Generalization of the Dybvig–Ingersoll–Ross Theorem and Asymptotic Minimality

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Key Publications for our Work

- Philip H. Dybvig, Jonathan E. Ingersoll, and Stephen A. Ross: *Long forward and zero-coupon rates can never fall,* The Journal of Business, Vol. 69, No. 1 (Jan. 1996), pp. 1–25. (proof in the appendix!)

- Friedrich Hubalek, Irene Klein, and Josef Teichmann: *A general proof of the Dybvig–Ingersoll–Ross theorem: long forward rates can never fall,* Mathematical Finance, Vol. 12, No. 4 (2002), pp. 447–451. (arXiv:0901.2080)

Probabilistic Model and Zero-Coupon Rates

For every maturity $T \in \mathbb{N}$ or $T \in (0, \infty)$, let a strictly positive, $\mathbb{F}$-adapted, zero-coupon bond price process $P(t,T)$ with $t \in \{0, 1, \ldots, T\}$ or $t \in [0,T]$, respectively, be given with normalization $P(T,T) = 1$.

Define zero-coupon rates (investment yields):

- Discrete case: For $T \in \mathbb{N}$ and $t \in \{0, \ldots, T-1\}$

$$R(t,T) := P(t,T)^{-1/(T-t)} - 1$$

- Continuous case: For $T > 0$ and $t \in [0,T)$

$$R(t,T) := -\frac{\log P(t,T)}{T-t}$$

Definition of Arbitrage-Free Forward Rates

The arbitrage-free forward rate $F(s,t,T)$ for a loan over the future time period $[t,T]$, contracted at time s:

- Discrete case: For $T \in \mathbb{N}$ and $s \leq t$ in $\{0,\ldots,T-1\}$

$$F(s,t,T) := \left(\frac{P(s,t)}{P(s,T)}\right)^{1/(T-t)} - 1$$

- Continuous case: For $T > 0$ and $s \leq t$ in $[0,T)$

$$F(s,t,T) := \frac{1}{T-t}\log\frac{P(s,t)}{P(s,T)}$$

Representation of Zero-Coupon Bond Prices

- Discrete-time case:
  For $T \in \mathbb{N}$ and $s \leq t$ in $\{0, \ldots, T-1\}$

$$P(t,T) = \frac{1}{(1 + R(t,T))^{T-t}}$$

$$P(s,T) = P(s,t)\frac{1}{(1 + F(s,t,T))^{T-t}}$$

- Continuous-time case:
  For $T > 0$ and $s \leq t$ in $[0, T)$

$$P(t,T) = \exp\big(-(T-t)R(t,T)\big)$$

$$P(s,T) = P(s,t)\exp\big(-(T-t)F(s,t,T)\big)$$

Dybvig–Ingersoll–Ross Theorem: Long Forward and Zero-Coupon Rates Can Never Fall

Theorem: Assume that the zero-coupon bond market is "arbitrage-free".

- If for $s < t$ the long-term spot rates

  $$l(s) := \lim_{T\to\infty} R(s,T) \quad \text{and} \quad l(t) := \lim_{T\to\infty} R(t,T)$$

  exist almost surely, then $l(s) \le l(t)$ almost surely.
- If for $s \le t$ the long-term forward rate

  $$l_F(s,t) := \lim_{T\to\infty} F(s,t,T)$$

  exist a. s., then $l_F(s,t) \overset{\text{a.s.}}{=} l(s)$ and corresponding results hold.

Why Should the Theorem Be True?

From time s to a later time t the information increases from $\mathcal{F}_s$ to $\mathcal{F}_t$, so a more informed decision concerning the best zero-coupon bonds for long-term investment can be made. This should give $l(s) \leq l(t)$, because the earnings during $[s,t]$ are negligible in the limit $T \to \infty$.

Necessity of absence of arbitrage for investments in long-term zero-coupon bonds: Suppose that

- $P(s,T) = e^{-(T-s)}$ for all $T \geq s$ and
- $P(t,T) = 1$ for all $T \geq t$.

Then $l(s) = 1$ and $l(t) = 0$, hence the assertion does not hold. Indeed, there is arbitrage: at time s, sell one t-maturity bond and buy e^{T-t} bonds with maturity T.

Why Is the Theorem Relevant?

- Long-term investment returns are important for life insurers and pension funds.
- The theorem gives conditions which arbitrage-free bond price models have to satisfy.
- The theorem can be used to constrain the parameters of factor models to avoid arbitrage.
- It's mathematically interesting to investigate the notion of "arbitrage-free" in case of infinitely many assets.

Definitions of "Arbitrage-Free"

Problem: Infinitely many assets!

- Hubalek et al.: There exists a bank account process and an equivalent measure $\mathbb{Q}$ such that every discounted zero-coupon bond price process is a $\mathbb{Q}$-martingale.
- Dybvig et al.: There does not exist a sequence of net trades (allowing free disposal) such that either
  - (i) the price tends to zero but the payoff tends uniformly to a nonnegative random variable that is positive with positive probability or
  - (ii) the price tends to a negative number but the payoff tends uniformly to a non-negative random variable.

Disadvantage of Dybvig–Ingersoll–Ross Theorem

- Existence of the limit for the long-term spot and forward rates has to be shown in advance.
- There exist models where these limits do not exist!

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Solution: Use Limit Superior!

For $t \geq 0$ define

$$l(t) := \limsup_{T\to\infty} R(t,T) = \lim_{n\to\infty} \operatorname*{ess\,sup}_{T>n\vee t} R(t,T).$$

and for $0 \leq s \leq t$ define

$$\begin{aligned} l_F(s,t) &= \limsup_{T\to\infty} F(s,t,T) \\ &= \lim_{n\to\infty} \operatorname*{ess\,sup}_{T>n\vee t} F(s,t,T). \end{aligned}$$

Limit Superior is Economically Meaningful

Lemma (G. & S.): Given $t \geq 0$, there exists a sequence of $\mathcal{F}_t$-measurable random maturities $T_n : \Omega \longrightarrow (n \vee t, \infty)$, each one taking only a finite number of values, such that

$$l(t) \overset{\text{a.s.}}{=} \lim_{n \to \infty} R(t, T_n).$$

Remark: To approximate the supremum of the possible long-term investment returns at time t, the investor can therefore choose an appropriate bond maturity based on the information at time t.

Generalization of the Dybvig–Ingersoll–Ross Theorem

Theorem (G. & S.): If, for $0 \leq s < t$, there exists a probability measure $\mathbb{Q}_{s,t}$ on $(\Omega, \mathcal{F}_t)$, equivalent of $\mathbb{P}|_{\mathcal{F}_t}$, such that for all sufficiently large $T > t$

$$P(s,T) \geq P(s,t)\,\mathbb{E}_{\mathbb{Q}_{s,t}}[P(t,T)\,|\,\mathcal{F}_s] \quad \text{a. s.}$$

then

- $l(s) \leq l(t)$ a. s. and
- $l_F(s,s') \leq l_F(t,t')$ a. s. for all $s' \geq s$ and $t' \geq t$.

Remarks:

- If $\mathbb{Q}_{s,t}$ is the forward (time s) risk neutral probability measure for maturity t, then equality holds.
- For equality, this corresponds to the version of Hubalek et al., their method of proof can be adapted.

A Model Class with Forward Risk Neutral Measures

Bank account B_t with $t \in \mathbb{N}_0$ or $t \in [0, \infty)$, strictly positive, $\mathbb{F}$-adapted, $B_0 = 1$. Assume that $1/B_T$ is $\mathbb{Q}$-integrable for every $T > 0$. Define

$$P(t,T) = \mathbb{E}_{\mathbb{Q}}\Big[\frac{B_t}{B_T}\Big|\mathcal{F}_t\Big], \quad t \in [0,T],$$

and

$$\frac{d\mathbb{Q}_{s,t}}{d\mathbb{Q}} = \frac{B_s}{P(s,t)B_t}, \quad s \in [0,t).$$

Then by Bayes' formula

$$\mathbb{E}_{\mathbb{Q}_{s,t}}[P(t,T)\,|\,\mathcal{F}_s] \overset{\text{a.s.}}{=} \mathbb{E}_{\mathbb{Q}}\bigg[\frac{B_s}{P(s,t)B_t}\,\mathbb{E}_{\mathbb{Q}}\Big[\frac{B_t}{B_T}\Big|\mathcal{F}_t\Big]\bigg|\mathcal{F}_s\bigg]$$

$$\overset{\text{a.s.}}{=} P(s,T)/P(s,t)$$

Short-Rate Models

For $\mathbb{F}$-progressive interest rate intensity process $\{r_t\}_{t \geq 0}$ with locally integrable paths define

$$B_t = \exp\Big(\int_0^t r_u \, du\Big), \quad t \in [0, \infty).$$

If $1/B_T$ is $\mathbb{Q}$-integrable, then, for all $0 \leq t \leq T$,

$$P(t,T) = \mathbb{E}_{\mathbb{Q}}\Big[\exp\Big(-\int_t^T r_u \, du\Big)\Big|\mathcal{F}_t\Big]$$

and if $t < T$

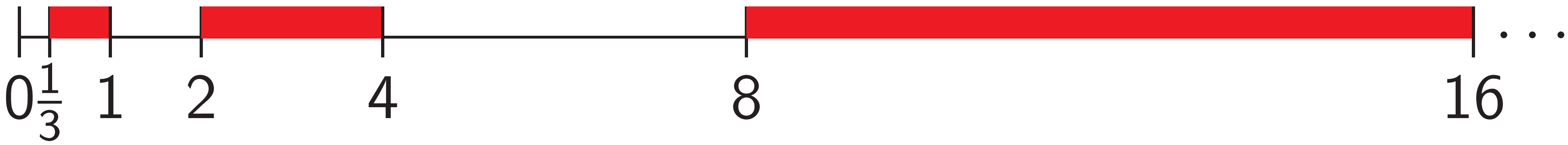
$$R(t,T) = -\frac{1}{T-t} \log \mathbb{E}_{\mathbb{Q}}\Big[\exp\Big(-\int_t^T r_u \, du\Big)\Big|\mathcal{F}_t\Big]$$

A Deterministic Short-Rate Model, where the Limit of the Zero-Coupon Rates Does Not Exist

Define càdlàg interest rate intensity process $r_t = 1_A(t)$ for $t \geq 0$, where

$$A = \left[\tfrac{1}{3}, 1\right) \cup \bigcup_{k=0}^{\infty} \left[2^{2k+1}, 2^{2k+2}\right).$$

Visualization of A:



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Then, for $0 \leq t < T$,

$$R(t,T) = \frac{1}{T-t} \int_t^T 1_A(u)\,du = \frac{\lambda(A \cap [t,T])}{T-t}$$

and $R(0, 2^{2n+1}) = \frac{1}{3}$ and $R(0, 2^{2n+2}) = \frac{2}{3}$ for $n \in \mathbb{N}$. More generally, every point in the interval $[\frac{1}{3}, \frac{2}{3}]$ is an accumulation point of $\{R(t,T)\}_{T>t}$ as $T \to \infty$.

Vasiček Model with Time-Dependent Volatility and Non-Existing Limit of the Zero-Coupon Rates

Let $\alpha > 0$, $\mu \in \mathbb{R}$, $\sigma : [0, \infty) \to \mathbb{R}$ deterministic and locally L^2, and $\{W_t\}_{t\geq 0}$ Brownian motion. Consider as interest rate intensity process $\{r_t\}_{t\geq 0}$ the solution of $dr_t = \alpha(\mu - r_t)\,dt + \sigma_t\,dW_t$. Then, for $0 \leq t < T$,

$$R(t,T) = \mu + (r_t - \mu)\frac{1 - e^{-\alpha(T-t)}}{\alpha(T-t)} - \frac{1}{2\alpha^2(T-t)}\int_t^T \left(1 - e^{-\alpha(T-s)}\right)^2 \sigma_s^2\,ds.$$

For $\sigma_s := 1_A(s)$, $s \geq 0$, with set A from previous slide, the limit of $\{R(t,T)\}_{T>t}$ as $T \to \infty$ does not exist.

Notions for Arbitrage in the Limit

Given $0 \leq s < t$, the zero-coupon bonds with maturity $T \geq t$ provide an arbitrage possibility in the limit, if there exist $\mathcal{F}_s$-measurable portfolios (φ_n, ψ_n) and maturities $T_n : \Omega \to (n \vee t, \infty)$ attaining only finitely many values such that

- $V_n(s) := \varphi_n P(s, T_n) + \psi_n P(s,t) \overset{\text{a.s.}}{=} 0$ for all $n \in \mathbb{N}$,
- $\mathbb{P}\big(\liminf_{n\to\infty} V_n(t) > 0\big) > 0$,
- $\liminf_{n\to\infty} V_n(t) \geq 0$ a. s.,

where $V_n(t) := \varphi_n P(t, T_n) + \psi_n$. The bonds provide an arbitrage opportunity in the limit with vanishing risk if, in addition, for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $V_n(t) \geq -\varepsilon$ a. s. for all $n \geq n_\varepsilon$.

Relations Between Notions of Absence for Arbitrage

Fix $0 \le s < t$.

- No arbitrage possibility in the limit
  $\Longrightarrow$ No arbitrage in the limit with vanishing risk
- If $\mathcal{F}_t$ is finite, then both notions are equivalent.
- There exists a forward (time s) risk neutral measure $\mathbb{Q}_{s,t}$ for maturity t
  $\Longrightarrow$ No arbitrage in the limit with vanishing risk

We have examples to show:

- No arbitrage possibility in the limit
  $\not\Longrightarrow$ Existence of $\mathbb{Q}_{s,t}$
- Existence of $\mathbb{Q}_{s,t}$
  $\not\Longrightarrow$ No arbitrage possibility in the limit

Generalization of the Dybvig–Ingersoll–Ross Theorem

Theorem (G. & S.): If, for $0 \le s < t$, there is no arbitrage possibility in the limit with vanishing risk by investing in the long-term zero-coupon bonds, then

- $l(s) \le l(t)$ a. s. and
- $l_F(s, s') \le l_F(t, t')$ a. s. for all $s' \ge s$ and $t' \ge t$.

Remarks:

- Without the weakening to only long positions, this corresponds to the version of Dybvig et al., the proof from their appendix can be adapted.
- This implies the previous version if we require there that $P(s,T) \overset{\text{a.s.}}{=} P(s,t)\,\mathbb{E}_{\mathbb{Q}_{s,t}}[P(t,T)\,|\,\mathcal{F}_s]$ for all $T > t$.

Lower Envelope and Asymptotic Minimality

Definition: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{G} \subset \mathcal{F}$ a sub-σ-algebra, and X an $\overline{\mathbb{R}}$-valued random variable. The lower $\mathcal{G}$-measurable envelope $X_{\mathcal{G}}$ of X is defined as the essential supremum of all $\overline{\mathbb{R}}$-valued, $\mathcal{G}$-measurable Z with $Z \leq X$ a. s.

Questions: Consider times $0 \leq s < t$.
If $l(s) \leq l(t)$ a. s., then $l(s) \leq l(t)_{\mathcal{F}_s}$ a. s. by definition.

- Under which conditions do we have $l(s) \overset{\text{a.s.}}{=} l(t)_{\mathcal{F}_s}$? (This is called asymptotic minimality.)
- Under which notions of absence of arbitrage can $\mathbb{P}\big(l(s) < l(t)_{\mathcal{F}_s}\big) > 0$ be possible?

Results for Asymptotic Minimality

Theorem (G. & S.): Consider $0 \le s < t$. If there is no arbitrage possibility in the limit by short-selling the long-term zero-coupon bonds, then for every $\mathcal{F}_s$-measurable sequence of maturities $T_n : \Omega \to (n \vee t, \infty)$ taking only finitely many values,

$$\Big(\liminf_{n\to\infty} R(t, T_n)\Big)_{\mathcal{F}_s} \le l(s) \quad \text{a. s.}$$

Corollary: If there is no arbitrage in the limit, then

$$\Big(\liminf_{T\to\infty} R(t, T)\Big)_{\mathcal{F}_s} \le l(s) \le \Big(\overbrace{\limsup_{T\to\infty} R(t, T)}^{=\,l(t)}\Big)_{\mathcal{F}_s} \qquad \text{a. s.}$$

If the limit exists a. s., then $l(s) \overset{\text{a.s.}}{=} l(t)_{\mathcal{F}_s}$.

An Example Without Arbitrage in the Limit, where Asymptotic Minimality Fails

Consider $X(\omega) = \omega$ for $\omega \in \Omega := \{0,1\}$. Take $\mathcal{F}_t = \{\emptyset, \Omega\}$ for $t \in [0, \frac{1}{3})$ and $\mathcal{F}_t = \mathcal{P}(\Omega)$ for $t \geq \frac{1}{3}$. Take the interest rate intensity process

$$r_t = X 1_A(t) + (1 - X) 1_{A^c \cap [1/3, \infty)}(t), \quad t \in [0, \infty).$$

with $A = [\frac{1}{3}, 1) \cup \bigcup_{k \in \mathbb{N}_0} [2^{2k+1}, 2^{2k+2})$ as before.

By explicit calculation,

- $R(s,T) \leq \frac{1}{2}$ for $0 \leq s < \frac{1}{3} < T$, hence $l(s) \leq \frac{1}{2}$,
- and $l(t) = \limsup_{T \to \infty} R(t,T) = \frac{2}{3}$ for $t \geq \frac{1}{3}$.

An Example where Limits and Forward Risk Neutral Measures Exist, but Asymptotic Minimality Fails

Consider $\Omega = (0,1]$ with Lebesgue measure $\mathbb{Q}$, define $\mathcal{F}_s = \{\emptyset, \Omega\}$ for $s \in [0,1)$ and $\mathcal{F}_t$ for $t \geq 1$ as the Borel σ-algebra. Define $\tau(\omega) = 1/\omega$ and $r_t = 1_{[\tau,\infty)}(t)$ for $t \in [0,\infty)$. τ is $\mathcal{F}_1$-measurable, hence

$$R(1,T) = \frac{1}{T-1}\int_1^T r_u\,du = \frac{T-(T\wedge\tau)}{T-1} \xrightarrow{T\to\infty} 1$$

If $s \in [0,1)$ and $T \geq 1$, then $R(s,T)$ equals

$$-\frac{1}{T-s}\log \mathbb{E}_{\mathbb{Q}}\Bigg[\underbrace{\exp\bigg(-\int_1^T r_u\,du\bigg)}_{\geq 1_{\{\tau\geq T\}}}\Bigg] \leq -\frac{\log(1/T)}{T-s} \xrightarrow{T\to\infty} 0$$

Asymptotic Minimality is Not an Interval Property

Consider $\Omega = \mathbb{N}$, $\tau(\omega) = \omega$, $\mathbb{Q}(\{\omega\}) = \frac{1}{\omega} - \frac{1}{\omega+1}$,

$$\mathcal{F}_t = \begin{cases} \{\varnothing, \Omega\} & \text{for } t \in [0,1), \\ \{\varnothing, \{1\}, \Omega \setminus \{1\}, \Omega\} & \text{for } t \in [1,2), \\ \mathcal{P}(\Omega) & \text{for } t \in [2,\infty). \end{cases}$$

Define $r_t = (1 - \frac{1}{\tau})1_{[\tau,\infty)}(t)$ for $t \geq 0$. τ is $\mathcal{F}_2$-meas.
By explicit calculation, $R(2,T) \to 1 - \frac{1}{\tau}$ as $T \to \infty$.
Hence $l(2) = 1 - \frac{1}{\tau}$, $l(2)_{\mathcal{F}_0} = 0$, $l(2)_{\mathcal{F}_1} = \frac{1}{2} 1_{\Omega \setminus \{1\}}$.
By explicit calculation, $l(0) = l(1) = 0$, hence
asymptotic monotonicity holds for times 0 and 2,
but $l(1) = 0 \neq l(2)_{\mathcal{F}_1}$!

Reference

V. Goldammer and U. Schmock: *Generalization of the Dybvig–Ingersoll–Ross Theorem and Asymptotic Minimality.* Mathematical Finance (27 pages, to appear)

Slides and preprint available via
www.fam.tuwien.ac.at/~schmock/Dybvig-Ingersoll-Ross.html

Recent work (arXiv:0901.2080):
C. Kardaras and E. Platen:
On the Dybvig–Ingersoll–Ross Theorem

Determination of the maximal order that long-term rates at earlier dates can dominate those at later dates

Thank you for your attention!