A Generalization of Panjer's Recursion and Numerically Stable Risk Aggregation

Workshop on Credit Risk
Université d'Evry Val d'Essonne, June 25–27, 2008

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Workshop on Portfolio Risk Management

Mo., Sept. 29, 2008, Vienna, Austria

Organized by

Research Group for
Financial and Actuarial Mathematics and
CD-Laboratory for Portfolio Risk Management
Institute for Mathematical Methods in Economics
Vienna University of Technology
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www.fam.tuwien.ac.at/prisma2008

Motivation: The Collective Model

Task: Calculate the distribution of the random sum

$$S = X_1 + \cdots + X_N$$

of N losses, where the loss sizes $\{X_i\}_{i \in \mathbb{N}}$ are i. i. d. and independent of N.

Applications:

- Claims in a homogeneous insurance portfolio
- Losses in a credit portfolio ($\rightarrow$ extended CreditRisk$^+$)
- Operational losses (Basel II), aggregation for every line of business and loss type.

Standard tool: Panjer's recursion for specific distributions of N, when the X_i are $\mathbb{N}_0$-valued.

Loss Number Distributions in the Panjer Class

Definition: A probability distribution $\{q_n\}_{n \in \mathbb{N}_0}$ is said to belong to the Panjer(a, b, k) class with $a, b \in \mathbb{R}$ and $k \in \mathbb{N}_0$ if $q_0 = q_1 = \cdots = q_{k-1} = 0$ and

$$q_n = \left(a + \frac{b}{n}\right) q_{n-1} \quad \text{for all } n \in \mathbb{N} \text{ with } n \geq k+1.$$

Important examples: (all distributions are known)

- Poisson$(\lambda) \in$ Panjer$(0, \lambda, 0)$ with intensity $\lambda > 0$
- NegBin$(\alpha, p) \in$ Panjer$(q, (\alpha - 1)q, 0)$ with $\alpha > 0$, probability $p \in (0, 1)$ and $q := 1 - p$
- Log$(q) \in$ Panjer$(q, -q, 1)$ with $q \in (0, 1)$ and $q_n = -\frac{q^n}{n \log(1-q)}$ for all $n \in \mathbb{N}$

Extended Logarithmic Distribution

For $k \in \mathbb{N}\setminus\{1\}$ and $q \in (0,1]$ define $q_0 = \cdots = q_{k-1} = 0$,

$$q_n = \frac{\binom{n}{k}^{-1} q^n}{\sum_{l=k}^{\infty} \binom{l}{k}^{-1} q^l} \quad \text{for } n \geq k.$$

ExtLog(k, q) is in the Panjer$(q, -kq, k)$ class.

Extended Negative Binomial Distribution

For $k \in \mathbb{N}$, $\alpha \in (-k, -k+1)$ and $p \in [0,1)$ define $q = 1 - p$, $q_0 = \cdots = q_{k-1} = 0$ and

$$q_n = \frac{\binom{\alpha+n-1}{n} q^n}{p^{-\alpha} - \sum_{j=0}^{k-1} \binom{\alpha+j-1}{j} q^j} \quad \text{for } n \geq k.$$

ExtNegBin(α, k, p) is in the Panjer$(q, (\alpha-1)q, k)$ class.

Extended Panjer Recursion

If $\mathcal{L}(N) \in$ Panjer(a, b, k), independent of the i. i. d. $\mathbb{N}_0$-valued sequence $\{X_n\}_{n\in\mathbb{N}}$, and $a\mathbb{P}(X_1 = 0) \neq 1$, then $S := X_1 + \cdots + X_N$ satisfies

$$\mathbb{P}(S=0) = \varphi_N\big(\mathbb{P}(X_1=0)\big)$$

with φ_N the probability generating function of N, and

$$\mathbb{P}(S=n) = \frac{1}{1 - a\mathbb{P}(X_1=0)} \Bigg(\mathbb{P}(S_k=n)\mathbb{P}(N=k) + \sum_{j=1}^{n} \Big(a + \frac{bj}{n}\Big) \mathbb{P}(X_1=j)\mathbb{P}(S=n-j) \Bigg)$$

for all $n \in \mathbb{N}$, where $S_k = X_1 + \cdots + X_k$.

Example for Numerical Instability

Take $N \sim \text{ExtNegBin}(\alpha, k, p)$ with $k \in \mathbb{N}$, $\varepsilon, p \in (0,1)$ and $\alpha := -k + \varepsilon$. Consider the loss distribution $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = l) = 1/2$ with $l \geq 3$. Then

$$p_{k+l} = \mathbb{P}(S = k+l) = q\frac{k(l-1)+\varepsilon k}{k+l}\Big(\frac{q_k}{2^{k+1}} + \frac{q_{k+l-1}}{k2^{k+l}}\Big) - q\frac{k(l-1)-\varepsilon l}{k+l}\frac{q_k}{2^{k+1}}.$$

With $\varepsilon = 1/10\,000$, $k = 1$, $l = 5$, $p = 1/10$:

$$p_6 = 0.1499\,926 - 0.1499\,701 = 0.0000\,225\,.$$

Panjer recursion with five significant digits gives

$$p_6 = 0.0000\,400\ldots \qquad (\approx 78\% \text{ relative error}).$$

Panjer Recursion Replaced by Weighted Convolution

Fix $l \in \mathbb{N}$, consider $N \sim \{q_n\}_{n\in\mathbb{N}_0}$ and $\tilde{N}_i \sim \{\tilde{q}_{i,n}\}_{n\in\mathbb{N}_0}$, define $S = X_1 + \cdots + X_N \sim \{p_n\}_{n\in\mathbb{N}_0}$ and $\tilde{S}_{(i)} = X_1 + \cdots + X_{\tilde{N}_i} \sim \{\tilde{p}_{i,n}\}_{n\in\mathbb{N}_0}$ for $i \in \{1,\ldots,l\}$. Assume there exist $k \in \mathbb{N}_0$ and $a_1,\ldots,a_l,b_1,\ldots,b_l \in \mathbb{R}$ such that

$$q_n = \sum_{i=1}^{l}\Big(a_i + \frac{b_i}{n}\Big)\tilde{q}_{i,n-i} \quad \text{for } n \geq k+l$$

and $\tilde{q}_{i,0} = \cdots = \tilde{q}_{i,k+l-i-1} = 0$ for $i \in \{1,\ldots,l\}$. Then $p_0 = \varphi_N\big(\mathbb{P}(X_1 = 0)\big)$ and, for $n \in \mathbb{N}$,

$$p_n = \sum_{j=1}^{k+l-1}\mathbb{P}(S_j = n)q_j + \sum_{i=1}^{l}\sum_{j=0}^{n}\Big(a_i + \frac{b_i j}{in}\Big)\mathbb{P}(S_i = j)\tilde{p}_{i,n-j}.$$

Combination of Truncated Distributions

Fix $k \in \mathbb{N}_0$, $l \in \mathbb{N}$. For all $i \in \{1, \dots, l\}$ assume that $\alpha_i \geq 0$, $\beta_i \geq -i\alpha_i$ (at least one $\neq$) and that the $\mathbb{N}_0$-valued $\tilde{N}_i$ satisfies $\mathbb{P}(\tilde{N}_i < k + l - i) = 0$. Consider $q_0, \dots, q_{k+l-1} \geq 0$ with $q_0 + \cdots + q_{k+l-1} \leq 1$. Define

$$q_n = c \sum_{i=1}^{l} \Bigl(\alpha_i + \frac{\beta_i}{n}\Bigr) \mathbb{P}(\tilde{N}_i = n - i) \quad \text{for } n \geq k + l,$$

$$c = \Biggl(1 - \sum_{n=0}^{k+l-1} q_n\Biggr) \Bigg/ \sum_{i=1}^{l} \Biggl(\alpha_i + \beta_i \, \mathbb{E}\Biggl[\frac{1}{i + \tilde{N}_i}\Biggr]\Biggr).$$

Then $\{q_n\}_{n \in \mathbb{N}_0}$ is a probability distribution satisfying the recursion condition with $a_i = c\alpha_i$ and $b_i = c\beta_i$ and the calculation of $\{p_n\}_{n \in \mathbb{N}_0}$ is numerically stable.

Weighted Convolution for ExtLog

Let $k \in \mathbb{N}$ and $q \in (0, 1)$. Let $\tilde{N} \sim \text{ExtLog}(k, q)$ and $N \sim \text{ExtLog}(k + 1, q)$, where $\text{ExtLog}(1, q)$ means $\text{Log}(q)$. Define $\tilde{S} = X_1 + \cdots + X_{\tilde{N}}$ and $S = X_1 + \cdots + X_N$. Then, with an explicit $b_1 > 0$, the weighted convolution

$$\mathbb{P}(S = n) = \frac{b_1}{n} \sum_{j=1}^{n} j \, \mathbb{P}(X_1 = j) \mathbb{P}(\tilde{S} = n - j), \quad n \in \mathbb{N},$$

is numerically stable. Numerically stable algorithm:

- Panjer recursion for $\text{Log}(q)$
- $k - 1$ weighted convolutions: $\text{Log}(q) \to \text{ExtLog}(2, q) \to \cdots \to \text{ExtLog}(k - 1, q) \to \text{ExtLog}(k, q)$

Numerically Stable Algorithm for ExtLog(2,1)

Let $N \sim \text{ExtLog}(2,1)$. For $S = X_1 + \cdots + X_N$ we have

$$\mathbb{P}(S=0) = \mathbb{P}(X_1=0) + \mathbb{P}(X_1 \geq 1) \log \mathbb{P}(X_1 \geq 1)$$

with $0 \log 0 := 0$ and, in the case $\mathbb{P}(X_1 \geq 1) > 0$,

$$\mathbb{P}(S=n) = \frac{1}{n} \sum_{j=1}^{n} j\, \mathbb{P}(X_1 = j)\, r_{n-j}, \quad n \in \mathbb{N},$$

where $r_0 = -\log \mathbb{P}(X_1 \geq 1)$ and, recursively for $n \in \mathbb{N}$,

$$r_n = \frac{1}{\mathbb{P}(X_1 \geq 1)} \Big(\mathbb{P}(X_1 = n) + \frac{1}{n} \sum_{j=1}^{n} j\, \mathbb{P}(X_1 = n-j)\, r_j \Big)$$

Weighted Convolution for ExtNegBin

Let $k \in \mathbb{N}_0$, $\alpha \in (-k, -k+1)$ and $p \in (0,1)$. Let $\tilde{N} \sim \text{ExtNegBin}(\alpha, k, p)$ and $N \sim \text{ExtNegBin}(\alpha-1, k+1, p)$, where $\text{ExtNegBin}(\alpha, 0, p)$ means $\text{NegBin}(\alpha, p)$. Define $\tilde{S} = X_1 + \cdots + X_{\tilde{N}}$ and $S = X_1 + \cdots + X_N$. Then

$$\mathbb{P}(S=n) = \frac{b_1}{n} \sum_{j=1}^{n} j\, \mathbb{P}(X_1 = j)\, \mathbb{P}(\tilde{S} = n-j), \quad n \in \mathbb{N},$$

with an explicit $b_1 > 0$ is numerically stable. Algorithm:

- Panjer recursion for $\text{NegBin}(\alpha + k, p)$
- k weighted convolutions:
  $\text{NegBin}(\alpha+k, p) \to \text{ExtNegBin}(\alpha+k-1, 1, p) \to \cdots$
  $\to \text{ExtNegBin}(\alpha+1, k-1, p) \to \text{ExtNegBin}(\alpha, k, p)$

Stable Algorithm for ExtNegBin$(\alpha-1,1,0)$

Let $N \sim \text{ExtNegBin}(\alpha-1,1,0)$ with $\alpha \in (0,1)$. For $S = X_1 + \cdots + X_N$ we have

$$\mathbb{P}(S=0) = 1 - \big(\mathbb{P}(X_1 \ge 1)\big)^{1-\alpha}$$

and in the case $\mathbb{P}(X_1 \ge 1) > 0$

$$\mathbb{P}(S=n) = \frac{1-\alpha}{n} \sum_{j=1}^{n} j\,\mathbb{P}(X_1 = j)\, r_{n-j}, \quad n \in \mathbb{N},$$

where $r_0 = \big(\mathbb{P}(X_1 \ge 1)\big)^{-\alpha}$ and, recursively for $n \in \mathbb{N}$,

$$r_n = \frac{1}{\mathbb{P}(X_1 \ge 1)} \sum_{j=1}^{n} \frac{n-j+\alpha j}{n} \mathbb{P}(X_1 = j)\, r_{n-j}\,.$$

Tempered α-Stable Distributions on $[0,\infty)$

Let Y be α-stable on $[0,\infty)$ with Laplace transform

$$\mathcal{L}_Y(s) = \mathbb{E}[\exp(-sY)] = \exp(-\gamma_{\alpha,\sigma} s^{\alpha}), \quad s \ge 0,$$

where $\alpha \in (0,1)$, $\sigma > 0$ and $\gamma_{\alpha,\sigma} := \sigma^{\alpha}/\cos\big(\frac{\alpha\pi}{2}\big)$.
For $\tau \ge 0$ define τ-tempered α-stable distribution

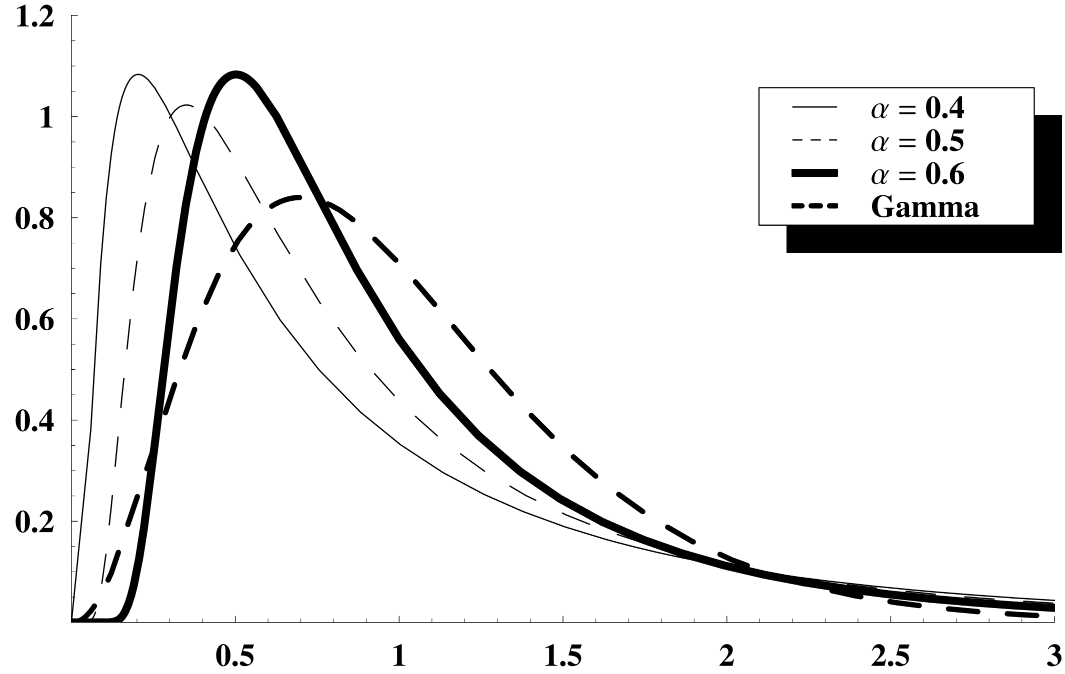
$$F_{\alpha,\sigma,\tau}(y) := \mathbb{E}[e^{-\tau Y} 1_{\{Y \le y\}}]/\mathbb{E}[e^{-\tau Y}], \quad y \in \mathbb{R}.$$

Let $\Lambda \sim F_{\alpha,\sigma,\tau}$. Then for $\tau > 0$

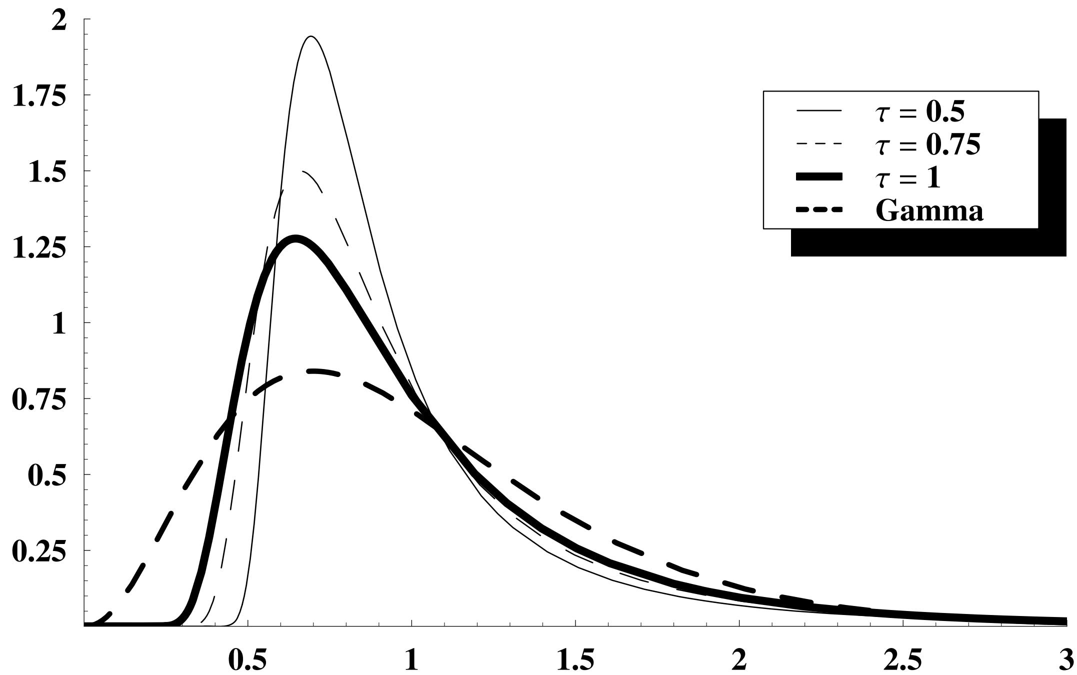
$$\mathcal{L}_\Lambda(s) = \exp\big(-\gamma_{\alpha,\sigma}\big((s+\tau)^{\alpha} - \tau^{\alpha}\big)\big), \quad s \ge -\tau,$$

$$\mathbb{E}[\Lambda] = -\mathcal{L}'_\Lambda(0) = \alpha\gamma_{\alpha,\sigma}\tau^{\alpha-1},$$

$$\text{Var}(\Lambda) = -\mathcal{L}''_\Lambda(0) - (\mathcal{L}'_\Lambda(0))^2 = \alpha(1-\alpha)\gamma_{\alpha,\sigma}\tau^{\alpha-2}.$$



Density of $\Lambda \sim F_{\alpha,\sigma,\tau}$ in comparison with gamma distribution, where $\alpha \in (0,1)$ and $\sigma, \tau > 0$ satisfy $\mathbb{E}[\Lambda] = 1$ and $\mathrm{Var}(\Lambda) = 0.3$.



Density of $\Lambda \sim F_{\alpha,\sigma,\tau}$ in comparison with gamma distribution, where $\alpha \in (0,1)$ and $\sigma, \tau > 0$ satisfy $\mathbb{E}[\Lambda] = 1$ and $\mathrm{Var}(\Lambda) = 0.3$.

Application: Poisson–Tempered α-Stable Mixtures

For $\alpha \in (0,1)$, $\lambda, \sigma > 0$ and $\tau \geq 0$ consider $\Lambda \sim F_{\alpha,\sigma,\tau}$ and the Poisson mixture $\mathcal{L}(N|\Lambda) \overset{\text{a.s.}}{=} \text{Poisson}(\lambda\Lambda)$.

Representation as compound Poisson distribution:

If

$$M \sim \text{Poisson}(\gamma_{\alpha,\sigma}((\lambda+\tau)^\alpha - \tau^\alpha))$$

and the i. i. d. sequence $\{N_m\}_{m\in\mathbb{N}}$ with

$$N_m \sim \text{ExtNegBin}\Big(-\alpha, 1, \frac{\tau}{\lambda+\tau}\Big)$$

are independent, then $N \overset{\mathbf{d}}{=} N_1 + \cdots + N_M$.

Numerically Stable Algorithm for Poisson–Tempered α-Stable Mixtures

- Apply Panjer's recursion for $\tilde{S} = X_1 + \cdots + X_{\tilde{N}}$

$$\tilde{N} \sim \text{NegBin}(1-\alpha, p) \qquad \text{with} \qquad p = \frac{\tau}{\lambda+\tau}.$$

- Use weighted convolution to pass from

$$\tilde{N} \sim \text{NegBin}(1-\alpha, p) \;\rightarrow\; N \sim \text{ExtNegBin}(-\alpha, 1, \text{p}).$$

- Take the previous calculated distribution of $S = X_1 + \cdots + X_N$ as new claim size distribution and apply Panjer's recursion for

$$M \sim \text{Poisson}(\gamma_{\alpha,\sigma}((\lambda+\tau)^\alpha - \tau^\alpha)).$$

Example: Poisson–Lévy Mixture

Special case for $\alpha = 1/2$ and $\tau = 0$.

Lévy distribution: Given $\sigma > 0$, assume that

$$\Lambda = Y \sim f_{\mathsf{L}}(x) = \left(\frac{\sigma}{2\pi x^3}\right)^{1/2} \exp\left(-\frac{\sigma}{2x}\right), \quad x > 0.$$

Let $\mathcal{L}(N|\Lambda) \stackrel{\text{a.s.}}{=} \text{Poisson}(\lambda\Lambda)$ with $\lambda > 0$.

Then $N \stackrel{\text{d}}{=} N_1 + \cdots + N_M$ with independent

$$M \sim \text{Poisson}\big(\sqrt{\lambda\sigma/2}\big)$$

and

$$N_m \sim \text{ExtNegBin}(-1/2, 1, 0), \quad m \in \mathbb{N},$$

and our numerically stable recursion is again applicable.

Example: Poisson–Inverse Gaussian Mixture

Fix $\mu, \tilde{\sigma} > 0$, define $\sigma = \mu^2/\tilde{\sigma}^2$ and $\tau = 1/(2\tilde{\sigma}^2)$.

Inverse Gaussian distribution: $\Lambda \sim F_{1/2,\sigma,\tau}$ has density

$$f_{\mathsf{IG}}(x) = \frac{\mu}{\sqrt{2\pi\tilde{\sigma}^2 x^3}} \exp\left(-\frac{(x-\mu)^2}{2\tilde{\sigma}^2 x}\right), \quad x > 0.$$

Let $\mathcal{L}(N|\Lambda) \stackrel{\text{a.s.}}{=} \text{Poisson}(\lambda\Lambda)$ with $\lambda > 0$.

Then $N \stackrel{\text{d}}{=} N_1 + \cdots + N_M$ with independent

$$M \sim \text{Poisson}\big(\sqrt{\sigma/2}(\sqrt{\lambda+\tau} - \sqrt{\tau})\big)$$

and

$$N_m \sim \text{ExtNegBin}\Big(-1/2, 1, \frac{\tau}{\lambda+\tau}\Big), \quad m \in \mathbb{N}.$$

Poisson–Reciprocal Inverse Gaussian Mixture

Fix $\mu, \tilde{\sigma} > 0$, define $\sigma = \mu^2/\tilde{\sigma}^2$ and $\tau = 1/(2\tilde{\sigma}^2)$.

Reciprocal inverse Gaussian distribution: Assume

$$\Lambda \sim f_{\mathsf{RIG}}(x) = \frac{1}{\sqrt{2\pi\tilde{\sigma}^2 x}} \exp\biggl(-\frac{(x-\mu)^2}{2\tilde{\sigma}^2 x}\biggr), \quad x > 0.$$

Let $\mathcal{L}(N|\Lambda) \stackrel{\text{a.s.}}{=} \text{Poisson}(\lambda\Lambda)$ with $\lambda > 0$.

Then $N \stackrel{\text{d}}{=} N_0 + N_1 + \cdots + N_M$ with independent

$$\begin{aligned} N_0 &\sim \text{NegBin}(1/2, p), \quad p = \tau/(\lambda+\tau), \\ M &\sim \text{Poisson}\bigl(\sqrt{\sigma/2}(\sqrt{\lambda+\tau} - \sqrt{\tau})\bigr), \\ N_m &\sim \text{ExtNegBin}(-1/2, 1, p), \quad m \in \mathbb{N}. \end{aligned}$$

Further Applications of the Ideas

- Generalization of De Pril's recursion to calculate higher moments of $S = X_1 + \cdots + X_N$.
- Generalization of Panjer's recursion to claim sizes with mixed support (density and an atom at zero) $\longrightarrow$ Integral equations for the density of S.

Reference

S. Gerhold, U. Schmock, R. Warnung:
A Generalization of Panjer's Recursion and Numerically Stable Risk Aggregation,
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`http://www.fam.tuwien.ac.at/~schmock/Stable_Panjer_Recursion.html`