Allocation of Risk Capital and Performance Measurement

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The allocation problem

originated from an audit
on RAC methods used by
a large Swiss insurance company.

Given risk bearing capital $C > 0$
for a financial institution, how to
allocate it to business units for

- measurement of risk contributions (for risk management),
- performance measurement (for steering the company),
- determination of bonuses for the management?

Further financial applications:

- Portfolios of defaultable bonds
- Portfolios of credit risks

Allocation principles
for risk capital:

Criteria:

- Respect dependencies (insurance/reinsurance, windstorm in several countries)
- Additive
- Computable for a portfolio of thousands of contracts
- "Fair" distribution of diversification effect

Investigated examples:

- Euler principle, Covariance principle
- Expected shortfall principle

Notation for Euler Principle

- Dependent risks $Z = (Z_1, \dots, Z_n)$
- Volumes $V = (V_1, \dots, V_n)$
- $R_i \equiv V_i Z_i$ result of business unit i
- Company result $R \equiv \langle V, Z \rangle = \sum_{i=1}^n V_i Z_i$
- Risk measure $\varrho \colon \mathbb{R}^n \ni V \mapsto \varrho(V)$
- Expected risk-adjusted return $r(\varrho, V) = \mathbb{E}[\langle V, Z \rangle] / \varrho(V)$
- $\alpha_i(\varrho, V)$ fraction of capital allocated to unit i, $\sum_{i=1}^n \alpha_i(\varrho, V) = 1$
- Expected risk-adjusted return of unit i:

$$r_i(\alpha, \varrho, V) \equiv \frac{\mathbb{E}[V_i Z_i]}{\alpha_i(\varrho, V) \varrho(V)}$$

Euler Principle

Def.: An allocation $A = (\alpha_1, \ldots, \alpha_n)$ is called consistent, if for the optimal portfolio $V = (V_1, \ldots, V_n)$ all individual returns are equal to the optimal company return.

Thm.: If the risk measure ϱ is differentiable and positively homogeneous, then an optimal portfolio exists and

$$\alpha_i(\varrho, V) \equiv \frac{V_i}{\varrho(V)} \frac{\partial \varrho}{\partial V_i}(V)$$

for V with $\varrho(V) \neq 0$ is consistent.

For V optimal and
$\varrho(V) \equiv -\mathbb{E}[\langle V, Z\rangle] + \kappa \sqrt{\mathrm{Var}[\langle V, Z\rangle]}$
$\Longrightarrow$ covariance principle

Expected shortfall principle:

- Stochastic gains of the business units: $R_1, R_2, \ldots, R_n \in L^1(\mathbb{P})$
- Profit and loss of the financial institution: $R \equiv R_1 + \cdots + R_n$
- Capital loss threshold c (for example α-quantile r_α of R)

Capital allocation:

$$\mathbb{E}[-R \,|\, R \leq c] = \sum_{i=1}^{n} \mathbb{E}[-R_i \,|\, R \leq c],$$

where

- $\mathbb{E}[-R \,|\, R \leq c]$ is the risk capital of the entire financial institution,
- $\mathbb{E}[-R_i \,|\, R \leq c]$ is the risk capital assigned to business unit i.

Calculating expected shortfall:

- $X \in L^1(\mathbb{P})$, $F_X(c) > 0$:

$$\mathbb{E}[X \,|\, X \le c] = \frac{1}{F_X(c)} \int_{-\infty}^{c} x \, F_X(dx)$$

- $X_1, \ldots, X_n \in L^1(\mathbb{P})$ exchangeable, $X \equiv X_1 + \cdots + X_n$, $F_X(c) > 0$:

$$\begin{aligned} \mathbb{E}[X_i \,|\, X \le c] &= \mathbb{E}[X_j \,|\, X \le c] \\ &= \frac{\mathbb{E}[X \,|\, X \le c]}{n} \end{aligned}$$

- $X, Y \in L^1(\mathbb{P})$ independent, $F_{X+Y}(c) = (F_X * F_Y)(c) > 0$:

$$\begin{aligned} &\mathbb{E}[X \,|\, X + Y \le c] \\ &\quad = \frac{1}{F_{X+Y}(c)} \int_{\mathbb{R}} x \, F_Y(c - x) \, F_X(dx) \end{aligned}$$

Generalises to indep. $X_1, \ldots, X_n$.

- $X, Y \in L^1(\mathbb{P})$ comonoton, $F_{X+Y}(c) > 0$:

  For $Z \equiv X + Y$ there exist cont., non-decreasing $u, v : \mathbb{R} \to \mathbb{R}$ such that $X = u(Z)$, $Y = v(Z)$ and $u(z) + v(z) = z$ for all $z \in \mathbb{R}$. Then

  $$\mathbb{E}[X \,|\, X + Y \leq c] = \mathbb{E}[u(Z) \,|\, Z \leq c] = \frac{1}{F_Z(c)} \int_{-\infty}^{c} u(z)\, F_Z(dz).$$

  Generalises to jointly comonoton $X_1, \ldots, X_n \in L^1(\mathbb{P})$.

- $\{(X_i, Y_i)\}_{i\in\mathbb{N}} \subset L^1(\mathbb{P})$ i. i. d.,
  X_i, Y_i comonoton,
  $N \sim \text{Poisson}(\lambda)$,
  N independent of $\{(X_i, Y_i)\}_{i\in\mathbb{N}}$:

  Write $X_i = u(Z_i)$ and $Y_i = v(Z_i)$ with $Z_i \equiv X_i + Y_i$,

  $$S_n \equiv X_1 + \cdots + X_n,$$
  $$T_n \equiv Z_1 + \cdots + Z_n.$$

  If $F_{T_N}(c) > 0$, then

  $$\mathbb{E}[S_N \,|\, T_N \le c]$$
  $$= \frac{\lambda}{F_{T_N}(c)} \int_{\mathbb{R}} u(z)\, F_{T_N}(c - z)\, F_Z(dz).$$

  F_Z discrete
  $\Longrightarrow F_{T_N}$ computable with Panjer algorithm

Advantages of expected shortfall:

- Takes frequency *and* severity of financial losses into account (contrary to VaR)
- Respects dependencies
- Additive
- One-sided risk measure (no capital required for a free lottery ticket)
- $\mathbb{E}[R_i \,|\, R \leq c]$ is in the convex hull of the possible values of R_i

Problems of expected shortfall:

- Dependence on tails, which are difficult to estimate in practice.
- Delicate dependence on the loss threshold c for small portfolios and discrete distributions.

Recent related work:

On the Coherent Allocation of Risk Capital

by Michel Denault
RiskLab, ETH Zürich

Combination of
– coherent risk measures (ADEH)
– ideas from game theory

Risk Contributions and Performance Measurement

by Dirk Tasche
TU Munich, Germany

Conditions on a vector field (for improving risk adjusted return) to be suitable for performance measurement with a risk measure ϱ

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$\longrightarrow$ Next talk

Risk Contributions and Performance Measurement

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$\longrightarrow$ Talk at ETH: Nov. 18, 1999