Dependence Properties of Dynamic Credit Risk Models

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Outline of Presentation

- Reduced-form portfolio credit risk model with default feedback (contagion)
- Concept of association and its properties
- Association of default intensities and implications for default times
- Properties of associated default times
- Association of accumulated hazard processes
- Applications to credit default swaps

Reduced-Form Portfolio Credit Risk Model

On a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ consider for every obligor $i \in \{1, \dots, d\}$

- an adapted, increasing, right-continuous, accumulated hazard process $\Lambda_i = \{\Lambda_i(t)\}_{t\geq 0}$ with $\Lambda_i(0) = 0$
- a standard exponentially distributed threshold E_i,
- the default time $\tau_i = \inf\{t \geq 0 \mid \Lambda_i(t) \geq E_i\}$ with

$$\mathbb{P}(\tau_i > t \,|\, \Lambda_i(t)) \overset{\text{a.s.}}{=} e^{-\Lambda_i(t)}, \qquad t \geq 0,$$

- the default indicator process $Y_i(t) = 1_{[E_i,\infty)}(\Lambda_i(t))$,
- possibly a default intensity process λ_i satisfying

$$\Lambda_i(t) = \int_0^t \lambda_i(s)\, ds, \qquad t \geq 0.$$

Model 1: Default Feedback by Default Intensities

- $\Psi = \{\Psi_t\}_{t\geq 0}$ an $\mathbb{R}^m$-valued environment process (contains relevant economic information like interest rates, stock price indices, economic indices, etc.)
- Thresholds $E = (E_1, \dots, E_d)$ independent of Ψ
- Default intensity $\lambda_i(t, \Psi_t, Y_t)$ of obligor $i \in \{1, \dots, d\}$

Aim: Investigate and control how dependence through environment process Ψ and previous defaults, given by the default indicator process $Y_t = (Y_1(t), \dots, Y_d(t))$, transfers to dependence of default times $\tau_1, \dots, \tau_d$.

Definition of Association

- An $\mathbb{R}^d$-valued random vector $X = (X_1, \dots, X_d)$ and its distribution $\mathcal{L}(X)$ are called associated, if

$$\mathrm{Cov}(f(X), g(X)) \ge 0$$

  for all measurable, componentwise increasing functions $f, g \colon \mathbb{R}^d \to \mathbb{R}$ for which $f(X)$, $g(X)$ and the product $f(X)g(X)$ are integrable.

- An $\mathbb{R}^d$-valued process $\{X_t\}_{t\ge 0}$ is called associated if for all $k \in \mathbb{N}$ and times $0 \le t_1 < \cdots < t_k$ the $\mathbb{R}^{dk}$-valued vector $(X(t_1), \dots, X(t_k))$ is associated.

Association is Notion for Positive Dependence

Let (X, Y) be an $\mathbb{R}^2$-valued random vector with marginal distributions F and G.

Definition: Kendall's τ

$$\tau_{\mathrm{K}}(X, Y) := \mathbb{E}[\mathrm{sign}(X - X')\,\mathrm{sign}(Y - Y')],$$

with (X', Y') an independent copy of (X, Y).

Definition: Spearman's ϱ

$$\varrho_{\mathrm{S}}(X, Y) := \mathrm{Corr}[F(X)G(Y)]$$

Lemma:* If (X, Y) is associated, then $\tau_{\mathrm{K}}(X, Y) \ge 0$ and $\varrho_{\mathrm{S}}(X, Y) \ge 0$.

*cf. Nelsen, An Introduction to Copulas, Springer (1999)

Properties of Association*

- If $X_1, \dots, X_d$ are independent, then the random vector $X = (X_1, \dots, X_d)$ is associated.
- If $X = (X_1, \dots, X_d)$ and $Y = (Y_1, \dots, Y_k)$ are associated random vectors, which are independent, then $(X_1, \dots, X_d, Y_1, \dots, Y_k)$ is associated.
- If $X = (X_1, \dots, X_d)$ is associated, then the vector $(f_1(X), \dots, f_k(X))$ is associated for every $k \in \mathbb{N}$ and every choice of measurable increasing (or decreasing) functions $f_1, \dots, f_k \colon \mathbb{R}^d \to \mathbb{R}$.
- If $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of associated $\mathbb{R}^d$-valued random vectors and $X_n \xrightarrow{\mathrm{d}} X$, then X is associated.

*see Esary, Proschan, Walkup (1967), Ann. Math. Statist. 38

Association is a Copula Property

Let $X = (X_1, \dots, X_d)$ be an $\mathbb{R}^d$-valued random vector with marginal distributions $F_1, \dots, F_d$. Define the copula $C_X : [0,1]^d \to [0,1]$ of X as distribution function of $(F_1(X_1), \dots, F_d(X_d))$.

Lemma: X is associated $\iff$ C_X is associated.

Proof: "$\Longrightarrow$" By property of association.
"$\Longleftarrow$" Use the lower quantile functions

$$F_i^{\leftarrow}(t) := \inf\{x \in \mathbb{R} \mid F_i(x) \ge t\}, \qquad t \in [0,1],$$

for $i \in \{1, \dots, d\}$ to see that

$$(X_1, \dots, X_d) \overset{\text{a.s.}}{=} (F_1^{\leftarrow}(F_1(X_1)), \dots, F_d^{\leftarrow}(F_d(X_d))).$$

Application to Defaultable Zero-Coupon Bonds

Let R_t denote the integrated stochastic interest intensity, i.e., e^{-R_t} is the factor for discounting from t to 0. Let Λ_t denote the accumulated hazard for default up to t.

Lemma: For a defaultable payment of 1 at time t, assume that (R_t, Λ_t) is associated under an equivalent pricing measure $\mathbb{P}$. Then for the price at time 0:

$$\mathbb{E}[e^{-R_t}1_{\{\tau>t\}}] \geq \mathbb{E}[e^{-R_t}]\,\mathbb{P}(\tau > t).$$

Proof: The vector $(e^{-R_t}, e^{-\Lambda_t})$ is associated. Since $\{\tau > t\} = \{\Lambda_t < E\}$ and $\mathbb{P}(\Lambda_t < E\,|\,\Lambda_t, R_t) = e^{-\Lambda_t}$, the definition of association implies

$$\mathbb{E}\big[e^{-R_t}1_{\{\tau>t\}}\big] = \mathbb{E}\big[e^{-R_t}e^{-\Lambda_t}\big] \geq \mathbb{E}\big[e^{-R_t}\big]\,\mathbb{E}\big[e^{-\Lambda_t}\big].$$

Conditional Increasing in Sequence and Association

Definition: $X = (X_1, \dots, X_d)$ is called conditional increasing in sequence (CIS) if for every $k \in \{2, \dots, d\}$ and bounded increasing $f\colon \mathbb{R} \to \mathbb{R}$

$$(x_1, \dots, x_{k-1}) \mapsto \mathbb{E}[f(X_k)\,|\,X_1 = x_1, \dots, X_{k-1} = x_{k-1}]$$

is increasing in every $x_1, \dots, x_{k-1}$.

Lemma:* If X is conditional increasing in sequence, then X is associated.

Remark: CIS is convenient for Markov processes.

*cf. A. Müller & D. Stoyan, *Comparison Methods for Stochastic Models and Risks,* Wiley (2002), Theorem 3.10.11.

Examples of Associated (Environment) Processes

- $\mathbb{R}^d$-valued process $\{X_t\}_{t\geq 0}$ with independent, associated increments $X_t - X_s$, $0 \leq s < t$. This includes deterministic time changes of 1-dim. Lévy processes.
- Interest rate process $\{r_t\}_{t\geq 0}$ in Vasicek's model is CIS because for all $0 \leq s < t$

$$r_t = m + (r_s - m)\,e^{-\kappa(t-s)} + \sigma \int_s^t e^{-\kappa(t-u)}\,dW_u.$$

- Birth-and-death processes are CIS.
- Interest rate process $\{r_t\}_{t\geq 0}$ in Cox-Ingersoll-Ross model is CIS.
- Volatility processes of GARCH(1,1) processes

Monotone Mixtures and Association

Definition: $X = (X_1, \dots, X_d)$ is called a monotone mixture of $\Theta = (\Theta_1, \dots, \Theta_k)$ if for every measurable, bounded and componentwise increasing $f : \mathbb{R}^d \to \mathbb{R}$ there exists a measurable, componentwise increasing $h\colon \mathbb{R}^k \to \mathbb{R}$ such that

$$h(\Theta) \overset{\text{a.s.}}{=} \mathbb{E}\big[f(X)\,|\,\Theta\big].$$

Lemma:* If the conditional distribution $\mathcal{L}(X|\Theta)$ is associated, Θ is associated and X is a monotone mixture of Θ, then the vector (X, Θ) is associated.

*see K. Jogdeo (1978), Ann. Statist. 6, 232–234.

Implication of Associated (Integrated) Intensities

Write $\lambda_t = (\lambda_1(t), \ldots, \lambda_d(t))$ for the joint $\mathbb{R}^d$-valued intensity process and $\Lambda_t = (\Lambda_1(t), \ldots, \Lambda_d(t))$ for the integrated version (accumulated hazard) at time $t \geq 0$.

Lemma: (B. & S.)

- If $\{\lambda_t\}_{t\geq 0}$ is associated and càdlàg, then $\{\Lambda_t\}_{t\geq 0}$ is associated.
- If $\{\Lambda_t\}_{t\geq 0}$ is associated, right-continuous and $\Lambda_i(t) \nearrow \infty$ a.s. as $t \to \infty$ for every $i \in \{1, \ldots, d\}$, and the thresholds $(E_1, \ldots, E_d)$ are associated and independent of $\{\Lambda_t\}_{t\geq 0}$, then the default times $(\tau_1, \ldots, \tau_d)$ are associated.

Association in Model 1 with Default Intensities

Theorem: (B. & S.) If

- environment process Ψ is associated,
- $\lambda_i(t, \Psi_t, Y_t)$ is increasing in 2nd and 3rd argument,
- $\int_0^\infty \lambda_i(t, \Psi_t, y)\, dt \overset{\text{a.s.}}{=} \infty$ for every $y \in \{0,1\}^d$, $y_i = 0$,
- technical conditions (suitable meas. & continuity),

then the accumulated hazard process

$$\Lambda_t = \left(\int_0^t \lambda_i(s, \Psi_s, Y_s)\, ds\right)_{i=1,\ldots,d}, \quad t \geq 0,$$

is associated, and the default times $\tau = (\tau_1, \ldots, \tau_d)$ are associated, too.

Association and Positive Supermodular Dependence

Definition: $f : \mathbb{R}^d \to \mathbb{R}$ is called supermodular if

$$f(x) + f(y) \leq f(x \vee y) + f(x \wedge y), \qquad \forall x, y \in \mathbb{R}^d.$$

Definition: Let $X = (X_1, \ldots, X_d)$ be a random vector and $X^\perp = (X_1^\perp, \ldots, X_d^\perp)$ a copy with independent components. Then X and its distribution $\mathcal{L}(X)$ are called positive supermodular dependent (PSD) if

$$\mathbb{E}[f(X^\perp)] \leq \mathbb{E}[f(X)]$$

for all measurable, supermodular $f : \mathbb{R}^d \to \mathbb{R}$ for which the expectations exist.

Lemma:* X is associated $\implies$ X is PSD.

*cf. Christofides, Vaggelatou (2004), J. Multivariate Anal. 88.

Implications of Associated Default Times

If $(\tau_1, \ldots, \tau_d)$ are associated, then, for all non-void $I \subset \{1, \ldots, d\}$ and $\{t_i\}_{i\in I} \subset [0, \infty)$,

$$\mathbb{P}(\tau_i^\perp > t_i \text{ for all } i \in I) \leq \mathbb{P}(\tau_i > t_i \text{ for all } i \in I),$$
$$\mathbb{P}(\tau_i^\perp \leq t_i \text{ for all } i \in I) \leq \mathbb{P}(\tau_i \leq t_i \text{ for all } i \in I),$$

because the indicator functions are supermodular.

Definition: A r.v. X is smaller in usual stochastic order than Y, if $\mathbb{P}(X > t) \leq \mathbb{P}(Y > t)$ for all $t \in \mathbb{R}$.

Consequence: With $\leq_{\text{st}}$ for usual stochastic order,

$$\min_{i\in I} \tau_i^\perp \leq_{\text{st}} \min_{i\in I} \tau_i \qquad \text{and} \qquad \max_{i\in I} \tau_i \leq_{\text{st}} \max_{i\in I} \tau_i^\perp.$$

Associated Default Times and Concordance Order

Notation:
$F_X(t) = \mathbb{P}(X_1 \le t_1, \dots, X_d \le t_d)$ distribution function
$\bar{F}_X(t) = \mathbb{P}(X_1 > t_1, \dots, X_d > t_d)$ survival function

Definition: An $\mathbb{R}^d$-valued random vector X is called smaller than Y in concordance order ($X \le_c Y$), if $F_X(t) \le F_Y(t)$ and $\bar{F}_X(t) \le \bar{F}_Y(t)$ for all $t \in \mathbb{R}^d$.

Remark: $X \le_c Y$ implies equality of one-dimensional marginal distributions.

Previous observation: If $(\tau_1, \dots, \tau_d)$ is associated, then $(\tau_1^\perp, \dots, \tau_d^\perp) \le_c (\tau_1, \dots, \tau_d)$

Associated Default Times and Order Statistics

For default times $\tau_1, \dots, \tau_d$ let $\tau_{1:d} \le \dots \le \tau_{d:d}$ denote the order statistics.

Lemma: $(\tau_1, \dots, \tau_d)$ is associated
$\implies (\tau_{1:d}, \dots, \tau_{d:d})$ is associated.

Proof: Since for $k \in \{1, \dots, d\}$

$$\tau_{k:d} = \min_{\substack{I \subset \{1,\dots,d\} \\ |I|=k}} \max_{i \in I} \tau_i \,,$$

every $\tau_{k:d}$ is an increasing function of $(\tau_1, \dots, \tau_d)$.

Model 2: Accumulated Hazard Processes

For every obligor $i \in \{1, \dots, d\}$ and time $t \ge 0$ put

$$\Lambda_i(t) = \Psi_i(t) + \sum_{j \in \{1,\dots,d\} \setminus \{i\}} 1_{\{\tau_j \le t\}} \Gamma_{i,j}(t - \tau_j).$$

Theorem: (B. & S.) Assume that for $i, j \in \{1, \dots, d\}$

- Ψ_i and $\Gamma_{i,j}$ are positive processes with increasing paths,
- $\{\Gamma_t\}_{t \ge 0}$ with $\Gamma_t = (\Gamma_{1,2}(t), \dots, \Gamma_{d,d-1}(t))$, thresholds $(E_1, \dots, E_d)$, and the environment process $\{\Psi_t\}_{t \ge 0}$ are associated and independent,
- technical conditions (suitable continuity, $\Lambda_i(t) \nearrow \infty$).

Then the accumulated hazard processes $\{\Lambda_t\}_{t \ge 0}$ as well as the default times $\tau_1, \dots, \tau_d$ are associated.

Application to Credit Default Swaps (CDS)

- Reference party R issues a bond with maturity T^*.
- Party A buys the bond and pays swap rate c continuously to B until swap maturity $T \le T^*$ or τ_A (A pays even if B or R have already defaulted).
- Party B pays 1 € at time T to A if $\tau_R \le T$ and $\tau_B > T$.

With pricing measure $\mathbb{P}$ and spot rate process $\{r_t\}_{t \in [0,T]}$, the fair swap rate at time 0 is

$$c = \frac{\mathbb{E}\big[\exp\big(-\int_0^T r_s\,ds\big) 1_{\{\tau_B > T, \tau_R \le T\}}\big]}{\mathbb{E}\big[\int_0^T \exp\big(-\int_0^t r_s\,ds\big) 1_{\{\tau_A > t\}}\,dt\big]}.$$

Application to Credit Default Swaps (Cont.)

Lemma: Assume that (τ_B, τ_R) is associated and independent of the spot rate process. Then $c \le c^{\perp}$, where $c^{\perp}$ denotes the fair swap rate when $\tau_B^{\perp}$ and $\tau_R^{\perp}$ are independent with $\tau_B \overset{\mathsf{d}}{=} \tau_B^{\perp}$ and $\tau_R \overset{\mathsf{d}}{=} \tau_R^{\perp}$.

Proof: Note that

$$1_{\{\tau_B > T,\, \tau_R \le T\}} = 1_{\{\tau_R \le T\}} - 1_{\{\tau_B \le T,\, \tau_R \le T\}}.$$

Association of (τ_B, τ_R) implies positive supermodular dependence, hence $(\tau_B^{\perp}, \tau_R^{\perp}) \le_{\mathsf{c}} (\tau_B, \tau_R)$ and

$$\mathbb{P}(\tau_B^{\perp} \le T, \tau_R^{\perp} \le T) \le \mathbb{P}(\tau_B \le T, \tau_R \le T),$$

which yields the statement.

Generalization to k^{th}-to-Default Credit Swaps

Suppose A buys a collaterized debt obligation (CDO), which defaults if the k^{th} default happens in a portfolio of d obligors with default times $\tau_1, \ldots, \tau_d$. Fair swap rate:

$$c_k = \frac{\mathbb{E}\big[\exp\big(-\int_0^T r_s\,ds\big) 1_{\{\tau_B > T,\, \tau_{k:d} \le T\}}\big]}{\mathbb{E}\big[\int_0^T \exp\big(-\int_0^t r_s\,ds\big) 1_{\{\tau_A > t\}}\,dt\big]}.$$

Lemma: Assume that $(\tau_B, \tau_1, \ldots, \tau_d)$ is associated and independent of the spot rate process $\{r_t\}_{t\in[0,T]}$. Then $c_k \le c_k^{\perp}$, where $c_k^{\perp}$ denotes the fair swap rate when $\tau_B^{\perp}, \tau_1^{\perp}, \ldots, \tau_d^{\perp}$ are independent.

Proof: Note that $(\tau_B, \tau_{k:d})$ is an increasing function of $(\tau_B, \tau_1, \ldots, \tau_d)$, hence associated. Use previous lemma.