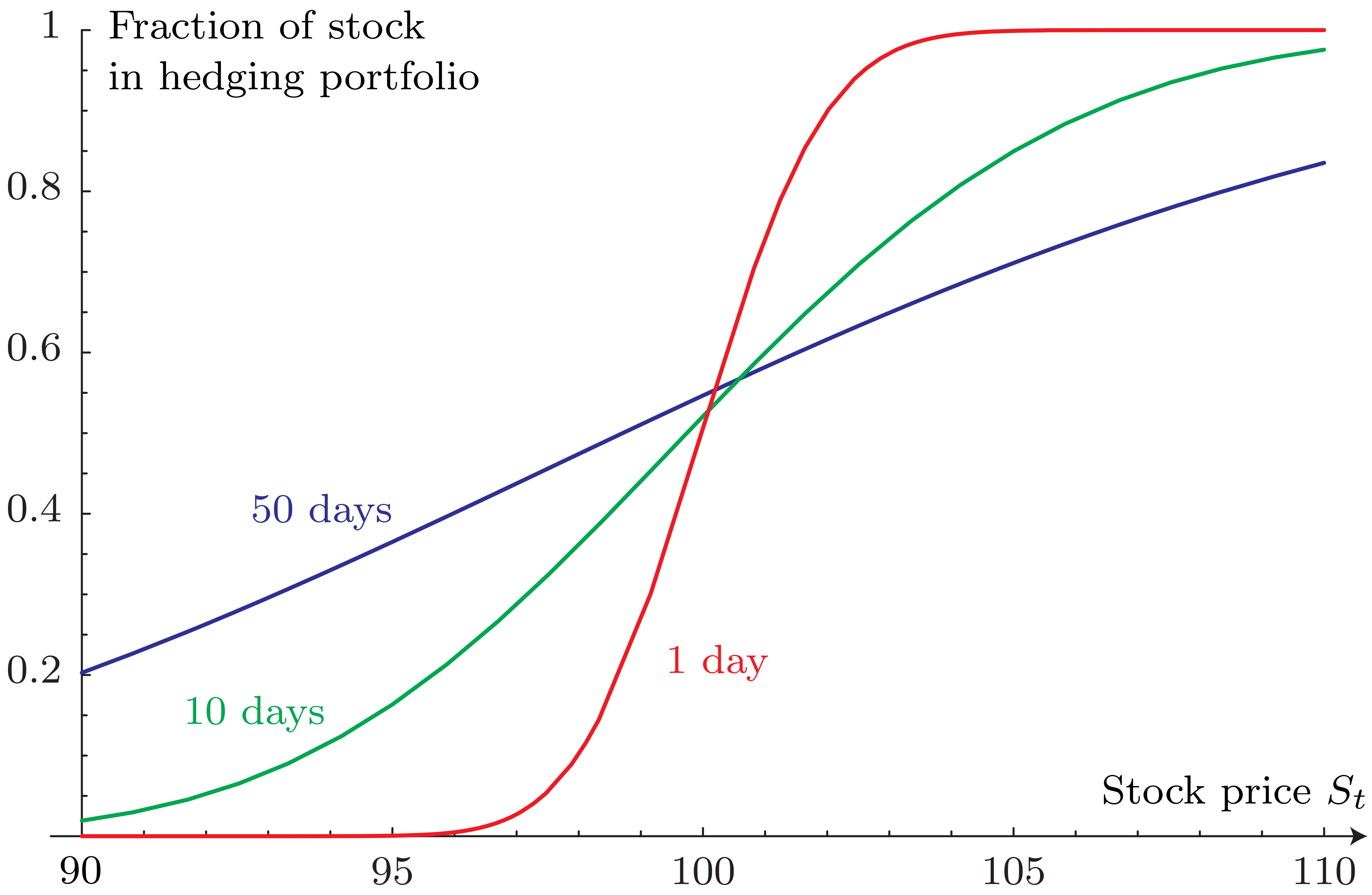
In the Samuelson model:

Calculation of the fraction $\in (0, 1)$ at time t by differentiation of the Black–Scholes formula w. r. t. S_t

$$\mathcal{N}\left(\frac{\log(S_t/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}\right)$$



Price of a European call option for three different maturities, interest rate $r = 5\%$, volatility $\sigma = 30\%$, strike price $K = 100$



Hedge for a European call option for three different maturities, interest rate $r = 5\%$, volatility $\sigma = 30\%$, strike price $K = 100$

Reverse up-and-out call

European call option with strike $K > 0$ and knock-out barrier $B > K$. Pay-off at maturity T

$$g(S) \triangleq (S_T - K)^+ 1_{\{\max_{t\in[0,T]} S_t < B\}}$$

No-arbitrage Black–Scholes price at time $t \in [0, T]$

$$v(t, x) = \mathbb{E}^{t,x}\big[e^{-r_d(T-t)}(S(T) - K)^+ 1_{\{\max_{u\in[t,T]} S_u < B\}}\big]$$

if $S_t = x > 0$ and no knock-out occurred before t.

Delta hedging:

– If S_t is well below B: Buy a fraction of the underlying.

– If S_t is just below B: Go short in the underlying.

Price of the reverse up-and-out call

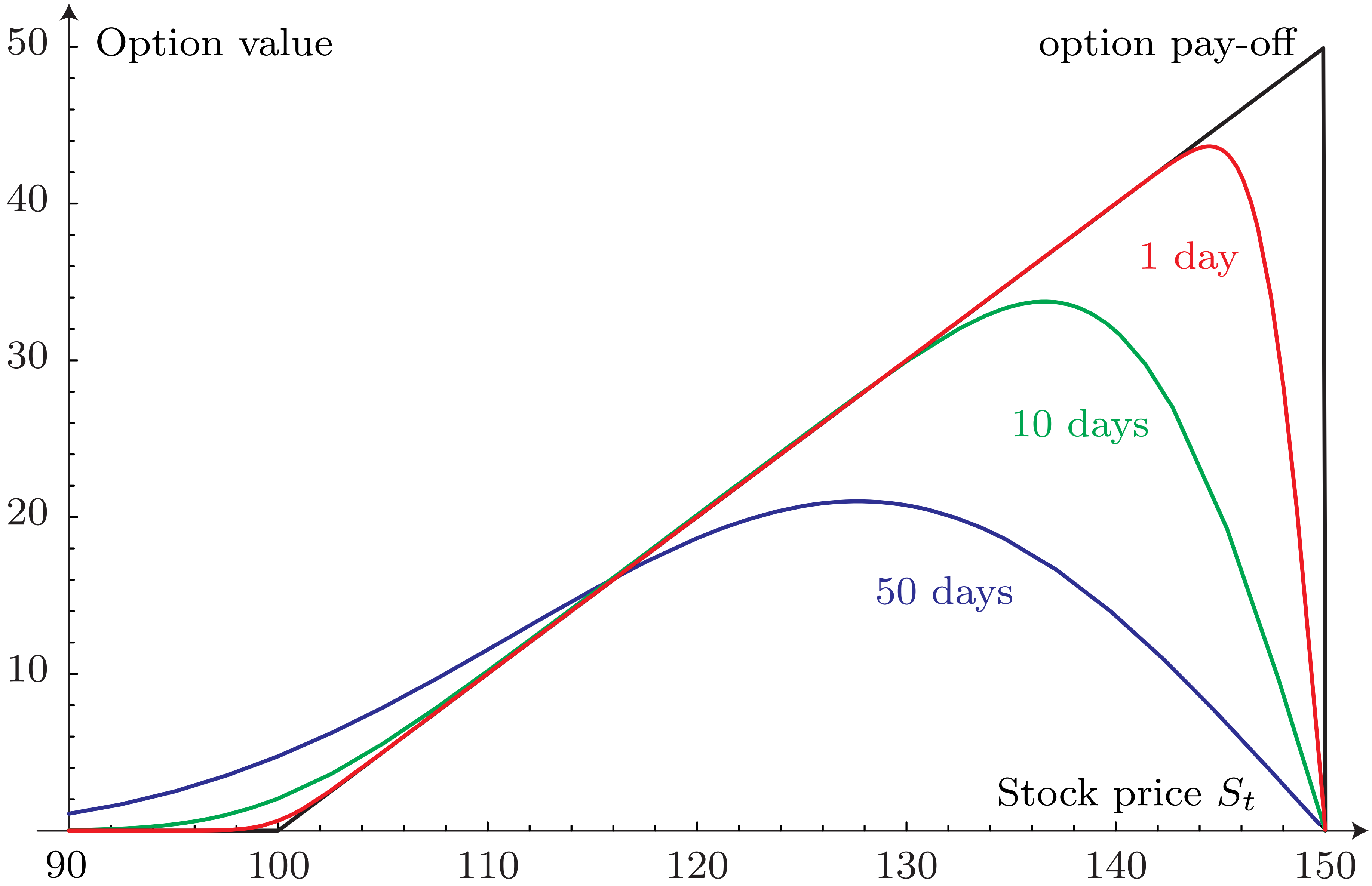
$$
\begin{aligned}
v(t,x) = xe^{-r_f\tau}&\big[\mathcal{N}(b-\theta_+) - \mathcal{N}(k-\theta_+)\big] \\
&+ xe^{-r_f\tau+2b\theta_+}\big[\mathcal{N}(b+\theta_+) - \mathcal{N}(2b-k+\theta_+)\big] \\
&- Ke^{-r_d\tau}\big[\mathcal{N}(b-\theta_-) - \mathcal{N}(k-\theta_-)\big] \\
&- Ke^{-r_d\tau+2b\theta_-}\big[\mathcal{N}(b+\theta_-) - \mathcal{N}(2b-k+\theta_-)\big]
\end{aligned}
$$

where $\mathcal{N}$ is the standard normal distribution function,

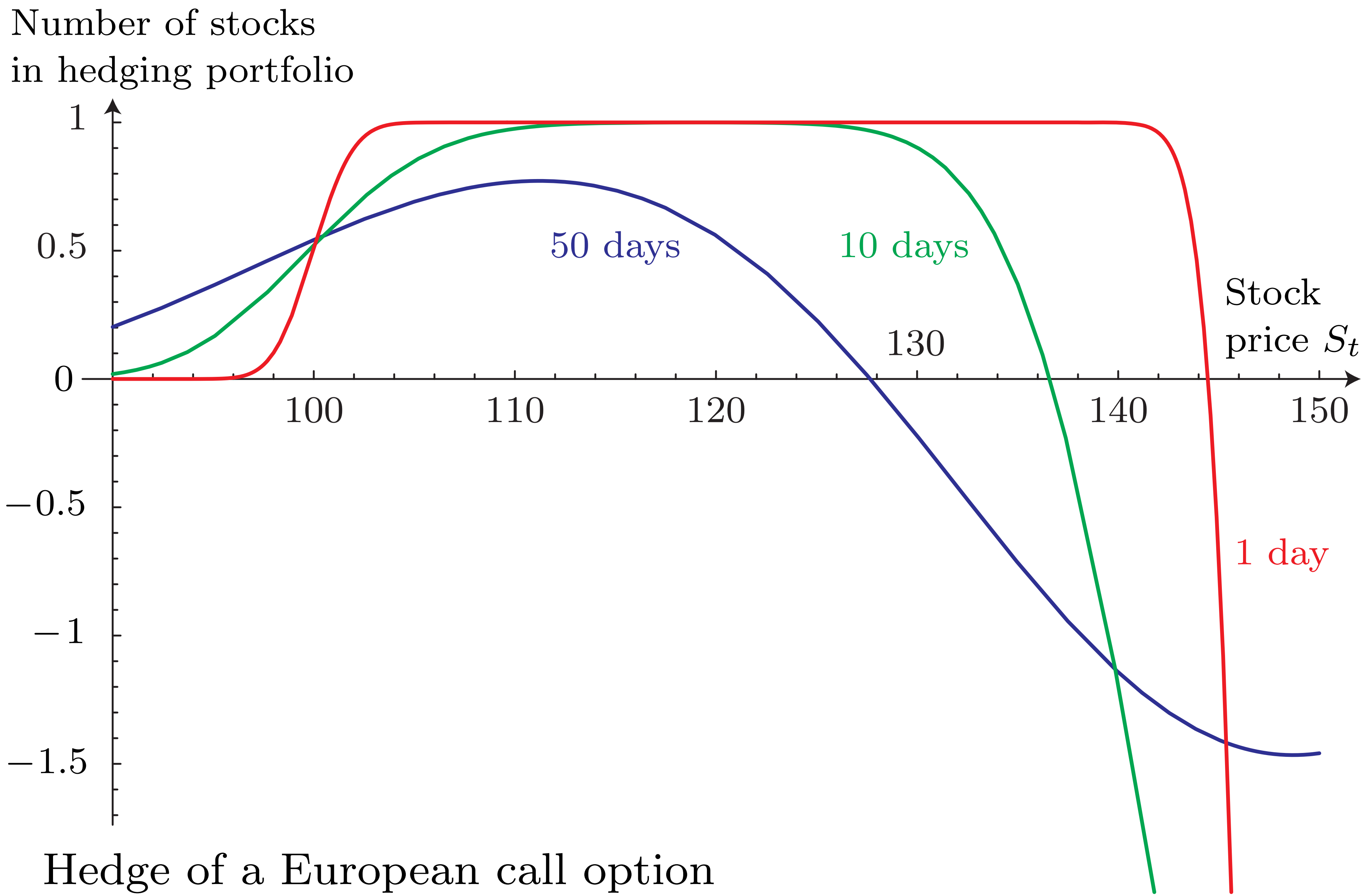
$$
\tau \triangleq T - t, \quad \theta_\pm \triangleq \left(\frac{r}{\sigma} \pm \frac{\sigma}{2}\right)\sqrt{\tau}
$$

and

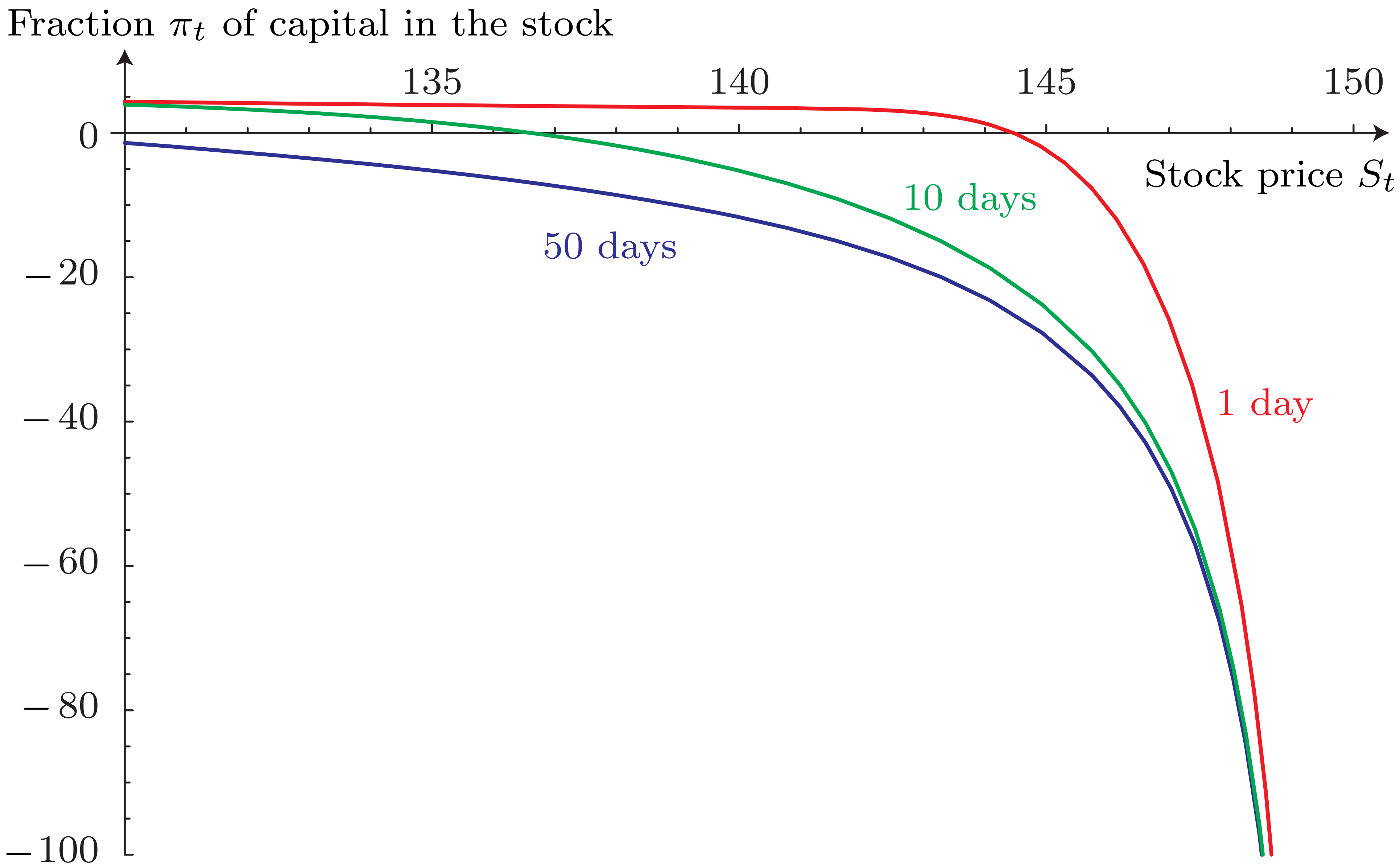
$$
b \triangleq \frac{1}{\sigma\sqrt{\tau}} \log\frac{B}{x}, \quad k \triangleq \frac{1}{\sigma\sqrt{\tau}} \log\frac{K}{x}.
$$



Price of a European call option with knock-out barrier $B = 150$ for three different maturities together with the option pay-off, interest rate $r = 5\%$, volatility $\sigma = 30\%$, strike price $K = 100$



Hedge of a European call option
with knock-out barrier $B = 150$ for three different maturities,
interest rate $r = 5\%$, volatility $\sigma = 30\%$, strike price $K = 100$



Fraction π_t of capital invested in the stock to replicate a European call option with knock-out barrier $B = 150$ for three different maturities, interest rate $r = 5\%$, volatility $\sigma = 30\%$, strike price $K = 100$

Problems with large FX positions

- Large exposure for one sold barrier option
- Liquidity risk and transaction costs
- High model risk!

Possible solutions

- Pay a rebate at maturity or at the first hitting time of the barrier, when the option knocks out.
- Modify the knock-out regulation (soft barrier option, step option, Parisian option).
- Impose constraint for the hedge portfolio. $\longrightarrow$ incomplete market, superhedge the option

Evolution of the hedge capital X_t

π_t	fraction of X_t in foreign currency (adapted)
$1-\pi_t$	fraction of X_t in domestic currency
C_t	capital consumed in $[0,t]$

$$\begin{aligned} dX_t &= \frac{\pi_t X_t}{S_t} dS_t + r_f \pi_t X_t \, dt + r_d (1-\pi_t) X_t \, dt - dC_t \\ &= r_d X_t \, dt + \sigma \pi_t X_t \, dW_t - dC_t \end{aligned}$$

Option pay-off

Lower semi-continuous function $g \colon C_+[0,T] \to [0,\infty)$

Short-selling constraint for foreign currency

$$\pi_t \geq -\alpha \text{ for all } t \in [0,T] \text{ with } \alpha \geq 0$$

Upper hedging price

$$v(0, S_0; \alpha) \triangleq \inf\{\, X_0 \mid \exists\ (\pi, C) \text{ with } X_T \geq g(S) \\ \text{and } \pi_t \geq -\alpha\ \forall\, t \in [0, T]\,\}$$

Dual maximization problem

Theorem: (Cvitanić & Karatzas 1993, El Karoui & Quenez 1995)

$$v(0, S_0; \alpha) = \sup_{\lambda \in \mathcal{L}} \mathbb{E}_\lambda \left[\, e^{-r_d T - \alpha \lambda_T} g(S) \right]$$

$\mathcal{L}$ contains all adapted, non-decreasing λ with $\lambda(0) = 0$, which are Lipschitz-continuous in t, uniformly in ω.

$$\frac{d\mathbb{P}_\lambda}{d\mathbb{P}} = \exp\left(-\frac{1}{\sigma} \int_0^T \lambda'_t \, dW_t - \frac{1}{2\sigma^2} \int_0^T (\lambda'_t)^2 \, dt \right)$$

Simplification for dependence on final value

Theorem: (Broadie, Cvitanić & Soner 1998)

If $g(S) = \varphi(S_T)$, then $v(0, S_0; \alpha) = \mathbb{E}\big[\, e^{-r_d T} \widehat{\varphi}_\alpha(S_T) \big]$ with face-lift

$$\widehat{\varphi}_\alpha(x) \triangleq \sup_{\lambda \geq 0} e^{-\alpha\lambda} \varphi(x e^{-\lambda}), \quad x \geq 0.$$

Aim of our work

- Generalization to path-dependent options by conversion of the dual maximization problem to a stochastic control problem.
- Explicit computation of the upper hedging price for several examples.

Idea behind Broadie–Cvitanić–Soner theorem

$(t, x) \mapsto v(t, x; \alpha)$ is the smallest function which

- satisfies the Black–Scholes PDE

$$v_t + rxv_x + \frac{1}{2}\sigma^2 x^2 v_{xx} - r_d v = 0,$$

- dominates the final pay-off, i.e. $v(T, x; \alpha) \geq \varphi(x)$,
- satisfies the constraint $\alpha v(t, x; \alpha) + x v_x(t, x, \alpha) \geq 0$.

$\widehat{\varphi}_\alpha$ is the smallest function dominating the pay-off and satisfying the constraint $\alpha\widehat{\varphi}_\alpha(x) + x\widehat{\varphi}'_\alpha(x) \geq 0$.

$\longrightarrow$ Solve Black–Scholes PDE with pay-off $\widehat{\varphi}_\alpha$.
Pleasant surprise: Solution satisfies constraint!

Extension to path-dependent up-and-out call

Observation: If $v(t, x; \alpha)$ solves the Black–Scholes PDE, then $w \triangleq \alpha v + x v_x$ solves the PDE, too.

Strategy:

- Boundary conditions for v give boundary conditions for w.
- Require $w = 0$ at the boundary where the unconstrained value function violates the constraint.
- Solve Black–Scholes PDE for w.
- Solve $w \triangleq \alpha v + x v_x$ for v.

Formulation of the dual problem as singular stochastic control problem

Theorem (Schmock/Shreve/Wystup):

$$v(0, S_0; \alpha) = \sup_{\lambda \in \mathcal{C}} \mathbb{E}\big[e^{-r_d T - \alpha \lambda_T} g(S e^{-\lambda})\big]$$

where $\mathcal{C} \triangleq \{\lambda \,|\, \lambda$ adapted, non-decreasing, continuous process, $\lambda(0) = 0\}$.

Remarks:

- Maximization w. r. t. processes is easier.
- Maximizing process can be found in many examples.
- Maximizing processes can be singularly continuous.
- Since g is lower instead of upper semi-continuous, maximizing processes need not exist.

Application to a European call option with strike K and knock-out barrier $B > K$

Obligation at maturity T:

$$g(S) \triangleq (S_T - K)^+ 1_{\{\max_{t\in[0,T]} S_t < B\}}$$

Maximization problem:

$$\sup_{\lambda\in\mathcal{C}} \mathbb{E}\Big[e^{-r_d T - \alpha\lambda_T}\Big(S_T e^{-\lambda_T} - K\Big)^+ 1_{\{\max_{t\in[0,T]} S_t e^{-\lambda_t} < B\}}\Big]$$

Supremum unchanged for $< \to \leq$.

Maximizing process:

$$S_t e^{-\lambda_t} \leq B \iff \lambda_t \geq \log S_t - \log B$$

$$\implies \lambda_t \geq \lambda_t^* \triangleq \max_{u\in[0,t]} (\log S_u - \log B)^+$$

Upper hedging price

$$
\begin{aligned}
v^*(t,x,\alpha) &= xe^{-r_f\tau}\Big[\mathcal{N}(b-\theta_+)-\mathcal{N}(k-\theta_+)\\
&+e^{\frac{1}{2}s(s-2\theta_+)}\times\left\{e^{sb}\mathcal{N}(-b+\theta_+-s)-e^{sk}\mathcal{N}(-k+\theta_+-s)\right\}\Big]\\
&+\frac{sxe^{-r_f\tau+2b\theta_+}}{s-2\theta_+}\Big[\mathcal{N}(b+\theta_+)-\mathcal{N}(\ell+\theta_+)+e^{\frac{1}{2}s(s-2\theta_+)}\\
&\quad\times\left\{e^{(s-2\theta_+)b}\mathcal{N}(-b+\theta_+-s)-e^{(s-2\theta_+)\ell}\mathcal{N}(-\ell+\theta_+s)\right\}\Big]\\
&-Ke^{-r_d\tau}\Big[\mathcal{N}(b-\theta_-)-\mathcal{N}(k-\theta_-)\\
&\quad+e^{\frac{1}{2}\tilde{s}(\tilde{s}-2\theta_-)}\left\{e^{\tilde{s}b}\mathcal{N}(-b+\theta_--\tilde{s})-e^{\tilde{s}k}\mathcal{N}(-k+\theta_--\tilde{s})\right\}\Big]\\
&-\frac{\tilde{s}Ke^{-r_d\tau+2b\theta_-}}{\tilde{s}-2\theta_-}\Big[\mathcal{N}(b+\theta_-)-\mathcal{N}(\ell+\theta_-)+e^{\frac{1}{2}\tilde{s}(\tilde{s}-2\theta_-)}\\
&\quad\times\left\{e^{(\tilde{s}-2\theta_-)b}\mathcal{N}(-b+\theta_--\tilde{s})-e^{(\tilde{s}-2\theta_-)\ell}\mathcal{N}(-\ell+\theta_--\tilde{s})\right\}\Big]
\end{aligned}
$$

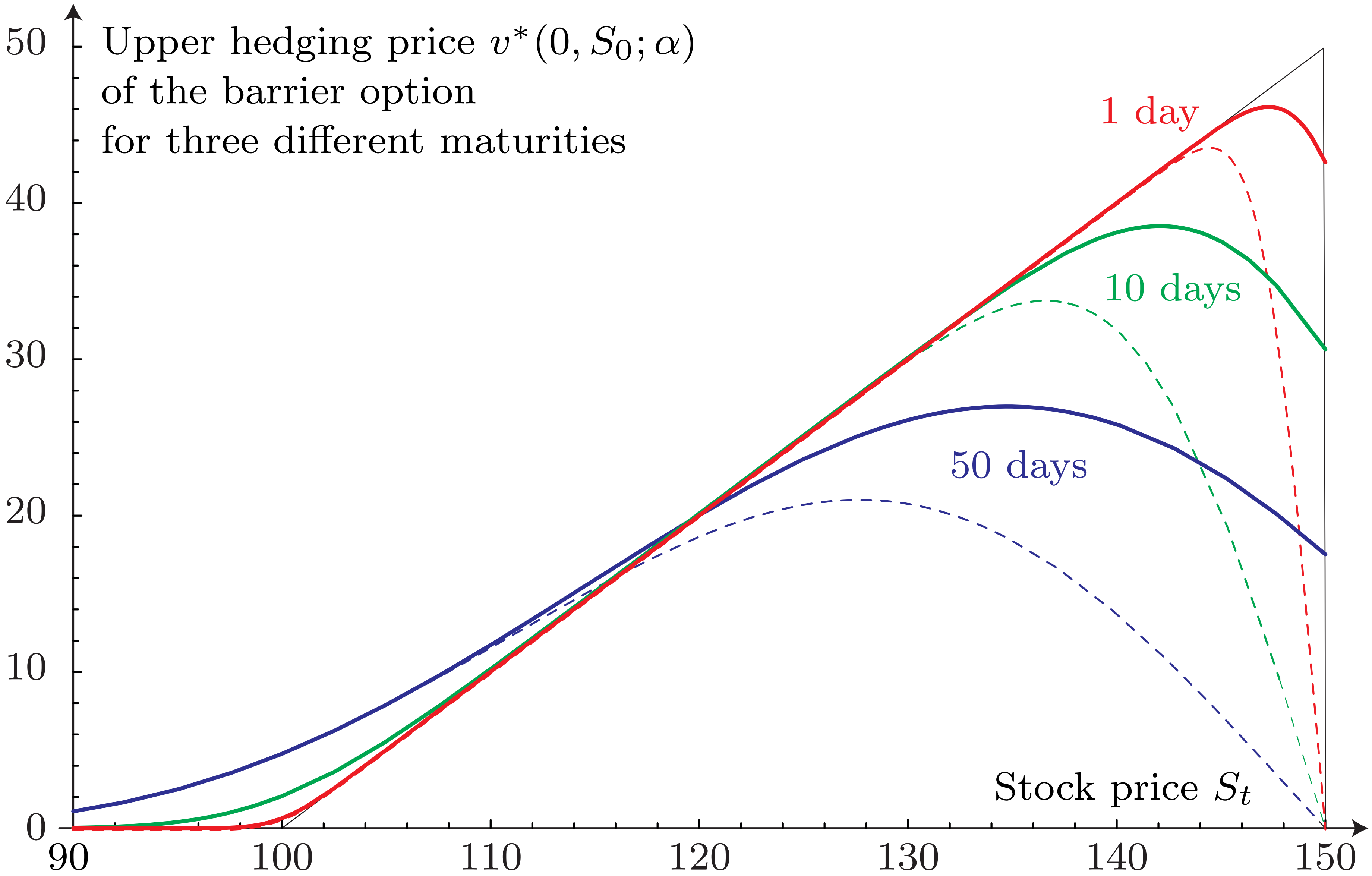
. . . with the abbreviations

$$\tau = T - t \qquad b = \frac{1}{\sigma\sqrt{\tau}} \log \frac{B}{x}$$

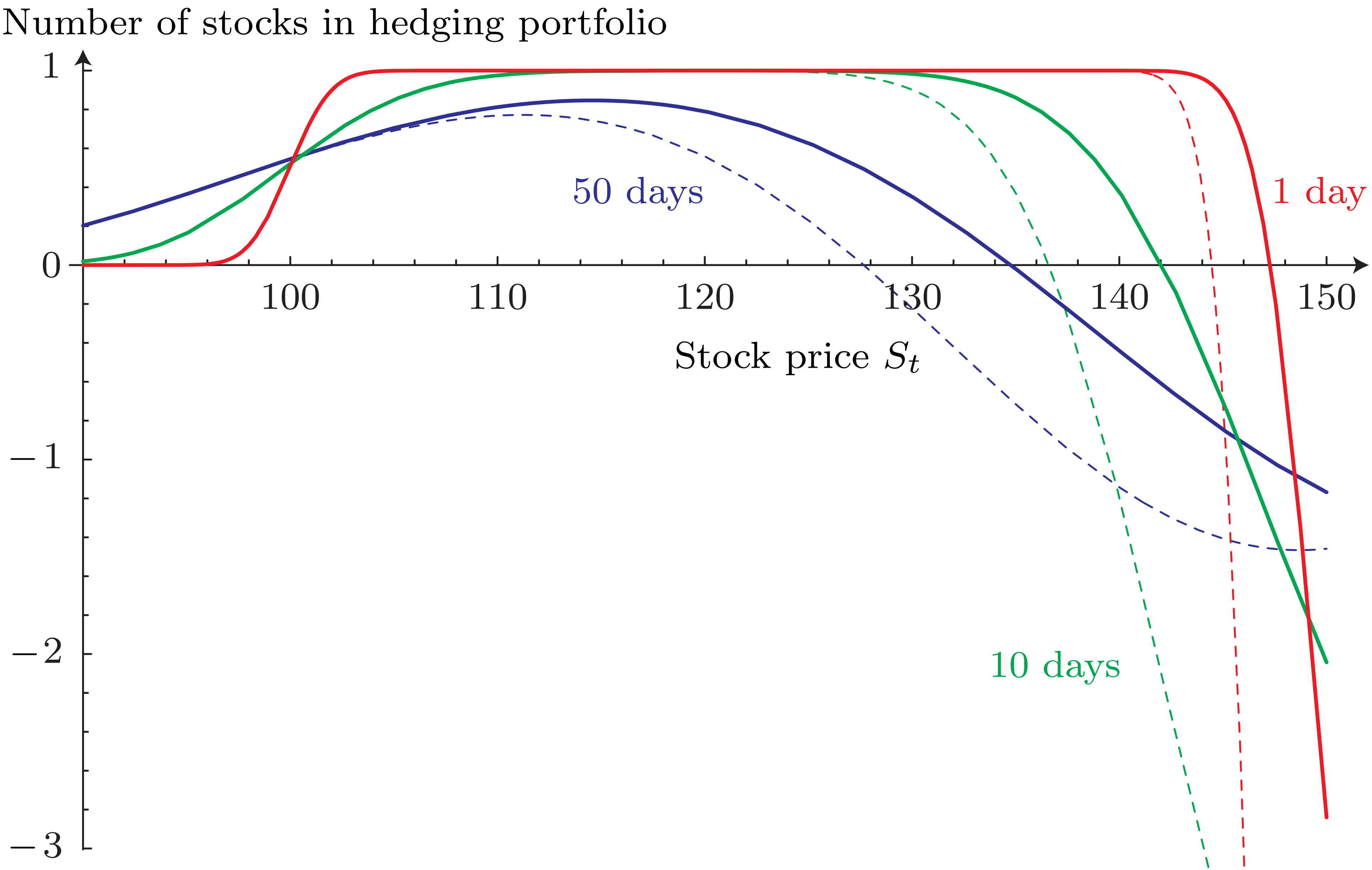
$$s = (1+\alpha)\sigma\sqrt{\tau} \qquad k = \frac{1}{\sigma\sqrt{\tau}} \log \frac{K}{x}$$

$$\tilde{s} = \alpha\sigma\sqrt{\tau} \qquad \theta_{\pm} = \left(\frac{r}{\sigma} \pm \frac{\sigma}{2}\right)\sqrt{\tau}$$

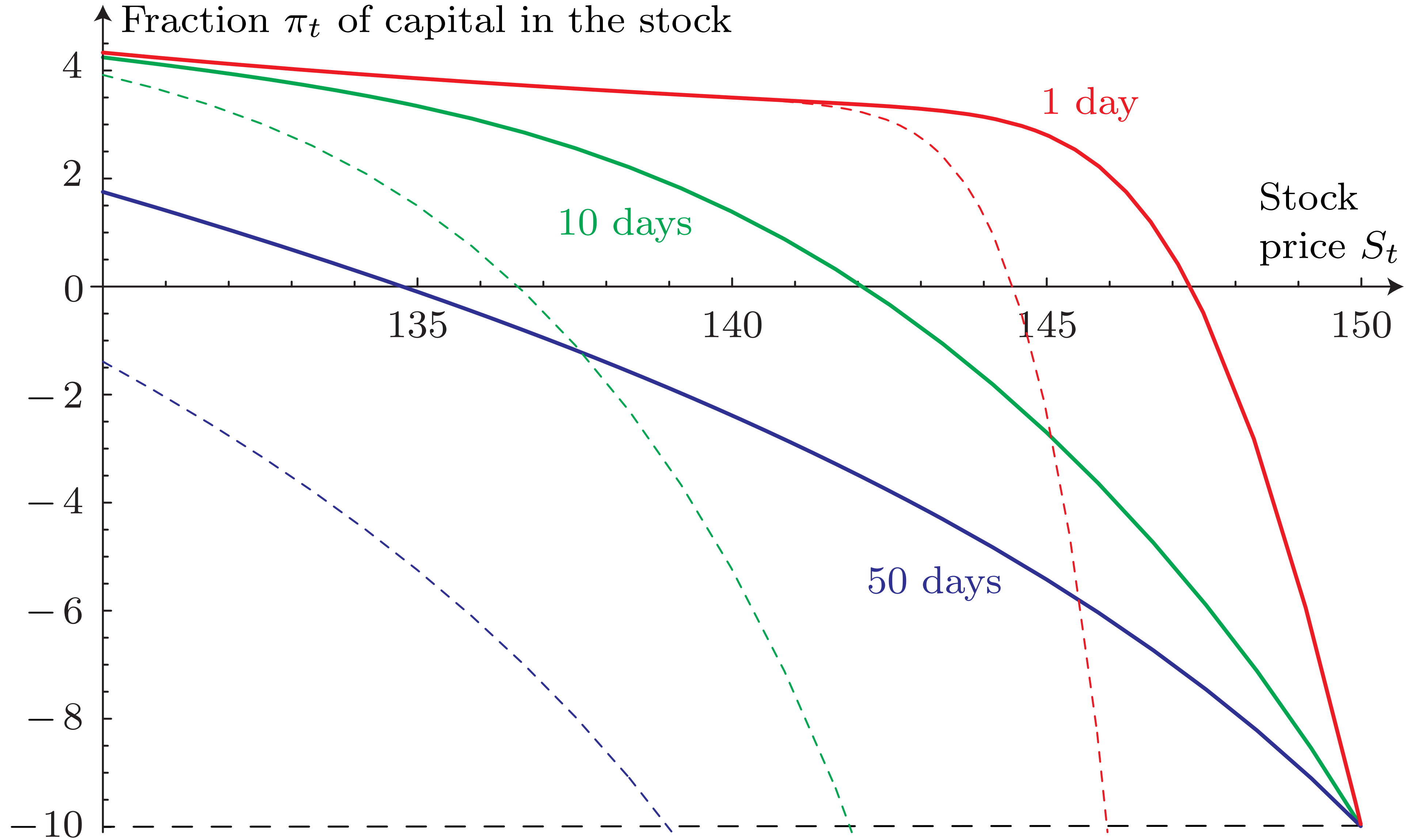
$$\ell = 2b - k \qquad \mathcal{N}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} \exp(-u^2/2)\, du$$



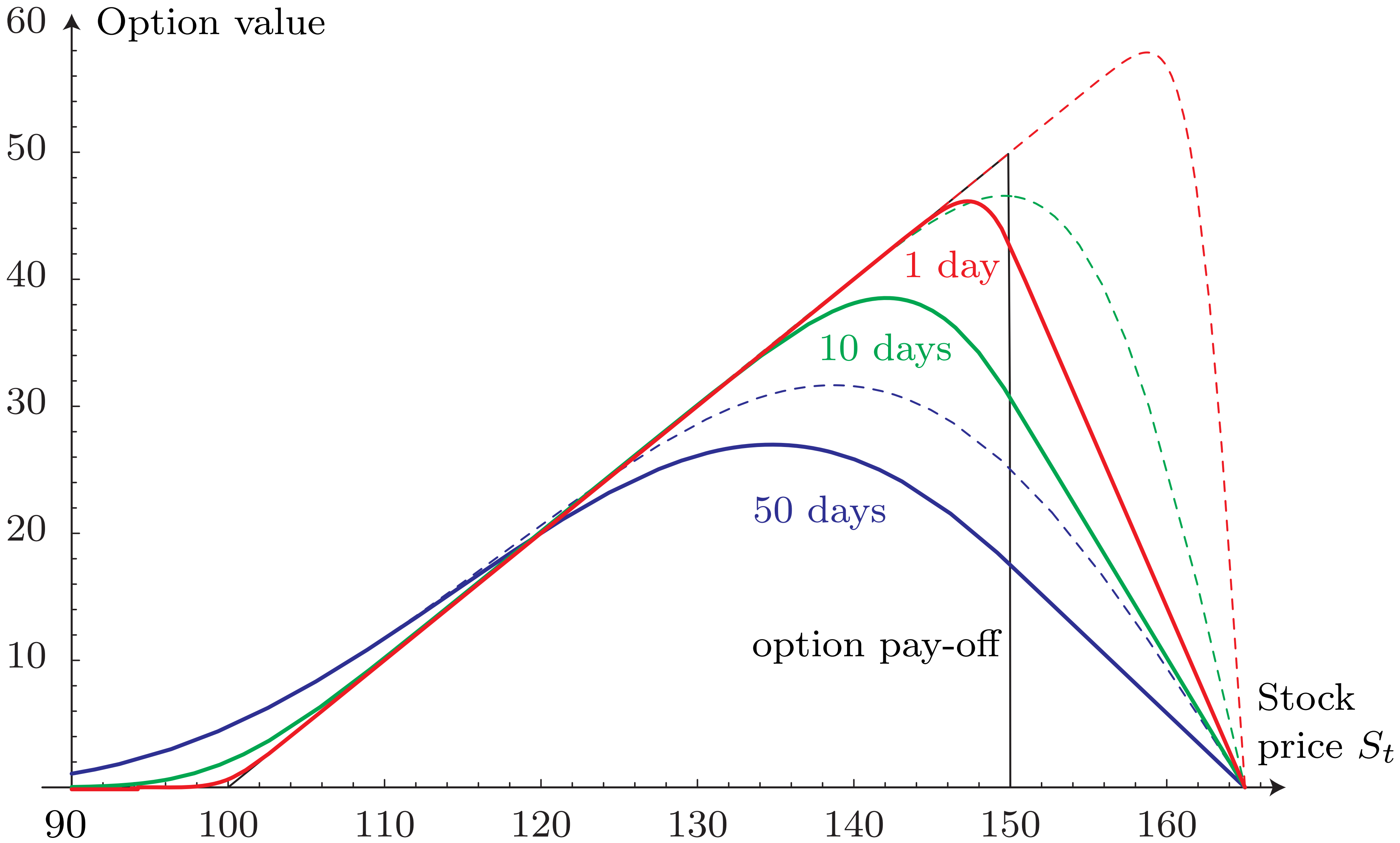
Knock-out barrier $B = 150$, portfolio constraint $\alpha = 10$,
interest rate $r = 5\%$, volatility $\sigma = 30\%$, strike price $K = 100$



Super-replication of a European call option with knock-out barrier $B = 150$, hedge-portfolio constraint $\alpha = 10$, interest rate $r = 5\%$, volatility $\sigma = 30\%$, strike price $K = 100$



Fraction π_t of capital invested in the stock to super-replicate a European call option with knock-out barrier $B = 150$, portfolio constraint $\alpha = 10$, interest rate $r = 5\%$, volatility $\sigma = 30\%$, strike price $K = 100$



Price of a European call option with knock-out barrier $B = 150$ for three different maturities, portfolio constraint $\alpha = 10$, interest rate $r = 5\%$, volatility $\sigma = 30\%$, strike price $K = 100$, and linear extrapolation. The dashed lines show the price without portfolio constraint but a barrier moved to $B' = B(1 + 1/\alpha) = 165$.

Stochastic impulsive control problem

- $0 < t_1 < t_2 < \cdots < t_I \leq T$ fixed dates for impulses
- $R[0,T]$ set of non-decreasing, on $[0,T]\setminus\{t_1,\ldots,t_I\}$ continuous, in $t_1,\ldots,t_I$ right-continuous functions which start in the origin

Theorem: (Schmock/Shreve/Wystup)

$$v(0,S_0;\alpha) = \sup_{\lambda\in\mathcal{R}} \mathbb{E}\Big[e^{-r_d T-\alpha\lambda_T} g_*(S,\lambda)\Big]$$

where $\mathcal{R} \triangleq \{\,\lambda \mid \lambda \text{ adapted process, paths in } R[0,T]\}$,

$$g_*(S,\lambda) \triangleq \inf_{\{\lambda_n\}_{n\in\mathbb{N}}} \liminf_{n\to\infty} g(Se^{-\lambda_n}),$$

Infimum over all $\{\lambda_n\}_{n\in\mathbb{N}}$ in $\mathcal{C}$ with $\lambda_n \to \lambda$ pointwise.

Applications of the stochastic impulsive control problem

European call option with knock-out barrier $B > K$, which is checked only at times $0 < t_1 < \cdots < t_I \leq T$.

Pay-off at maturity:

$$g(S) = \big(S_T - K\big)^+ \prod_{i=1}^{I} 1_{\{S_{t_i} < B\}}$$

Maximizing process (for $< \to \leq$):

$$\lambda_t^* = \max_{\{i;t_i \leq t\}} \big(\log S_{t_i} - \log B\big)^+$$

Upper hedging price $v(0, S_0, \alpha)$:

$$\mathbb{E}\Big[e^{-r_d T - \alpha \lambda_T^*} \big(S_T e^{-\lambda_T^*} - K\big)^+\Big]$$

2. Example: Asian put option

Pay-off at maturity T:

$$g(S) = \big(A(S) - S_T\big)^+$$

with arithmetic average

$$A(S) \triangleq \frac{1}{T}\int_0^T S_t\,dt$$

Maximizing process for $\alpha > 0$:

$$\lambda_t^* = \left(\log\frac{(1+\alpha)S_T}{\alpha A(S)}\right)^+ 1_{\{t=T\}}$$

Upper hedging price $v(0, S_0, \alpha)$:

$$\mathbb{E}\Big[e^{-r_d T - \alpha\lambda_T^*}\big(A(S) - S_T e^{-\lambda_T^*}\big)^+\Big]$$

3. Example: Lookback put

Pay-off at maturity T:

$$g(S) = \big(M(S) - S_T\big)^+$$

with maximal stock price

$$M(S) \triangleq \max_{t\in[0,T]} S_t$$

Maximizing process for $\alpha > 0$:

$$\lambda_t^* = \left(\log \frac{(1+\alpha)S_T}{\alpha M(S)}\right)^+ 1_{\{t=T\}}$$

Upper hedging price $v(0, S_0, \alpha)$:

$$\mathbb{E}\big[e^{-r_d T - \alpha\lambda_T^*}\big(M(S) - S_T e^{-\lambda_T^*}\big)^+\big]$$

Further application:
Pricing and hedging a book of options

Advantages:

- More realistic
- Elimination of opposite risks
- Higher value of the book
  $\longrightarrow$ hedge-portfolio constraint less severe
- Lower option prices, hence more competitive pricing in the financial market

Challenge:

Evaluation of the stochastic impulsive control problem is more complicated.

Example of a book: Two European call options with knock-out barriers $U > L > K$

Decision at time

$$t^* \triangleq T \wedge \min\{t \geq 0 \mid \forall\, s \in [t, T] \quad 2v^L(s, L; \alpha) \geq v^U(s, L; \alpha)\}$$

Maximizing process: For $t \in [0, t^*]$

$$\lambda_t^* = \max_{u \in [0,t]} \left(\log S_u - \log U\right)^+$$

and for $t \in [t^*, T]$

$$\lambda_t^* = \max_{u \in [0,t]} \left(\log S_u - \log U\right)^+ 1_{\{M_{t^*} > L\}} + \max_{u \in [t^*,t]} \left(\log S_u - \log L\right)^+ 1_{\{M_{t^*} \leq L\}}$$

Sketch of proof for the singular stochastic control problem

1. Step: Reduce the supremum in the result of Cvitanić & Karatzas, El Karoui & Quenez to piecewise linear processes $\lambda \in \mathcal{L}$, i. e.,

$$\sup_{\lambda \in \mathcal{L}} \mathbb{E}_\lambda \left[e^{-r_d T - \alpha \lambda_T} g(S) \right] = \sup_{\lambda \in \mathcal{L}_{\mathrm{pl}}} \mathbb{E}_\lambda \left[e^{-r_d T - \alpha \lambda_T} g(S) \right].$$

2. Step: Define for every process $\lambda \in \mathcal{L}_{\mathrm{pl}}$ a new process $\bar{\lambda} \in \mathcal{L}_{\mathrm{pl}}$ (and vice versa), so that Girsanov's theorem implies

$$\mathbb{E}_\lambda \left[e^{-r_d T - \alpha \lambda_T} g(S) \right] = \mathbb{E} \left[e^{-r_d T - \alpha \bar{\lambda}_T} g\left(S e^{-\bar{\lambda}} \right) \right].$$

Details for the 2. Step

Transformation of Trajectories:

Consider $\lambda \in \mathcal{L}_{\mathrm{pl}}$. Then there exist $m \in \mathbb{N}$ and $0 = t_0 < t_1 < \cdots < t_m = T$ such that

$$\lambda(t,\omega) = \sum_{i=0}^{m-1} a_i(\omega)\big((t_{i+1} \wedge t) - t_i\big)^+$$

for $t \in [0,T]$ with $\mathcal{F}^W(t_i)$-measurable $a_i : \Omega \to [0,\infty)$.
Define $\varphi_\lambda : \Omega \to \Omega$ by

$$\varphi_\lambda(\omega)(t) \triangleq \omega(t) + \frac{1}{\sigma}\lambda(t,\omega).$$

φ_λ is injective.

Surjectivity of the Transformation:

For surjectivity, take $\bar{\omega} \in \Omega$, set $\omega(0) = 0$ and inductively, for $i \in \{0, 1, \ldots, m-1\}$ and $t_i < t \leq t_{i+1}$,

$$\omega(t) \triangleq \bar{\omega}(t) - \frac{1}{\sigma} \sum_{j=0}^{i} a_j(\omega)\big((t_{j+1} \wedge t) - t_j\big)^+.$$

Then $\bar{\omega} = \varphi_\lambda(\omega)$. Define

$$\bar{\lambda}(\cdot, \bar{\omega}) \triangleq \lambda(\cdot, \varphi_\lambda^{-1}(\bar{\omega})) \quad \text{for all } \bar{\omega} \in \Omega.$$

By Girsanov's theorem:

$$\omega \sim \mathbb{P}_\lambda \iff \bar{\omega} = \varphi_\lambda(\omega) \sim \mathbb{P}$$

3. Step: Extension of the supremum to all $\lambda \in \mathcal{C}$, i. e.,

$$\sup_{\lambda \in \mathcal{L}_{\mathrm{pl}}} \mathbb{E}\big[e^{-r_d T - \alpha \lambda_T} g\big(S e^{-\lambda}\big)\big]$$

$$= \sup_{\lambda \in \mathcal{C}} \mathbb{E}\big[e^{-r_d T - \alpha \lambda_T} g\big(S e^{-\lambda}\big)\big].$$

Pointwise approximation of $\lambda \in \mathcal{C}$ by piecewise linear processes $\lambda_n \in \mathcal{L}_{\mathrm{pl}}$; lower semi-continuity is required here for the application of Fatou's lemma.

Sketch of proof for the stochastic impulsive control problem

Extend to supremum from $\mathcal{C}$ to $\mathcal{R}$, i. e.,

$$\sup_{\lambda\in\mathcal{C}} \mathbb{E}\left[e^{-r_d T-\alpha\lambda_T} g\left(S e^{-\lambda}\right)\right] = \sup_{\lambda\in\mathcal{R}} \mathbb{E}\left[e^{-r_d T-\alpha\lambda_T} g_*(S,\lambda)\right].$$

Pointwise approximation of $\lambda \in \mathcal{R}$ by continuous processes $\lambda_n \in \mathcal{C}$. Turn the jumps at $t_1, \ldots, t_I$ into bounded, non-negative martingales:

$$M_{i,n}(t) = \mathbb{E}[(\lambda(t_i) - \lambda(t_i-)) \wedge n \,|\, \mathcal{F}_t].$$

$\{\mathcal{F}_t\}_{t\in[0,T]}$ is a Brownian filtration, hence there exist continuous modifications of these martingales.