June 2, 1997

ESTIMATING THE VALUE OF THE WINCAT COUPONS OF THE WINTERTHUR INSURANCE CONVERTIBLE BOND*

Uwe Schmock

Mathematical Finance
Department of Mathematics, ETH Zürich
CH-8092 Zürich, Switzerland
schmock@math.ethz.ch

Abstract. The three annual $2\,{}^1\!/_4\%$ interest coupons of the Winterthur Insurance convertible bond (face value Chf 4 700) will only be paid out if during their corresponding observation periods no major storm or hail storm on one single day damages more than 6 000 motor vehicles insured with Winterthur Insurance. Data for events, where storm or hail damaged more than 1 000 insured vehicles, are available for the last ten years. Using a constant-parameter model, the estimated discounted value of the three Wincat coupons together is Chf 263.29. A conservative evaluation, which accounts for the standard deviation of the estimate, gives a coupon value of Chf 238.25. However, fitting a model, which admits a trend in the expected number of events per observation period, leads to substantially higher knock-out probabilities of the coupons. The estimated discounted value of the coupons drops to Chf 214.44; a conservative evaluation as above leads to substantially lower values. Hence, the model uncertainty is in this case substantially higher than the standard deviations of the used estimators.

Contents

1991 *Mathematics Subject Classification.* 62P05 (primary); 90A09 (secondary).

Key words and phrases. Wincat coupon, Winterthur Insurance, storm, hail, (generalized) Pareto distribution, composite Poisson model, generalized linear model.

*Financial support by Credit Suisse is gratefully acknowledged.

1. Introduction

The Swiss insurance company Winterthur Insurance has launched a three-year subordinated $2\,^1\!/_4\%$ convertible bond with so-called Wincat coupons, where Cat is an abbreviation for catastrophe. This bond with a face value of Chf 4 700 may be converted into five Winterthur Insurance registered shares at maturity (European-style option). The annual interest coupon of $2\,^1\!/_4\%$ will *not* be paid out if on any one calendar day during the corresponding observation period for the coupon more than 6 000 motor vehicles insured with Winterthur are damaged by hail or storm (wind speeds of 75 km/h and over). If the number of insured motor vehicles changes significantly, then the knock-out limit of 6 000 claims will be adjusted correspondingly.

Had Winterthur launched an identical fixed-rate convertible bond, then, according to Credit Suisse First Boston's brochure [2], the coupon rate would have been around 0.76 % lower (approximately 1.49 %). In other words, the investor receives an annual yield premium of 0.76 % for bearing a small portion of Winterthur's damage-to-vehicles risk. This convertible bond is intended as an instrument to diversify portfolios. It is suitable for this purpose, because storm and hail damages have only a very small correlation to traditional financial market risk. It is the intention of Winterthur Insurance to test the Swiss capital market for such a product.

This note will focus on estimating the risk arising from the Wincat coupons. Based on the available historic data, we shall present and work out several models and calculate the discounted value of the Wincat coupons in every case for an easy comparison of the various results. For the pricing of the European-style option for converting the bond into Winterthur Insurance registered shares, we refer to [2]. We just want to mention here, that the current value of the call option depends on the knock-out probability of the last coupon, because the exercise price of the call option is either Chf 4 805.75 (face value of the bond plus last coupon), if the last coupon is paid, or just the face value of Chf 4 700, if the last coupon is knocked out.

To estimate the risk of the Wincat coupons, a 10-year history of damage claims is provided in [2] and [6], see Table 1.1. During this period, a total of 17 events with more than 1 000 damaged vehicles were registered. Of these events, 15 happened during the summer and two were winter storms. Only two of the events, which happened on July 21^{st}, 1992, and July 5^{th}, 1993, caused more than 6 000 claims. Without any sophisticated modelling, this suggests a knock-out probability of 20 %, i. e., the expectation of the annual coupon payment would be 80 % of the $2\,^1\!/_4\%$ Wincat coupon, which is an expected annual yield of 1.8 %. Of course, as mentioned in [2, p. 11], this estimate has little statistical significance.

In Section 2 of this note, we present and briefly discuss the available historic data. Section 3 contains a critical review of a simple binomial model. In Section 4 we give a review of the constant-intensity model to estimate the discounted value of the Wincat coupons. We discuss several distributions to obtain an estimate for the probability, that an event causing more than 1 000 adjusted claims actually leads to the knock-out of the coupon. These distributions include the Bernoulli distribution, the Pareto distribution (used in [2]) and finally, suggested by extreme value theory, the generalized Pareto distribution. According to Winterthur's web page [6], the length of the observation period for the first coupon is not an entire

Year	Date	Event	Number of claims	Vehicles insured index	Adjusted claims
1987				1.248	
1988				1.204	
1989				1.161	
1990	27. Feb.	Storm	1 646	1.127	1 855
	30. June	Hail	1 395		1 572
1991	23. June	Hail	1 333	1.104	1 472
	6. July	Hail	1 114		1 230
1992	21. July	Hail	8 798	1.098	**9 660**
	31. July	Hail	1 085		1 191
	20. Aug.	Hail	1 253		1 376
	21. Aug.	Hail	1 733		1 903
1993	5. July	Hail	6 589	1.099	**7 241**
1994	2. June	Hail	4 802	1.086	5 215
	24. June	Hail	940		1 021
	18. July	Hail	992		1 077
	6. Aug.	Hail	2 460		2 672
	10. Aug.	Hail	2 820		3 063
1995	26. Jan.	Storm	1 167	1.067	1 245
	2. July	Hail	1 290		1 376
1996	20. June	Hail	1 262	1.000	1 262

TABLE 1.1. Claim numbers of past events which caused over 1 000 adjusted claims as provided in [2] and [6]. Since the number of motor vehicles insured with Winterthur tends to increase, former actual claim numbers are set into relation with the number of insured vehicles to obtain the number of adjusted claims.

year as assumed in [2]; therefore we recalculate the discounted value of the WINCAT coupons also for the cases already considered in [2].

In Section 5 we present and discuss various models with a time-dependent parameter for the number of events with more than 1 000 adjusted claims. We shall give several reasons why there might be a trend in the data. An investor, who wants to take a possible trend into account, might use one of these models to estimate the discounted value of the WINCAT coupons. These models will lead the investor to substantially lower estimates for the values of the WINCAT coupons. A short discussion of these values is given at the end of Subsection 5.4, see Table 5.5 for a comparison. These substantially different values indicate that the model uncertainty is the dominating one for the evaluation of the WINCAT coupons.

2. PRESENTATION AND DISCUSSION OF THE DATA

Whether a WINCAT coupon is paid on February 28$^{\text{th}}$ depends on the events happening during the corresponding observation period. These are specified on Winterthur's web page [6], see Table 2.1. The first observation period is shorter than a

Coupon date	Relevant observation period
February 28, 1998	February 28, 1997 – October 31, 1997
February 28, 1999	November 1, 1997 – October 31, 1998
February 28, 2000	November 1, 1998 – October 31, 1999

TABLE 2.1. Observation periods for the WINCAT coupons according to the web page of Winterthur Insurance [6].

year so that there are always four months left between the end of the observation period and the coupon payment date. This provides enough time to count the number of claims and to determine whether the corresponding coupon is knocked out. In the 10-year history of damage claims provided in [2] and [6], see Table 1.1, two events are not within the period from February 28$^{\text{th}}$ to October 31$^{\text{st}}$. This is relevant for the first coupon, we shall therefore always reduce the knock-out probability for the first coupon in a deterministic way (see Table 3.2) using the formula

$$P_{\text{CAT}} = 1 - (1 - \tilde{P}_{\text{CAT}})^{15/17}, \tag{2.1}$$

where $\tilde{P}_{\text{CAT}}$ denotes here the knock-out probability if the observation period were a full year. Formula (2.1) is motivated by the Poisson models used in Sections 4 and 5. It corresponds to reducing the Poisson parameter by the factor $^{15}/_{17}$, see the discussion in the introduction of Section 4 and the one of formula (4.8). By using (2.1), we neglect the fact that the number of events not occurring in the period from February 28$^{\text{th}}$ to October 31$^{\text{st}}$ is random as well. This simplification, however, is suggested by the lack of data and can be justified by the small influence of this $^{15}/_{17}$-correction (CHF 2.21 for $\tilde{P}_{\text{CAT}} = 20\%$, for example) when compared with the model uncertainty to be discussed. Furthermore, when analysing the numbers of adjusted claims in Section 4, we assume that the two numbers arising from the winter storms come from the same underlying distribution as the numbers arising from the hail storms. Again, this simplifying assumption is suggested by the small historic data set.

The number of claims arising from damage by storm or hail have to be set into relation with the number of vehicles insured with Winterthur. The statistical basis is 773 600 insured motor vehicles in 1996. This number already includes the motor vehicles insured with Neuenburger Schweizerische Allgemeine Versicherungsgesellschaft, which merges with Winterthur in 1997. The column *Vehicles insured index* in Table 1.1 gives the number of insured vehicles in 1996 divided by the number of insured vehicles for the respective year. The column *Adjusted claims* in Table 1.1 contains the claim numbers multiplied with the insured-vehicles index. Only events with more than 1 000 adjusted claims are shown in Table 1.1, because other data is not provided by Winterthur Insurance.

3. A critical review of a binomial model

To extract the relevant information from the historic data given in Table 1.1, we could use a simple model consisting of ten Bernoulli random variables X_{1987}, X_{1988}, $\ldots$, X_{1996}, where $X_y = 1$ means that least one event with more than 6 000 adjusted claims happened in the observation period ending at October 31$^{\text{st}}$ of the year y. We set $X_y = 0$ otherwise. For the model we assume that these ten random variables

Coupon	Discount rate	Discount factor
1.	1.87%	0.9816%
2.	2.33%	0.9550%
3.	2.57%	0.9267%

TABLE 3.1. Assumptions regarding the interest-rate structure taken from [2]. The discount rates correspond to the zero-coupon yield on Confederation bonds plus a spread of 35 basis points.

are independent and identically distributed. We are interested in estimating the probability $p = \mathbb{P}(X_y = 1)$. An unbiased estimator of p is the empirical mean

$$\hat{p} = \frac{1}{10} \sum_{y=1987}^{1996} X_y. \tag{3.1}$$

The data of Table 1.1 leads to $\hat{p} = 0.2$, because there were two observation periods out of ten where an event with more than 6 000 adjusted claims happened. Using coupon knock-out probabilities of $P_{\text{CAT}}(1997) = 1 - (1 - 0.2)^{15/17} \approx 0.179$ for the first observation period and $P_{\text{CAT}}(1998) = P_{\text{CAT}}(1999) = 0.2$ for the following two years, and using the interest rate structure of Table 3.1, the discounted value of the three coupons is calculated in Table 3.2.

Of course the estimator in in (3.1) can only lead to one of the eleven values in the set $\{0.0, 0.1, 0.2, \dots, 0.9, 1.0\}$. Hence, to be realistic, we should not favour any specific value within the interval $[0.15, 0.25]$. A recalculation of Table 3.2 with the knock-out probabilities 15% and 25% gives CHF 259.08 and CHF 229.78, respectively, for the discounted value of the three WINCAT coupons.

From a statistical point of view we should also consider the standard deviation of the estimator in (3.1). This will give an impression of the quality of the estimator. Since the variance is given by

$$\sigma^2(\hat{p}) = \operatorname{Var}\biggl(\frac{1}{10} \sum_{y=1987}^{1996} X_y\biggr) = \frac{p(1-p)}{10}, \tag{3.2}$$

Coupon	Principle	Interest	Discount factor	P_{CAT}	Value
1.	4 700	2 1/4%	0.9816%	17.9%	CHF 85.25
2.	4 700	2 1/4%	0.9550%	20%	CHF 80.79
3.	4 700	2 1/4%	0.9267%	20%	CHF 78.40
Discounted value of the three WINCAT coupons:					CHF 244.44

TABLE 3.2. Calculation of the discounted value of the three WINCAT coupons for the estimate $\hat{p} = 0.2$. The three discount factors are taken from Table 3.1. The product of the principle, the coupon interest rate and the discount factor is multiplied with the probability $(1 - P_{\text{CAT}})$ that the corresponding coupon is not knocked out. The 15/17-correction according to (2.1) was applied to the knock-out probability of the first coupon to take care of its shorter observation period given in Table 2.1.

we could follow statistical practice and use the estimated value $\hat{p} = 0.2$ for p to obtain an estimate $\hat{\sigma}^2(\hat{p})$ for the variance $\sigma^2(\hat{p})$. This would mean to use the estimator $\hat{p}(1 - \hat{p})/10$. In this binomial model, however, a short calculation shows that

$$\mathbb{E}\left[\frac{\hat{p}(1-\hat{p})}{10}\right] = \frac{10-1}{10} \cdot \frac{p(1-p)}{10},$$

which means that we would underestimate the variance in (3.2) by a factor 9/10. Therefore, we use the unbiased estimator

$$\hat{\sigma}^2(\hat{p}) = \frac{10}{9} \cdot \frac{\hat{p}(1-\hat{p})}{10} \tag{3.3}$$

for the variance of $\hat{p}$ to obtain the estimate

$$\hat{\sigma}(\hat{p}) = \sqrt{\frac{10}{9} \cdot \frac{0.2 \cdot 0.8}{10}} \approx 0.13 \tag{3.4}$$

for the standard deviation of $\hat{p}$. For a conservative estimate we may use $P_{\text{CAT}} = \hat{p} + \hat{\sigma}(\hat{p}) \approx 0.33$ as the knock-out probability. A recalculation of Table 3.2 with this knock-out probability leads to CHF 205.24.*

The empirical mean in (3.1) is a minimal sufficient estimator for p *in this model* [4, Chapter 1, Problem 17], hence we have done our best within this model. We cannot expect more from this model, because it uses the data of Table 1.1 very ineffectively. Already in the first step, the data is reduced to ten yes/no decisions (10 bit of information). By taking the mean in (3.1), this information is further reduced by ignoring the order of the ten yes/no decisions, leading to one out of eleven possible numbers. This is less then 4 bit of information. Having gone through this bottleneck, not much can be done with a statistical examination afterwards.

Instead of using the number $P_{\text{CAT}} = \hat{p} + \hat{\sigma}(\hat{p})$ as a conservative estimate of the knock-out probability, we could elaborate on this point by using the entire estimated distribution of $\hat{p}$. Taking investor-dependent utility functions and the current market price of risk into account, a more profound analysis would be possible than the one given above and in the following sections. Since the estimated knock-out probabilities and the corresponding standard deviations will vary substantially with the used models, the point of this note will already be made clear without such an analysis.

4. Composite Poisson models with constant parameters

To extract more data from Table 1.1 than in the previous section, we shall review several composite Poisson models. The one in Subsection 4.2 was used for the analysis in [2]. For every calendar day in an observation period there is a slight chance of a major storm or hail storm causing more than 1 000 adjusted claims. The data of Table 1.1 as well as common knowledge suggest that this slight chance varies with the season: In Switzerland, a storm is more likely to occur in late autumn or winter than in any other season while hail storms usually occur in summer. If the

*Most of the calculations in this note were done with the software package *Mathematica*. Only rounded numbers are given in the text, but for subsequent calculations machine precision of the numbers is used. Values in Swiss francs are given up to 1/100, although not all given digits are necessarily significant.

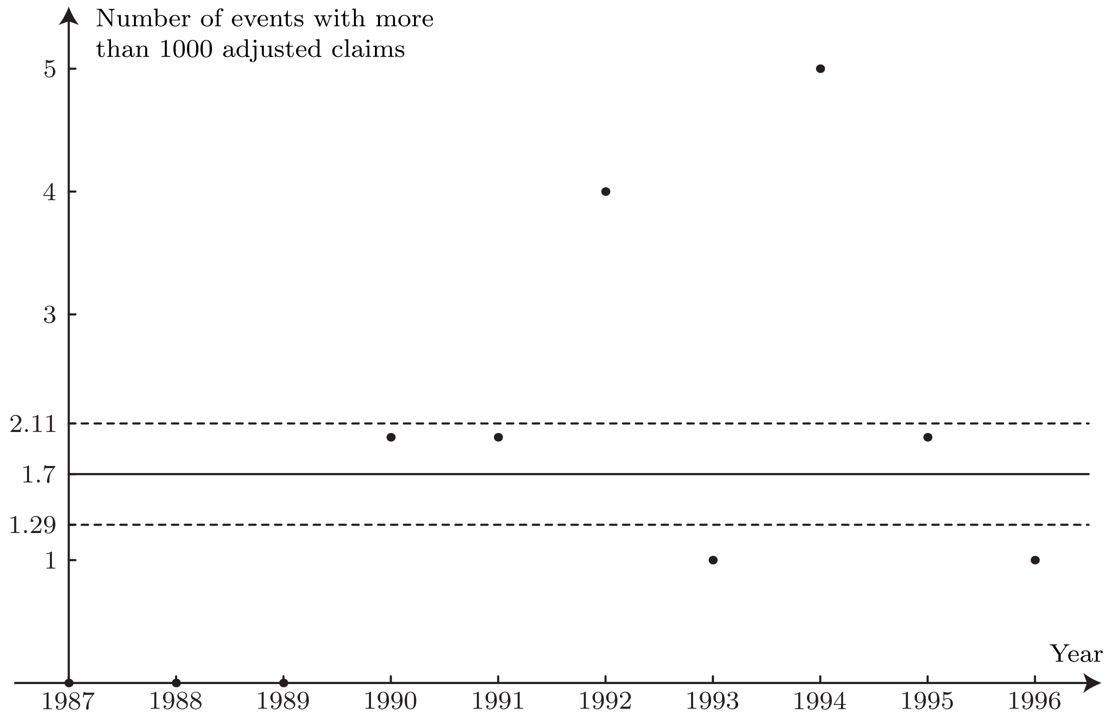

FIGURE 4.1. Observed number of events in the ten observation periods November 1st to October 31st causing more than 1 000 adjusted claims. The empirical mean of $\lambda_{1000}^{\text{const}} = 1.7$ events per observation period is also shown. The dashed lines indicate the standard deviation $\hat{\sigma}(\lambda_{1000}^{\text{const}})$.

dependence between the different days is sufficiently weak, then the Poisson limit theorem suggests that a Poisson random variable might be a good approximation for the number of those events within an observation period, which cause more than 1 000 adjusted claims. Note that Table 1.1 records hail storms for August 20^{th}, 1992, and the following day, hence the assumption of "sufficiently weak dependence" has to be kept in mind. Such two-day events can arise artificially from a single storm due to the dividing line at midnight, or they can arise due to weather conditions favouring a hail storm on two consecutive days. The use of a compound Poisson model however, which allows us to model such two-day events conveniently, does not seem to be appropriate here, because a single observation is not sufficient for a reliable estimate of the corresponding parameter.

The seasonal dependence mentioned above is also the reason why we have chosen the exponent $^{15}/_{17}$ in the correction formula (2.1). We think that this exponent based on the available data is more appropriate then the exponent $^{2}/_{3}$ based on the length of the shorter first observation period given in Table 2.1.

The Poisson distribution with parameter $\lambda > 0$ is defined by

$$\text{Poisson}_{\lambda}(k) = \frac{\lambda^k}{k!} e^{-\lambda} \qquad \text{for } k \in \mathbb{N}_0. \tag{4.1}$$

Let the random variable N_y describe the number of days within the observation period ending in year $y \in \{1987, \dots, 1996\}$, on which more than 1 000 adjusted claims arose from damage by storm or hail. We assume that these ten random variables are independent and that each of them has a Poisson distribution with the same parameter $\lambda > 0$. Since $\mathbb{E}[N_y] = \lambda$, the empirical mean

$$\lambda_{1000}^{\text{const}} = \frac{1}{10} \sum_{y=1987}^{1996} N_y \tag{4.2}$$

is an unbiased estimator for λ, which is also sufficient [4, Section 1.9, Example 16]. Table 1.1 contains $m = 17$ events within the $n = 10$ observation periods, hence

$$\lambda_{1000}^{\text{const}} = \frac{m}{n} = \frac{17}{10} = 1.7\ . \tag{4.3}$$

Figure 4.1 contains an illustration of the counting data and this empirical mean. Since $\mathrm{Var}(N_y) = \lambda$, the variance of the estimator $\lambda_{1000}^{\text{const}}$ in (4.2) is λ/n with $n = 10$, hence the estimated standard deviation of $\lambda_{1000}^{\text{const}}$ is

$$\hat{\sigma}(\lambda_{1000}^{\text{const}}) = \sqrt{\frac{\lambda_{1000}^{\text{const}}}{n}} = \sqrt{\frac{m/n}{n}} = \frac{\sqrt{m}}{n} = \frac{\sqrt{17}}{10} \approx 0.41\ . \tag{4.4}$$

It remains to determine the probability that an event, which causes more than 1 000 adjusted claims, actually causes more than 6 000 adjusted claims and therefore leads to the knock-out of the corresponding WINCAT coupon.

4.1. **Bernoulli distribution for the knock-out events.** In this subsection we shall introduce a simple model to describe events with more than 1 000 adjusted claims, which actually cause more than 6 000 adjusted claims; meaning that they lead to a knock-out of the WINCAT coupon. For this purpose we introduce Bernoulli random variables $X_1, \ldots, X_m$ for the $m = 17$ events, where $X_k = 1$ means that event number $k \in \{1, \ldots, m\}$ caused more than 6 000 adjusted claims. We set $X_k = 0$ otherwise. Proceeding as in Section 3, we can estimate the probability $p_{6000} = \mathbb{P}(X_k = 1)$ by the unbiased empirical mean

$$\hat{p}_{6000} = \frac{1}{m} \sum_{k=1}^{m} X_k.$$

The data of Table 1.1 leads to $\hat{p}_{6000} = 2/m = {}^2\!/_{17} \approx 0.118$. An analysis similar to (3.2), (3.3) and (3.4) gives the estimate

$$\hat{\sigma}(\hat{p}_{6000}) = \sqrt{\frac{m}{m-1} \, \frac{\hat{p}_{6000}(1 - \hat{p}_{6000})}{m}} = \sqrt{\frac{{}^2\!/_{17} \cdot (1 - {}^2\!/_{17})}{16}} \approx 0.081 \tag{4.5}$$

for the standard deviation of $\hat{p}_{6000}$.

If N is a random variable with a Poisson distribution given by (4.1) describing the number of events, and if independently of everything else we perform a Bernoulli experiment with success probability $p_{6000} \in [0, 1]$ for each of the N events, then an elementary exercise shows that the resulting number of successful events has a Poisson distribution with parameter $p_{6000}\lambda$. Therefore, under the above assumptions, the number of events per observation period leading to more than 6 000 adjusted claims has a Poisson distribution. An estimate for the corresponding Poisson parameter is

$$\lambda_{6000}^{\text{const}} = \hat{p}_{6000} \cdot \lambda_{1000}^{\text{const}} = \frac{2}{m} \cdot \frac{m}{n} = \frac{2}{10} = 0.2\ . \tag{4.6}$$

The probability that no such event happens, is given by $\exp(-\lambda_{6000}^{\text{const}})$, see (4.1) with $k=0$. Hence, the estimated knock-out probability is

$$P_{\text{CAT}} = 1 - \exp(-\lambda_{6000}^{\text{const}}) = 1 - \exp(-0.2) \approx 0.181\ . \tag{4.7}$$

A recalculation of Table 3.2 with this value of P_{CAT} leads to a discounted value of CHF 249.93 for the three WINCAT coupons.

To estimate the knock-out probability of the first WINCAT coupon, we have to replace $\lambda_{1000}^{\text{const}} = {}^{17}\!/\!_{10}$ from (4.3) by $\lambda_{1000}^{\text{const}} = {}^{15}\!/\!_{10}$, because only 15 events are recorded in Table 1.1 for the period from February 28th to October 31st. This leads via (4.6) and (4.7) to

$$\begin{aligned} P_{\text{CAT}} &= 1 - \exp\bigl(-\hat{p}_{6000} \cdot \lambda_{1000}^{\text{const}}\bigr) \\ &= 1 - \exp\Bigl(-\frac{2}{17}\cdot\frac{15}{10}\Bigr) = 1 - \exp\Bigl(-\frac{15}{17}\cdot 0.2\Bigr) \approx 0.162, \end{aligned} \tag{4.8}$$

which is exactly the same result as the one obtained by applying the correction formula (2.1) to the result of (4.7).

The variance of the estimator $\lambda_{6000}^{\text{const}}$ is not easily computable from the variances of $\hat{p}_{6000}$ and $\lambda_{1000}^{\text{const}}$, because these two estimators are dependent (knowing $\lambda_{1000}^{\text{const}}$ restricts the set of possible values for $\hat{p}_{6000}$). According to our model assumptions however, we have observations from $n = 10$ independent Poisson random variables available, which describe the number of events in each of the ten observation periods leading to more than 6 000 adjusted claims. Similar to (4.2) and (4.4), we therefore see that the estimator (4.6) for $\lambda_{6000}^{\text{const}}$ is unbiased and that

$$\hat{\sigma}(\lambda_{6000}^{\text{const}}) = \sqrt{\lambda_{6000}^{\text{const}}/n} = \sqrt{0.2/10} \approx 0.141\ .$$

For a conservative estimate of the knock-out probability we might use

$$P_{\text{CAT}} = 1 - \exp\bigl(-\lambda_{6000}^{\text{const}} - \hat{\sigma}(\lambda_{6000}^{\text{const}})\bigr) \approx 1 - \exp(-0.341) \approx 0.289\ .$$

A recalculation of Table 3.2 with this value of P_{CAT} leads to CHF 218.24 for the discounted value of the three WINCAT coupons.

There is a methodical problem with the approach in this subsection so far. We are mainly interested in an unbiased estimator for the knock-out probability P_{CAT}. The unbiasedness of the estimator $\lambda_{6000}^{\text{const}}$ for a model specific parameter is not of primary concern. To elaborate on this point, let $N_{6000,n}$ be the number of events with more than 6 000 adjusted claims within $n = 10$ observation periods. According to our model assumptions, $N_{6000,n}$ has a Poisson distribution with parameter $np\lambda$, where λ is the intensity for the number of events per observation period with more than 1 000 adjusted claims, and $p = p_{6000}$ is the "success" probability for the following Bernoulli experiment indicating whether actually more than 6 000 adjusted claims arise from the event. The estimator (4.7) corresponds to

$$P_{\text{CAT}} = 1 - \exp(-N_{6000,n}/n) \tag{4.9}$$

with $n = 10$. Calculating the expectation gives

$$\begin{aligned}\mathbb{E}[1-\exp(-N_{6000,n}/n)] &= 1-\sum_{k=0}^{\infty} e^{-k/n}\,\frac{(np\lambda)^k}{k!}e^{-np\lambda}\\ &= 1-\sum_{k=0}^{\infty}\frac{(np\lambda e^{-1/n})^k}{k!}e^{-np\lambda}\\ &= 1-\exp\bigl(-(1-e^{-1/n})np\lambda\bigr),\end{aligned}\tag{4.10}$$

which is different from $1-\exp(-p\lambda)$, hence (4.9) is biased. Multiplying $N_{6000,n}$ in (4.9) by the correction factor $\log\bigl(\frac{n}{n-1}\bigr)^n$ leads to the estimator

$$P_{\text{CAT}} = 1-\Bigl(1-\frac{1}{n}\Bigr)^{N_{6000,n}}\tag{4.11}$$

with expectation $1-\exp(-p\lambda)$ as a calculation similar to (4.10) shows. Hence the estimator (4.11) is unbiased. Since $n = 10$ and $N_{6000,10} = 2$ by Table 1.1, we obtain the estimate

$$P_{\text{CAT}} = 1-\Bigl(\frac{9}{10}\Bigr)^2 = 0.19\,.\tag{4.12}$$

The corresponding recalculation of Table 3.2 gives CHF 247.37 for the discounted value of the three WINCAT coupons.

For the variance of the estimator in (4.11) we obtain after a short calculation similar to (4.10)

$$\operatorname{Var}(P_{\text{CAT}}) = \mathbb{E}\biggl[\Bigl(1-\frac{1}{n}\Bigr)^{2N_{6000,n}}\biggr]-e^{-2p\lambda} = e^{-2p\lambda}(e^{p\lambda/n}-1).$$

Using the estimate $\lambda_{6000}^{\text{const}} = 0.2$ for $p\lambda$ from (4.6) and $(9/10)^2$ for $e^{-p\lambda}$ from (4.12), we obtain

$$\hat{\sigma}(P_{\text{CAT}}) = e^{-p\lambda}\sqrt{e^{p\lambda/n}-1} \approx e^{-p\lambda}\sqrt{p\lambda/n} \approx \Bigl(\frac{9}{10}\Bigr)^2\sqrt{0.02} \approx 0.115\,.\tag{4.13}$$

A recalculation of Table 3.2 with the conservative knock-out probability $P_{\text{CAT}} + \hat{\sigma}(P_{\text{CAT}}) \approx 0.305$ gives CHF 213.73 for the discounted value of the three WINCAT coupons.

The estimated standard deviation in (4.13) is slightly smaller than the one in the simple binomial model given by (3.4). This indicates that in our case the composite Poisson model of this subsection leads only to a slight improvement. Indeed, the estimator (4.12) for the knock-out probability uses only the information that two events within the ten years caused more than 6 000 adjusted claims. Since the model of this subsection allows these two events to happen in the same year, the estimated knock-out probability in (4.12) is 1% lower than the one in the binomial model. If the two events with more than 6 000 adjusted claims had actually happened in the same year and not in consecutive ones, the discrepancy in the estimated knock-out probabilities would be 9%, because the estimate in the binomial model of Section 3 would drop from 20% to 10%. This indicates that in this respect the composite Poisson model of this subsection is more robust than the binomial one.

4.2. **Pareto distribution for the knock-out events.** The binomial model of Section 3 and the corresponding composite Poisson model of Subsection 4.1 do not use the adjusted claim numbers recorded in Table 1.1. For the benefit of a better estimate of p_{6000}, let us incorporate these numbers into the model. The step function in Figure 4.2 is the empirical distribution function of the adjusted claim numbers from Table 1.1. A heavy-tailed distribution of common use is the Pareto distribution, its distribution function is given by

$$\text{Pareto}_{a,b}(x) = \begin{cases} 1-(a/x)^b & \text{for } x \geq a, \\ 0 & \text{for } x < a, \end{cases} \tag{4.14}$$

where a and b are strictly positive parameters. The Pareto distribution is used in [2] to model the number of adjusted claims per event given that more than 1 000 adjusted claims arise from the event. We choose $a = 1\,000$, because only those events with more than 1 000 adjusted claims are contained in Table 1.1.

To fit the empirical distribution with a Pareto distribution as in Figure 4.2, we need an estimator for the exponent b. If a random variable X has a Pareto distribution with parameters a and b, then $Y \equiv \log(X/a)$ satisfies

$$\mathbb{P}(Y \leq y) = \mathbb{P}(X \leq ae^y) = 1 - \Big(\frac{a}{ae^y}\Big)^b = 1 - e^{-by}, \qquad y \geq 0,$$

which means that Y has an exponential distribution with expectation $\mathbb{E}[Y] = 1/b$. Hence, if the independent random variables $X_1, \dots, X_m$ with a Pareto distribution given by (4.14) describe the adjusted number of claims for the m events, then the random variables $Y_1, \dots, Y_m$ with $Y_k \equiv \log(X_k/a)$ are independent and exponentially distributed. Their empirical mean

$$\frac{1}{m}\sum_{k=1}^m Y_k = \frac{1}{m}\sum_{k=1}^m \log\frac{X_k}{a}$$

is an unbiased estimator for $1/b$. This suggests using the reciprocal value

$$\frac{m}{\sum_{k=1}^m Y_k} = \frac{m}{\sum_{k=1}^m \log(X_k/a)} \tag{4.15}$$

as an estimator for b. Another way to derive this estimator is to consider the likelihood function

$$L_m(b) \equiv \prod_{k=1}^m \frac{b}{X_k}\Big(\frac{a}{X_k}\Big)^b, \qquad b > 0, \tag{4.16}$$

which is the product of the densities of the Pareto distribution (4.14) evaluated at $X_1, \dots, X_m$. By differentiating the logarithm of this likelihood function, we find that b given by (4.15) maximizes the likelihood function (4.16), hence (4.15) is also the maximum likelihood estimator for b.

Let us calculate the expectation of the estimator in (4.15). The sum $\sum_{k=1}^m Y_k$ has a gamma distribution with parameters m and b, meaning that

$$\mathbb{P}\Big(\sum_{k=1}^m Y_k \leq y\Big) = \int_0^y \frac{b}{\Gamma(m)}(bt)^{m-1}e^{-bt}\,dt, \qquad y \geq 0.$$

This fact is easily proved by an induction on m, because the convolution of the

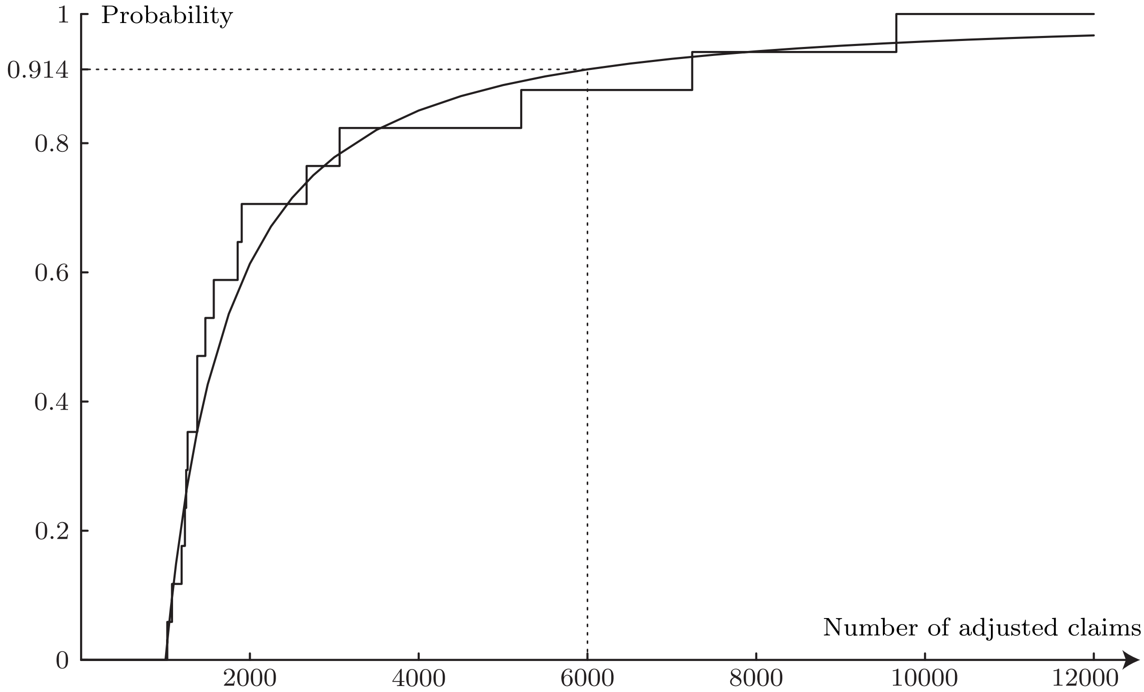

FIGURE 4.2. The empirical distribution of the number of (adjusted) claims per event, given that more than 1 000 claims arise from the event. Also shown is the Pareto distribution (4.14) with $a = 1\,000$ and $b = \hat{b}$, where $\hat{b} \approx 1.37$ is the maximum likelihood estimate, corrected with the factor $(m-1)/m$ for $m = 17$ to eliminate the bias. The estimated probability, that an event with at least 1 000 claims causes at most 6 000 claims, is around 0.914.

exponential density and the gamma density of parameter m leads to the gamma density of parameter $m+1$:

$$\int_0^\infty be^{-b(t-s)} \cdot \frac{b}{\Gamma(m)}(bs)^{m-1}e^{-bs}\,ds = \frac{b}{m\Gamma(m)}(bt)^m e^{-bt}, \qquad t \ge 0,$$

and the gamma function satisfies $\Gamma(m+1) = m\Gamma(m)$. Calculating the expectation of (4.15) for $m \ge 2$ shows that

$$\begin{aligned}\mathbb{E}\biggl[\frac{m}{\sum_{k=1}^m \log(X_k/a)}\biggr] &= \int_0^\infty \frac{m}{t} \cdot \frac{b}{\Gamma(m)}(bt)^{m-1}e^{-bt}\,dt \\ &= \frac{mb}{m-1}\int_0^\infty \frac{b}{\Gamma(m-1)}(bt)^{m-2}e^{-bt}\,dt = \frac{m}{m-1}b.\end{aligned} \tag{4.17}$$

This means that the estimator in (4.15) underestimates the tail of the Pareto distribution. To obtain an unbiased estimator for b, we therefore have to use

$$\hat{b} = \frac{m-1}{\sum_{k=1}^m \log(X_k/a)} \tag{4.18}$$

instead of (4.15). The data from the last column of Table 1.1 leads to

$$\hat{b} \approx 1.37\,. \tag{4.19}$$

The Pareto distribution with this value is shown in Figure 4.2.

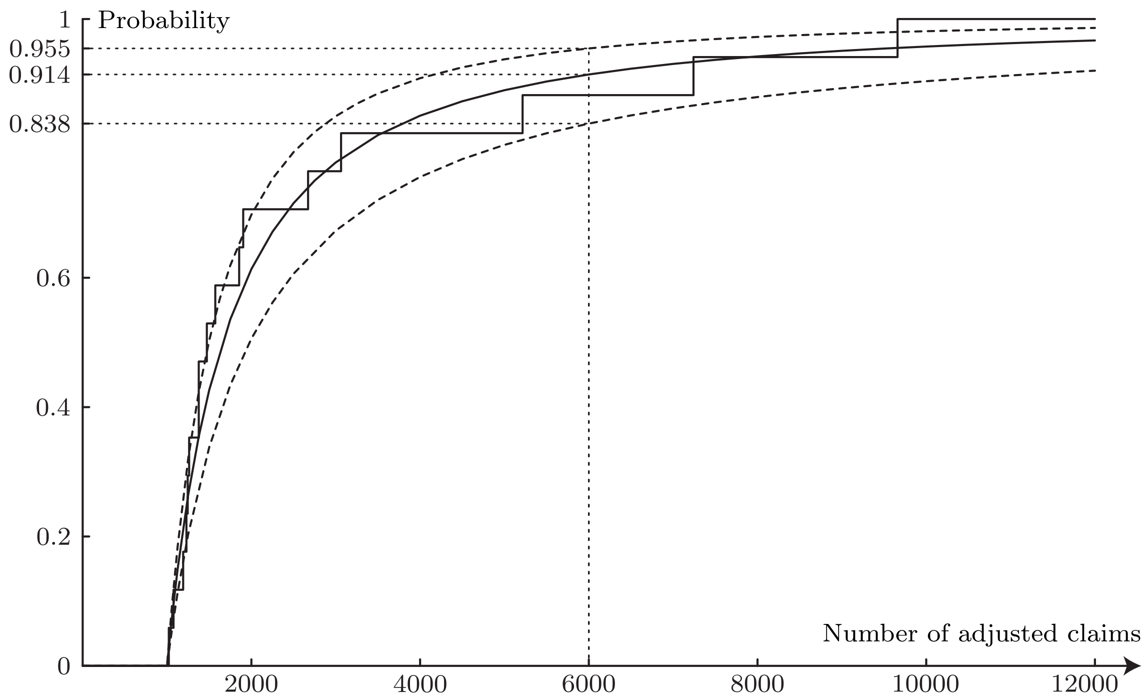

FIGURE 4.3. This is Figure 4.2 with two additional Pareto distributions illustrating the estimated standard deviation of the estimate $\hat{b} \approx 1.37$. The lower dashed curve corresponds to the parameter $\hat{b} - \hat{\sigma}(\hat{b}) \approx 1.02$, the upper one to the parameter $\hat{b} + \hat{\sigma}(\hat{b}) \approx 1.73$.

A calculation similar to (4.17) leads to

$$\mathrm{Var}(\hat{b}) = \frac{b^2}{m-2} \tag{4.20}$$

for $m \geq 3$. Therefore, $\hat{\sigma}(\hat{b}) = \hat{b}/\sqrt{m-2}$ is an unbiased estimator for the standard deviation; using the numerical value from (4.19) gives

$$\hat{\sigma}(\hat{b}) \approx 1.37/\sqrt{15} \approx 0.35\ . \tag{4.21}$$

The Pareto distributions corresponding to $\hat{b} \pm \hat{\sigma}(\hat{b})$ are both shown in Figure 4.3.

Using the estimator $\hat{b}$ for the parameter of the Pareto distribution (4.14), we obtain the estimator

$$\hat{p}_{6000} = 1 - \mathrm{Pareto}_{1000,\hat{b}}(6000) = 6^{-\hat{b}} \tag{4.22}$$

for the probability that an event, which causes more than 1 000 adjusted claims, actually causes more than 6 000 adjusted claims. The numerical value $\hat{b} \approx 1.37$ from (4.19) leads to

$$\hat{p}_{6000} \approx 6^{-1.37} \approx 0.086\ . \tag{4.23}$$

Considering the two Pareto distributions corresponding to $\hat{b} - \hat{\sigma}(\hat{b}) \approx 1.02$ and $\hat{b} + \hat{\sigma}(\hat{b}) \approx 1.73$ (see Figure 4.3), we obtain via (4.22) the asymmetric interval

$$[6^{-1.73}, 6^{-1.02}] \approx [0.045, 0.162] \tag{4.24}$$

around the estimate $\hat{p}_{6000} \approx 0.086$ as an indication of the standard deviation. This is an improvement compared to the interval $[0.037, 0.199]$ arising from the Bernoulli distribution via (4.5).

Following the approach in [2], we recalculate the estimate (4.6) for the Poisson parameter $\lambda_{6000}^{\text{const}}$ describing the number of knock-out events per observation period using $\lambda_{1000}^{\text{const}} = 1.7$ from (4.3) and $\hat{p}_{6000} \approx 0.086$ from (4.23). We obtain

$$\lambda_{6000}^{\text{const}} = \hat{p}_{6000} \cdot \lambda_{1000}^{\text{const}} \approx 0.146\,.$$

As in (4.7), the estimated knock-out probability is

$$\begin{aligned} P_{\text{CAT}} &= 1 - \exp\bigl(-\lambda_{6000}^{\text{const}}\bigr) \\ &= 1 - \exp\bigl(-\hat{p}_{6000} \cdot \lambda_{1000}^{\text{const}}\bigr) \approx 1 - \exp(-0.146) \approx 0.136\,. \end{aligned} \tag{4.25}$$

A recalculation of Table 3.2 with this value of P_{CAT} leads to a discounted value of CHF 263.29 for the three WINCAT coupons.

To get a rough estimate of the standard deviation of the knock-out probability in (4.25), consider this probability as a function of the two estimated parameters $\hat{b}$ and $\hat{\lambda} \equiv \lambda_{1000}^{\text{const}}$:

$$P_{\text{CAT}}(\hat{b}, \hat{\lambda}) = 1 - \exp\bigl(-6^{-\hat{b}}\hat{\lambda}\bigr).$$

Using the approximating plane in (b, λ) and thereby neglecting all higher order terms in the Taylor expansion, we get

$$P_{\text{CAT}}(\hat{b}, \hat{\lambda}) \approx P_{\text{CAT}}(b, \lambda) + \frac{\partial P_{\text{CAT}}}{\partial b}(b, \lambda)(\hat{b} - b) + \frac{\partial P_{\text{CAT}}}{\partial \lambda}(b, \lambda)(\hat{\lambda} - \lambda).$$

Since $\hat{b}$ and $\hat{\lambda}$ are unbiased, we obtain for the variance

$$\begin{aligned} \operatorname{Var}\bigl(P_{\text{CAT}}(\hat{b}, \hat{\lambda})\bigr) \approx{}& \Bigl(\frac{\partial P_{\text{CAT}}}{\partial b}(b, \lambda)\Bigr)^2 \operatorname{Var}(\hat{b}) + \Bigl(\frac{\partial P_{\text{CAT}}}{\partial \lambda}(b, \lambda)\Bigr)^2 \operatorname{Var}(\hat{\lambda}) \\ &+ \frac{\partial P_{\text{CAT}}}{\partial b}(b, \lambda)\,\frac{\partial P_{\text{CAT}}}{\partial \lambda}(b, \lambda)\,\mathbb{E}[(\hat{b} - b)(\hat{\lambda} - \lambda)]. \end{aligned}$$

The estimators $\hat{b}$ and $\hat{\lambda} = \lambda_{1000}^{\text{const}}$ are certainly not independent, because the observed number m of events determines $\lambda_{1000}^{\text{const}}$ via (4.3) and the variance of $\hat{b}$ via (4.20). Nevertheless, assuming that they are approximately uncorrelated, meaning that $\mathbb{E}[(\hat{b} - b)(\hat{\lambda} - \lambda)] \approx 0$, evaluating the partial derivatives of the knock-out probability P_{CAT} at the estimated point $(\hat{b}, \hat{\lambda})$ instead of (b, λ), and using the estimated standard deviations from (4.21) and (4.4) instead of $(\operatorname{Var}(\hat{b}))^{1/2}$ and $(\operatorname{Var}(\hat{\lambda}))^{1/2}$, we obtain the approximation

$$\hat{\sigma}(P_{\text{CAT}}(\hat{b}, \hat{\lambda})) \approx \sqrt{\Bigl(\frac{\partial P_{\text{CAT}}}{\partial b}(\hat{b}, \hat{\lambda})\Bigr)^2 \hat{\sigma}^2(\hat{\lambda}) + \Bigl(\frac{\partial P_{\text{CAT}}}{\partial \lambda}(\hat{b}, \hat{\lambda})\Bigr)^2 \hat{\sigma}^2(\hat{b})} \approx 0.086\,. \tag{4.26}$$

From (4.25) and (4.26) we obtain $P_{\text{CAT}}(\hat{b}, \hat{\lambda}) + \hat{\sigma}(P_{\text{CAT}}(\hat{b}, \hat{\lambda})) \approx 0.221$ as a conservative estimate of the knock-out probability. A recalculation of Table 3.2 leads to a discounted value of CHF 238.25 for the three WINCAT coupons. Due to these calculations, in [2] the rounded knock-out probability of 0.25 is considered to be a conservative estimate, leading to a discounted value of CHF 229.78.† This value is

†In [2] a discounted value of CHF 227.09 is actually derived, because the 15/17-correction for the first observation period is not taken into account.

supposed to include a risk premium for the investor because the standard deviation of the knock-out probability is added and the result rounded in a conservative way.

Before turning our attention to a generalized Pareto distribution for the knock-out events, let us conclude this subsection with some supplementary considerations concerning the biasedness of the estimators for p_{6000} and P_{CAT}. First note that $\lambda_{1000}^{\mathrm{const}}$ from (4.2) and $\hat{b}$ from (4.18) are unbiased estimators for the two model parameters λ and b, but this does not imply that $\hat{p}_{6000}$ and P_{CAT}, given by (4.22) and (4.25), respectively, are unbiased. The arguments leading to the unbiased estimator (4.11) in the case of the Bernoulli distribution for the knock-out probability in Subsection 4.1 suggest that the estimator

$$P_{\mathrm{CAT}} = 1 - \Big(1 - \frac{\hat{p}_{6000}}{n}\Big)^{N_{1000,n}} = 1 - \Big(1 - \frac{1}{6^{\hat{b}} n}\Big)^{N_{1000,n}} \tag{4.27}$$

is a small improvement, because this would be an unbiased estimator for P_{CAT} if $\hat{b}$ were non-random. Here the random variable $N_{1000,n}$ denotes the number of events with more than 1 000 adjusted claims within the n observation periods. Recall that $N_{1000,n}$ has a Poisson distribution with parameter $n\lambda$. Substituting our estimate $\hat{b} \approx 1.37$ from (4.19) and $N_{1000,n} = 17$ for the $n = 10$ observation periods into (4.27) leads to $P_{\mathrm{CAT}} \approx 0.136$, which gives a discounted value of Chf 263.13 for the three Wincat coupons. This is a decrease of only Chf 0.16 compared to the value arising from (4.25).

If we consider $N_{1000,10} = 17$ as non-random and replace $\hat{b} \approx 1.37$ from (4.19) by $\hat{b} - \hat{\sigma}(\hat{b}) \approx 1.02$ in the estimator (4.27) to find a conservative estimate, we get $P_{\mathrm{CAT}} \approx 0.242$, which via Table 3.2 leads to Chf 232.14 for the discounted value of the three Wincat coupons. Note that this knock-out probability is about 0.02 larger than the one obtained from (4.26) and is already very close to the conservatively rounded value of 0.25 from [2].

An examination of the above model reveals that the conditional distribution of the estimator $\hat{b}$ given $N_{1000,n}$ is only specified in the case $N_{1000,n} \geq 2$. Furthermore, (4.20) shows that the estimator $\hat{b}$ does not have a variance unless $N_{1000,n} \geq 3$. Therefore, the above approach of fitting the empirical distribution of the adjusted claim numbers by a Pareto distribution is applicable only in the case of appropriate data sets. Such an a priori exclusion of certain data sets already introduces a bias which suggests that unbiasedness for estimators like (4.25) or (4.27) is a problematical notion. Maybe a notion of conditional unbiasedness would be more appropriate. This means in our case that one would like to have estimators for P_{CAT} such that the conditional expectation given $\hat{p}_{6000}\, N_{1000,n} \geq 1$, for example, is the right one.

4.3. **Generalized Pareto distribution for the knock-out events.** In Subsection 4.2, we did not give a theoretical argument in favour of the Pareto distribution in addition to the desire to pick a heavy-tailed distribution. Let us use an idea from extreme value theory to overcome this deficiency. It will turn out that we should use a generalized version of the Pareto distribution defined in (4.14).

Let $X_1, \ldots, X_k$ denote the adjusted number of claims arising from k events. We shall assume that $X_1, \ldots, X_k$ are independent and distributed according to a heavy-tailed distribution function. We are only interested in those numbers which exceed a certain threshold a, which is 1 000 in our case. This means we are interested in the excess distribution function

$$F_a(x) = \mathbb{P}(X_1 - a \le x \,|\, X > a), \qquad x \in \mathbb{R}.$$

Extreme value theory essentially says the following in our case [3, Section 3.4]: If the original distribution function of $X_1, \ldots, X_k$ is heavy-tailed, then the excess distribution functions $\{F_a\}_{a>0}$ can be better and better approximated (with respect to the supremum norm) by generalized Pareto distributions of the form

$$G_{b,\tau_a}(x) = \begin{cases} 1 - (1 + \tau_a x)^{-b} & \text{for } x \ge 0, \\ 0 & \text{for } x < 0, \end{cases}$$

as the threshold a tends to infinity. Here b is a strictly positive parameter and $\tau_a > 0$ varies with the threshold a. This suggests that we should try to fit the empirical distribution function of the observations exceeding the threshold a by a distribution function of the form

$$G_{a,b,\tau}(x) = \begin{cases} 1 - (1 + \tau(x-a))^{-b} & \text{for } x \ge a, \\ 0 & \text{for } x < a. \end{cases} \tag{4.28}$$

We have chosen this parametrisation of the shifted generalized Pareto distributions, because we are only interested in the heavy-tailed case and because (4.28) with $\tau = 1/a$ reduces to the Pareto distribution (4.14). Compared to (4.14), the generalized Pareto distribution gives us the freedom of the additional scale parameter τ.

Since there are only $m = 17$ observations available in Table 1.1, there is no point in choosing a higher threshold than $a = 1000$. The corresponding log-likelihood function for the $m = 17$ observations is

$$l(b,\tau) = m \log b\tau - (b+1) \sum_{k=1}^{m} \log(1 + \tau(X_k - 1000)), \qquad b > 0,\ \tau > 0.$$

Inserting the data from the last column of Table 1.1 and calculating the maximum likelihood estimator $(\hat{b}, \hat{\tau})$ numerically, i.e., searching for the point $(\hat{b}, \hat{\tau})$ which maximizes the log-likelihood function l, we find that

$$\hat{b} \approx 1.38 \qquad \text{and} \qquad \hat{\tau} \approx 0.0011,$$

hence

$$\hat{p}_{6000} = 1 - G_{1000,\hat{b},\hat{\tau}}(6000) \approx 1 - 0.9243 = 0.0757\,. \tag{4.29}$$

The corresponding fit of the empirical distribution with a generalized Pareto distribution is shown in Figures 4.4 and 4.5. A calculation as in (4.27) gives the estimate

$$P_{\text{CAT}} = 1 - \Bigl(1 - \frac{\hat{p}_{6000}}{10}\Bigr)^{N_{1000,10}} \approx 0.121,$$

which leads via a recalculation of Table 3.2 to a discounted value of CHF 267.48 for the three WINCAT coupons.

For comparison with the earlier results on the standard deviation of $\hat{p}_{6000}$ in the case of the Bernoulli distribution for the knock-out events in (4.5) and for the corresponding case of the Pareto distribution in (4.24), we would like to give again an estimate for the standard deviation of $\hat{p}_{6000}$. This does not seem to be possible by analytical means, however. Therefore, we prefer to construct an interval $[\hat{p}_{6000}^-, \hat{p}_{6000}^+]$ around the estimated value $\hat{p}_{6000} \approx 0.0757$ from (4.29), which can serve

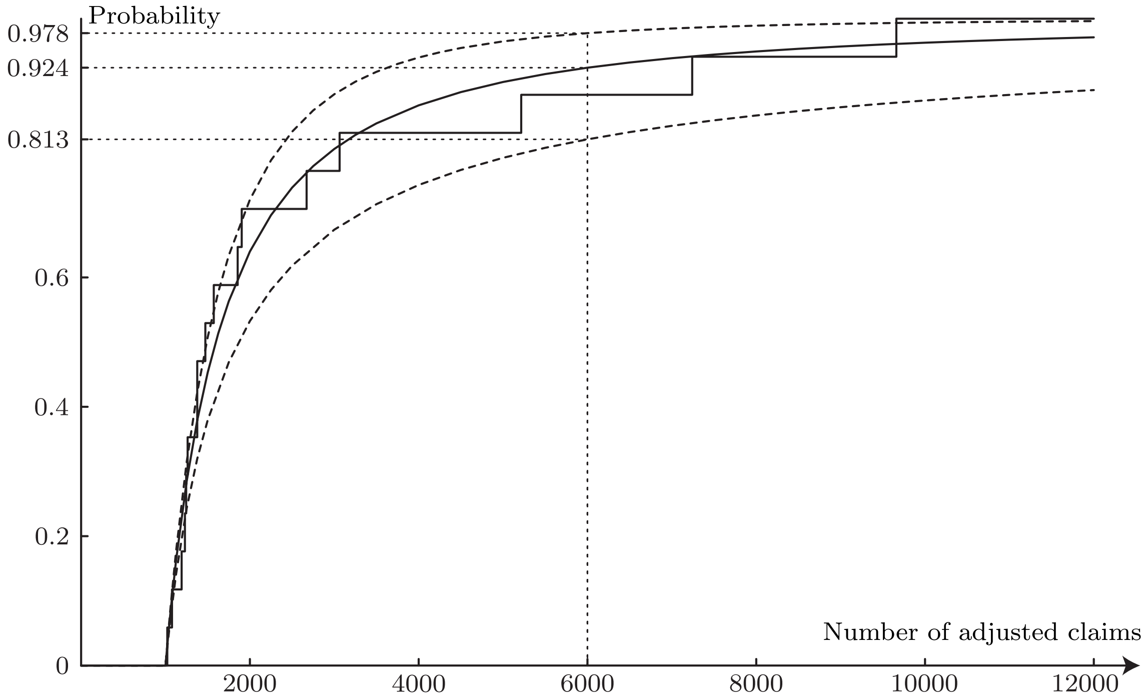

FIGURE 4.4. The empirical distribution of the number of adjusted claims per event (solid step function) and the fitted generalized Pareto distribution (solid curve) with threshold $a = 1000$, estimated exponent $\hat{b} \approx 1.38$ and estimated scale parameter $\hat{\tau} \approx 0.0011$. The estimated probability that an event with at least 1 000 claims causes at most 6 000 claims, is around 0.924, and $[0.813, 0.978]$ is an approximate 68 %-confidence interval for this probability. The two dashed curves are generalized Pareto distributions chosen such that they indicate the standard deviation of the estimated probability for at most 6 000 claims.

as the region for accepting the null hypothesis $p = \hat{p}_{6000}$ at a 68 %-confidence level when using the log-likelihood ratio statistic. We choose the 68 % level, because this is the probability that a normally distributed random variable with mean μ and variance $\sigma^2 > 0$ takes its value in the interval $[\mu - \sigma, \mu + \sigma]$. As log-likelihood ratio statistic, also called deviance, we use

$$D(b, \tau) = 2l(\hat{b}, \hat{\tau}) - 2l(b, \tau), \qquad b, \tau > 0. \tag{4.30}$$

We want to determine the smallest interval $[\hat{p}^-_{6000}, \hat{p}^+_{6000}]$ such that

$$\begin{aligned}\big\{ (b, \tau) \in (0, \infty)^2 \,\big|\, D(b, \tau) \le \chi^2_{2,0.32} \big\}& \\ \subset \big\{ (b, \tau) \in (0, \infty)^2 \,\big|\, 1 - G_{1000,b,\tau}(6000) &\in [\hat{p}^-_{6000}, \hat{p}^+_{6000}] \big\},\end{aligned} \tag{4.31}$$

where $\chi^2_{2,0.32} \approx 2.30$ denotes the 32 %-quantile of the chi-squared distribution with two degrees of freedom. In other words: We are looking for the smallest probability $\hat{p}^-_{6000}$ and the largest probability $\hat{p}^+_{6000}$, which can arise from generalized Pareto distributions with parameters (b, τ) close to $(\hat{b}, \hat{\tau})$ in the sense that the deviance $D(b, \tau)$ does not exceed the 32 %-quantile $\chi^2_{2,0.32}$ of the χ^2_2-distribution. This choice for the upper bound of the deviance $D(b, \tau)$ is based on the asymptotic normality of the maximum likelihood estimators, see for example [4, Section 8.8]. According

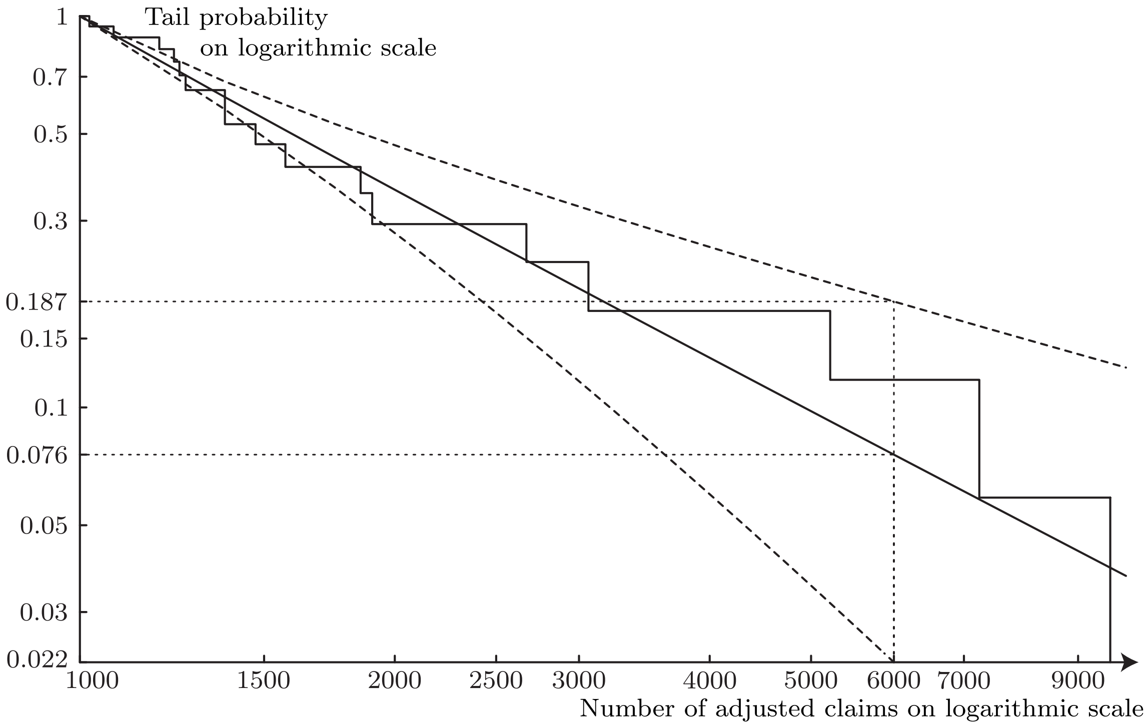

FIGURE 4.5. This is Figure 4.4 on log-log scale to magnify the important part. Instead of the distribution functions, the corresponding tail probabilities are shown. Pareto distibution functions defined by (4.14) would give strait lines in this log-log plot. The estimated generalized-Pareto fit $x \mapsto 1-G_{1000,\hat{b},\hat{\tau}}(x)$ is close to a strait line because $\hat{\tau} \approx 0.0011$ is close to $1/a = 0.001$. The estimates $\hat{p}_{6000}^{-} \approx 0.022$, $\hat{p}_{6000} \approx 0.0757$ and $\hat{p}_{6000}^{+} \approx 0.187$ are shown. This figure supports the model assumption, that the adjusted claim numbers follow a heavy-tailed distribution.

to [5, Appendix A], the approximation of the distribution of the deviance by the chi-squared distribution is often quite accurate for small numbers of observations, even when the normal approximation for the parameter estimates is unsatisfactory. When compared to methods using the second derivatives of the log-likelihood function at the estimated point $(\hat{b}, \hat{\tau})$, the log-likelihood ratio statistic has the advantage of being able to give asymmetric confidence intervals and thereby being less prejudiced. This is useful in our case, because we don't want to obtain negative estimates for $\hat{p}_{6000}^{-}$, for example.

Note that the interval $[\hat{p}_{6000}^{-}, \hat{p}_{6000}^{+}]$ in (4.31) does not depend on the parametrisation arising from $(b,\tau) \mapsto G_{1000,b,\tau}$ in (4.28). We can use this observation to change to an advantageous parametrisation which reduces the amount of numerical calculations necessary to determine the above acceptance interval. Since the equation $p = 1 - G_{1000,b,\tau}(6000)$ can be solved for τ yielding

$$\tau(b,p) = \frac{p^{-1/b} - 1}{5000},$$

we can use p itself as a parameter by changing the parametrisation from (4.28) to $(b,p) \mapsto G_{1000,b,\tau(b,p)}$. Rewriting the inclusion (4.31) with this parametrisation yields $\{\, (b,p) \in (0,\infty) \times (0,1) \mid D(b,\tau(b,p)) \le \chi^2_{2,0.32} \,\} \subset (0,\infty) \times [\hat{p}_{6000}^{-}, \hat{p}_{6000}^{+}]$.

Numerical calculations lead to

$$[\hat{p}^-_{6000}, \hat{p}^+_{6000}] \approx [0.022, 0.187],$$

the corresponding exponents $\hat{b}^- \approx 2.81$ and $\hat{b}^+ \approx 0.714$ are the only ones with a deviance less or equal to the quantile $\chi^2_{2,0.32}$. The shifted generalized-Pareto distribution functions

$$x \mapsto G_{1000,\hat{b}^-,\hat{\tau}^-}(x) \qquad \text{with} \qquad \hat{\tau}^- = \tau(\hat{b}^-, \hat{p}^-_{6000}) \approx 0.00057$$

and

$$x \mapsto G_{1000,\hat{b}^+,\hat{\tau}^+}(x) \qquad \text{with} \qquad \hat{\tau}^+ = \tau(\hat{b}^+, \hat{p}^+_{6000}) \approx 0.00190$$

are shown in Figures 4.4 and 4.5. If we consider the number $N_{1000,10} = 17$ as non-random and use $\hat{p}^+_{6000} \approx 0.187$ instead of p_{6000}, a calculation as in (4.26) leads to the conservative estimate $P_{\text{CAT}} \approx 0.274$. A recalculation of Table 3.2 gives a discounted value of CHF 222.75 for the three WINCAT coupons.

5. Composite Poisson models with a time-dependent parameter

The constant-parameter model of Section 4 is a static one. It gives equal weight to every recorded event and, by construction, does not allow to discover a trend in the data. Every redistribution of the 17 events in Table 1.1 to the ten observation periods would lead to the same result for the coupon values (if we disregard the $^{15}/_{17}$-correction in Table 3.2). An investor, however, might want to take a possible trend into account when estimating the discounted value of the WINCAT coupons. There are several reasons why there might be a trend, for example:

- The variability of the weather could change, due to human influence (increased CO_2-part in the atmosphere) or solar activity (11-year cycle of sun spots), for example.
- Winterthur might increase its market share in other regions like the French or Italian speaking parts of Switzerland; this can happen in particular when Winterthur merges with another insurance company (like merging with Neuenburger Schweizerische Allgemeine Versicherungsgesellschaft in 1997, for example). Due to the Swiss Alps, the local climate is in general quite different in different regions of Switzerland, so a change in Winterthur's engagement in a particular region can considerably increase or decrease the company's exposure to storm or hail damages.
- Severe damage caused by hail is a local event. If the density of motor vehicles insured with Winterthur increases (due to more cars per inhabitant, more inhabitants per area or a greater market share of Winterthur Insurance within an area), then more insured motor vehicles are likely to be damaged in every single event.
- The habits of the insured might change. They might buy a second or third car for the family without building or renting an additional garage to protect the car in case of bad weather. Or the insured are better off financially and they can afford the deductable, hence they take chances and don't drive the car to a secure place in case of a storm/hail forecast.

In any case—whatever the particular reason—it is a reasonable idea to consider a model which is flexible enough to take a possible trend in the data into account.

In the following subsections we shall use, for every year $y \in \{1987, \dots, 1996\}$, a random variable N_y describing the number of calendar days within the observation period ending in year y, during which more than 1 000 adjusted claims are caused by storm or hail. We assume that these random variables are independent and that every N_y has a Poisson distribution given by (4.1), but with a parameter $\lambda(y)$ depending on the year $y \in \{1987, \dots, 1996\}$. For the purpose of nicer graphics, we shall treat y as a continuous variable within the figures. We shall discuss four different choices for the dependence $y \mapsto \lambda(y)$.

5.1. **Linear trend of the parameter.** To start with the apparently simplest dependence, we assume that the Poisson parameter for the number of events with more than 1 000 adjusted claims depends linearly on the year y, namely

$$\lambda_{\alpha,\beta}(y) = \alpha + \beta(y - 1987), \tag{5.1}$$

where we subtract 1987 to get reasonable numbers for α. When using (5.1), we have to make sure that $\lambda_{\alpha,\beta}(y) \geq 0$ for all years under consideration. This will certainly be the case when $\alpha, \beta \geq 0$. The corresponding likelihood function arising from the ten observations $N_{1987}, \dots, N_{1996}$ is

$$L(\alpha, \beta) = \prod_{y=1987}^{1996} \mathrm{Poisson}_{\lambda_{\alpha,\beta}(y)}(N_y) \tag{5.2}$$

with the Poisson distribution given by (4.1) and the parameter $\lambda_{\alpha,\beta}(y)$ as in (5.1). When trying to calculate the maximum likelihood estimators for α and β numerically, it becomes clear that there is no solution of

$$\frac{\partial}{\partial \alpha} L(\alpha, \beta) = 0 \qquad \text{and} \qquad \frac{\partial}{\partial \beta} L(\alpha, \beta) = 0$$

satisfying $\alpha \geq 0$. As a pragmatic approach, let us set $\alpha = 0$. This means we consider the special case where the Poisson parameter $\lambda_\beta(y)$ depends on β in the form $\lambda_\beta(y) = \beta(y - 1987)$. Calculating the maximum likelihood estimator $\hat\beta$ maximizing $L(0, \beta)$, we get the numerical result $\hat\beta \approx 0.378$. The corresponding strait line is shown in Figure 5.1. Extrapolation to the three years following 1996 gives the estimated values for $\lambda_{\hat\beta}(y)$ contained in Figure 5.1 as well as Table 5.1. Using these extrapolated Poisson parameters $\lambda_{\hat\beta}(y)$ and the conditional probability $\hat p_{6000} \approx 0.0757$ from (4.29), which was estimated by a generalized Pareto distribution, the knock-out probabilities can be calculated as in (4.25) by the formula

$$P_{\mathrm{CAT}}(y) = 1 - \exp\bigl(-\hat p_{6000} \cdot \lambda_{\hat\beta}(y)\bigr)$$

for $y \in \{1997, 1998, 1999\}$. The results are given in the fourth column of Table 5.1. Applying the $^{15}/_{17}$-correction of (2.1) to $P_{\mathrm{CAT}}(1997)$ and inserting the resulting coupon-dependent knock-out probabilities into Table 3.2, a recalculation of this table leads to the discounted values of the three WINCAT coupons. These values are given in the last column of Table 5.1. The sum of these discounted values of the three WINCAT coupons is CHF 223.88.

This model with a linear trend in the Poisson parameter $y \mapsto \lambda_\beta(y)$ has a severe problem with the year 1987, because the estimate $\lambda_{\hat\beta}(1987) = 0$ is certainly wrong. The estimated standard deviations for the years 1987–1989, as shown in Figure 5.1,

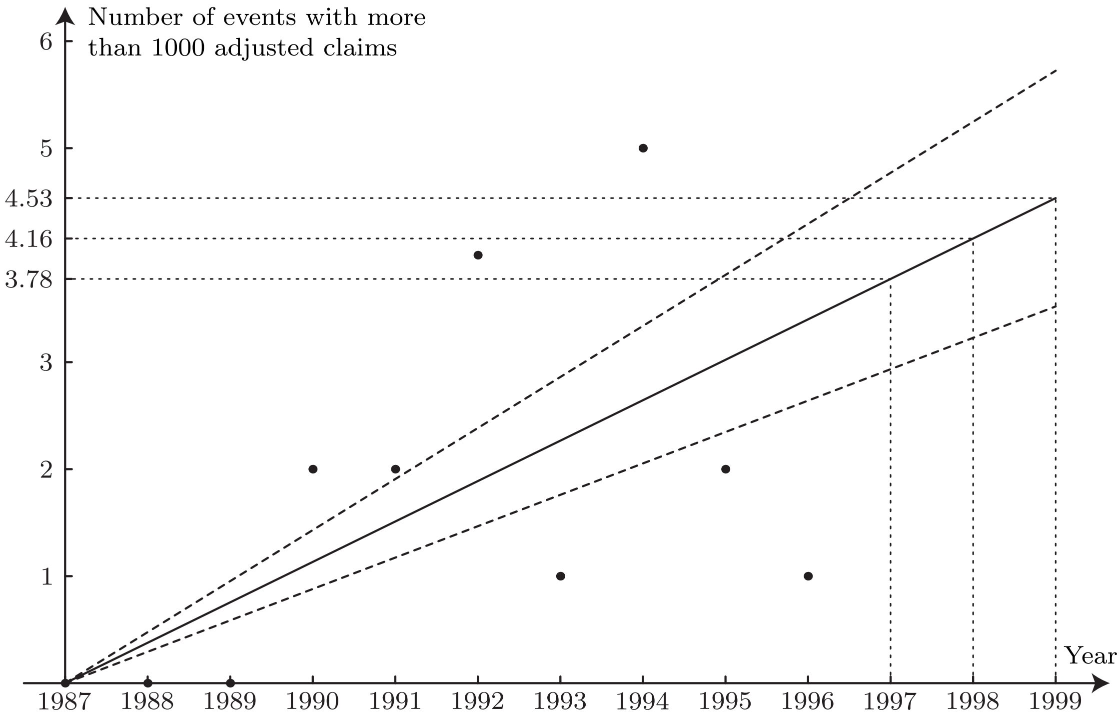

FIGURE 5.1. Observed number of events causing more than 1 000 adjusted claims. A linear fit of the intensity $\lambda_\beta(y) = \beta(y - 1987)$, using the maximum likelihood method, leads to $\hat\beta \approx 0.378$. The dashed lines indicate this estimated standard deviation of $\lambda_{\hat\beta}(y)$. This model has a problem with the years up to 1987 and it certainly underestimates the standard deviation in the first years.

are quite unrealistic, too. Model predictions for the years before 1987 are impossible, because negative values for $\lambda_{\hat\beta}(y)$ are unacceptable. Due to model deficiencies, we refrain from calculating a conservative estimate for the discounted value of the three WINCAT coupons. In the following subsections we shall discuss models which do not have these deficiencies.

Year y	$\lambda_{\hat\beta}(y)$	$\hat p_{6000}$	$P_{\text{CAT}}(y)$	Coupon value
1997	3.78	0.0757	24.9%	CHF 80.64
1998	4.16	0.0757	27%	CHF 73.72
1999	4.53	0.0757	29%	CHF 69.52
Discounted value of the three WINCAT coupons:				CHF 223.88

TABLE 5.1. Calculation of the discounted value of the three WINCAT coupons in the case of a linear dependence $\lambda_\beta(y) = \beta(y - 1987)$ of the Poisson parameter. The Poisson parameters $\lambda_{\hat\beta}(y)$ are the extrapolated values from Figure 5.1. The conditional probability $\hat p_{6000}$ for a knock-out event, given that an event occurs, is taken from (4.29). The forth column contains $P_{\text{CAT}}(y) = 1-\exp(-\hat p_{6000}\,\lambda_{\hat\beta}(y))$. The discounted coupon values are then calculated according to Table 3.2 taking into account the $^{15}/_{17}$-correction from (2.1) for the shorter first observation period.

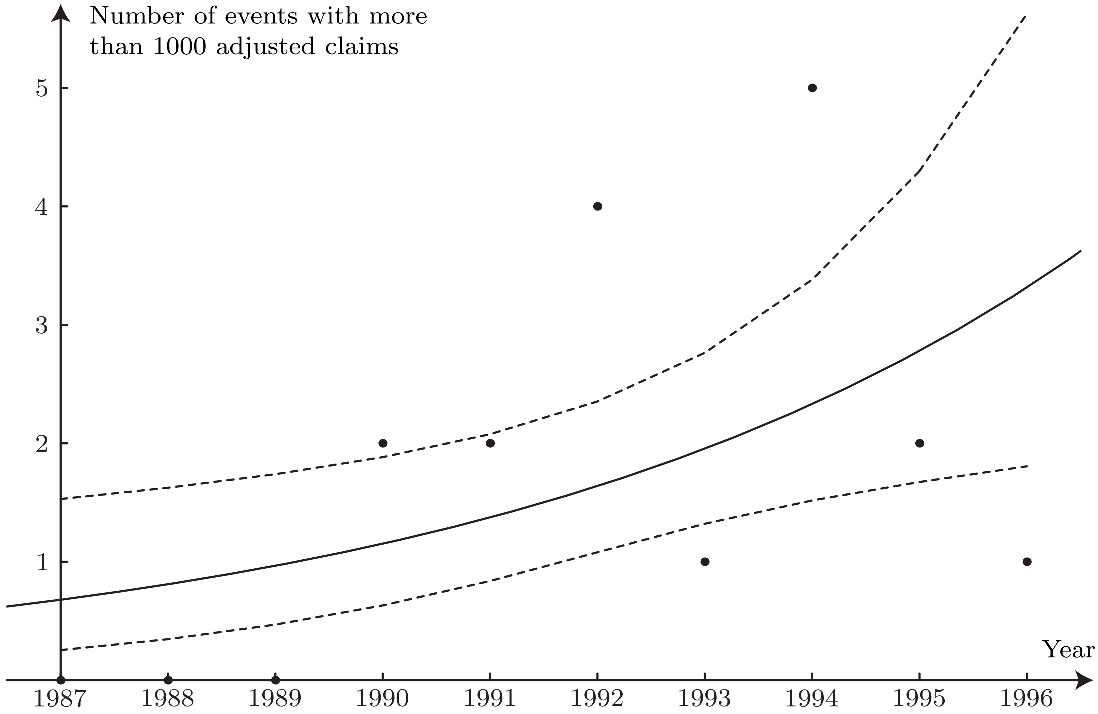

FIGURE 5.2. Log-linear dependence $\lambda_{\alpha,\beta}(y) = \exp(\alpha + \beta(y - 1992))$ of the Poisson parameter. The maximum likelihood method leads to $\hat{\alpha} \approx 0.494$ and $\hat{\beta} \approx 0.176$, the result is shown as a solid curve. The dashed piecewise linear curves indicate for every year y the estimated standard deviation of $\lambda_{\hat{\alpha},\hat{\beta}}(y)$ as derived from the log-likelihood ratio statistic.

5.2. **Log-linear trend of the parameter.** To avoid the problem of negative Poisson parameters, let us consider the prime example of a model where this cannot occur, namely a generalized linear model with a log-linear dependence of the Poisson parameter, meaning that

$$\lambda_{\alpha,\beta}(y) = \exp(\alpha + \beta(y - 1992)), \qquad \alpha, \beta, y \in \mathbb{R}. \tag{5.3}$$

We subtract 1992 from y in order to get approximately orthogonal parameters, meaning that the corresponding maximum likelihood estimators for α and β have only a small correlation (for the notion of orthogonal parameters, see for example [1, p. 182–185]). The corresponding likelihood function for this model is given by (5.2) with $\lambda_{\alpha,\beta}(y)$ as in (5.3). The maximum likelihood estimators for α and β, calculated numerically, are

$$\hat{\alpha} \approx 0.494 \qquad \text{and} \qquad \hat{\beta} \approx 0.176\,.$$

The corresponding curve $y \mapsto \lambda_{\hat{\alpha},\hat{\beta}}(y)$ is shown in Figure 5.2. It approximates quite well within the time span 1987–1996. The extrapolated values of the Poisson parameter $\lambda_{\hat{\alpha},\hat{\beta}}(y)$ for the years 1997–1999 can be read off from Figure 5.3, they are listed in Table 5.2. The calculations of Table 5.2 lead to a discounted value of CHF 214.37 for the three WINCAT coupons.

To obtain an estimate for the standard deviation of the estimated Poisson parameter $\lambda_{\hat{\alpha},\hat{\beta}}(y)$ for every year $y \in \{1987, \ldots, 1999\}$, we use the log-likelihood ratio

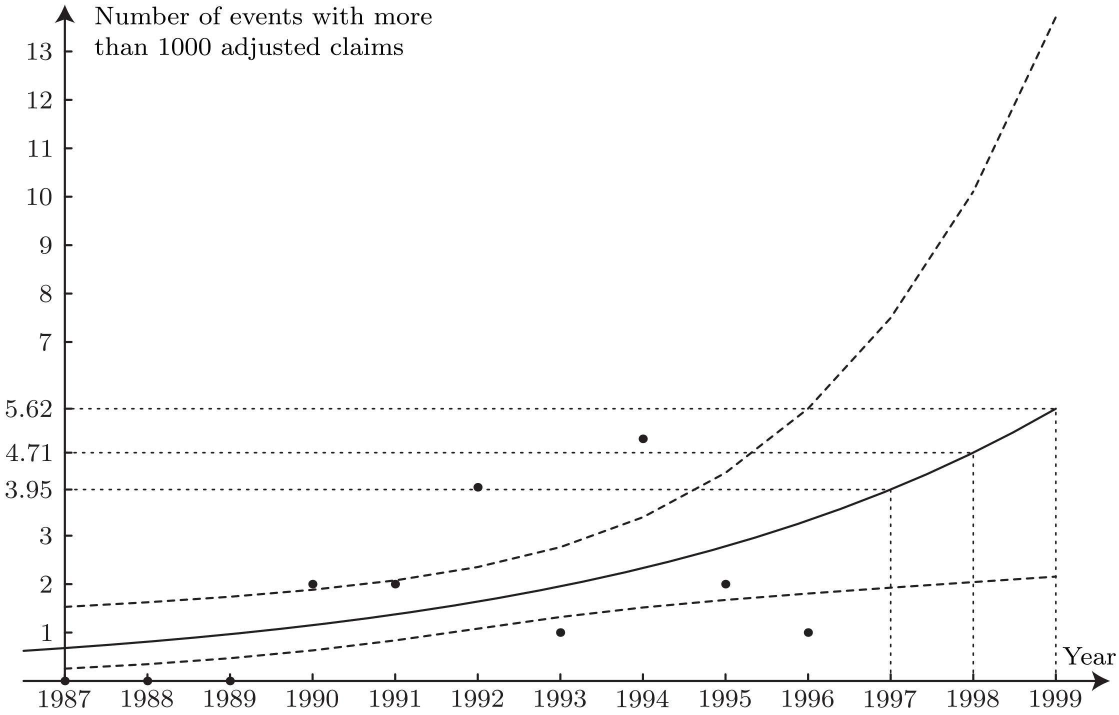

FIGURE 5.3. Extrapolation arising from the log-linear dependence of the Poisson parameter $\lambda_{\hat{\alpha},\hat{\beta}}(y)$ with the maximum likelihood estimates $\hat{\alpha} \approx 0.494$ and $\hat{\beta} \approx 0.176$. The uncertainty of the extrapolated values, indicated by the dashed piecewise linear curves, is very large for the years 1997–1999.

statistic, which we already applied in Subsection 4.3. The log-likelihood function arising from (5.2) with $\lambda_{\alpha,\beta}(y)$ as in (5.3) is given by

$$l(\alpha,\beta) = \sum_{y=1987}^{1996} \bigl(N_y(\alpha+\beta(y-1992)) - e^{\alpha+\beta(y-1992)} - \log N_y!\bigr).$$

Year y	$\lambda_{\hat{\alpha},\hat{\beta}}(y)$	$\hat{p}_{6000}$	$P_{\text{CAT}}(y)$	Coupon value
1997	3.95	0.0757	25.9%	CHF 79.70
1998	4.71	0.0757	30%	CHF 70.66
1999	5.62	0.0757	34.7%	CHF 64.01
Discounted value of the three WINCAT coupons:				CHF 214.37

TABLE 5.2. Calculation of the discounted value of the three WINCAT coupons in the case of a log-linear dependence of the Poisson parameter, see (5.3). The Poisson parameters $\lambda_{\hat{\alpha},\hat{\beta}}(y)$ are the extrapolated values from Figure 5.3. The conditional probability $\hat{p}_{6000}$ for an knock-out event, given an event occurs, is taken from (4.29). The knock-out probabilities are calculated using $P_{\text{CAT}}(y) = 1 - \exp(-\hat{p}_{6000}\,\lambda_{\alpha,\hat{\beta}})$. The coupon values are then obtained by a recalculation of Table 3.2, taking the $^{15}\!/_{17}$-correction for the shorter first observation period into account.

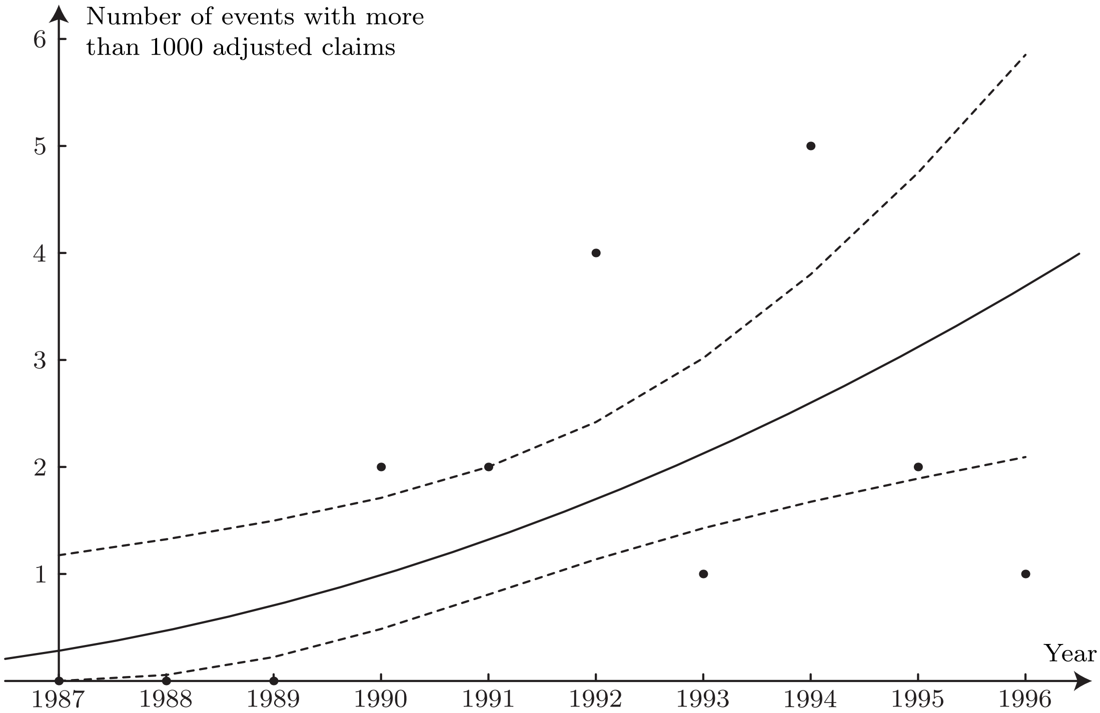

FIGURE 5.4. Root-linear dependence $\lambda_{\alpha,\beta}(y) = (\alpha + \beta(y - 1992))^2$ of the Poisson parameter. The maximum likelihood method leads to $\hat{\alpha} \approx 1.30$ and $\hat{\beta} \approx 0.154$. These estimates give the solid curve. The 68%-confidence bounds are indicated by the dashed piecewise linear curves.

Similarly to (4.30), we define the log-likelihood ratio statistic or deviance by

$$D(\alpha, \beta) = 2l(\hat{\alpha}, \hat{\beta}) - 2l(\alpha, \beta), \qquad \alpha, \beta \in \mathbb{R}.$$

Corresponding to the inclusion (4.31), for every year $\tilde{y} \in \{1987, \ldots, 1999\}$, we want to determine the smallest 68%-confidence interval $[\lambda_{\tilde{y}}^-, \lambda_{\tilde{y}}^+] \subset (0, \infty)$ such that

$$\left\{ (\alpha, \beta) \in \mathbb{R}^2 \,\middle|\, D(\alpha, \beta) \le \chi^2_{2,0.32} \right\} \subset \left\{ (\alpha, \beta) \in \mathbb{R}^2 \,\middle|\, \lambda_{\alpha,\beta}(\tilde{y}) \in [\lambda_{\tilde{y}}^-, \lambda_{\tilde{y}}^+] \right\}. \tag{5.4}$$

Solving $\lambda = \lambda_{\alpha,\beta}(\tilde{y}) = \exp(\alpha + \beta(\tilde{y} - 1992))$ for α yields

$$\alpha_{\tilde{y}}(\beta, \lambda) = \log \lambda - \beta(\tilde{y} - 1992).$$

With this reparametrisation, (5.4) reduces to

$$\left\{ (\beta, \lambda) \in \mathbb{R} \times (0, \infty) \,\middle|\, D(\alpha_{\tilde{y}}(\beta, \lambda), \beta) \le \chi^2_{2,0.32} \right\} \subset \mathbb{R} \times [\lambda_{\tilde{y}}^-, \lambda_{\tilde{y}}^+]$$

The numerical results are shown in Figures 5.2 and 5.3 as dashed piecewise linear curves above and below the solid curves.

The results in Figure 5.2 for the years 1987 up to 1996 look quite satisfactory, although the 68%-confidence intervals are larger than the estimated standard deviation $\hat{\sigma}(\lambda_{1000}^{\text{const}})$ in the constant-parameter model, see Figure 4.1. In Figure 5.3, the upper 68%-confidence bounds λ_{1997}^+, λ_{1998}^+ and λ_{1999}^+ for the future observation periods show a large uncertainty of the estimates. Of course, the small size of the historic data set is partially responsible for this uncertainty. The main contribution, however, comes from the log-linear model itself, because it blows up the unavoidable uncertainty of the maximum likelihood estimators $\hat{\alpha}$ and $\hat{\beta}$ in an exponential

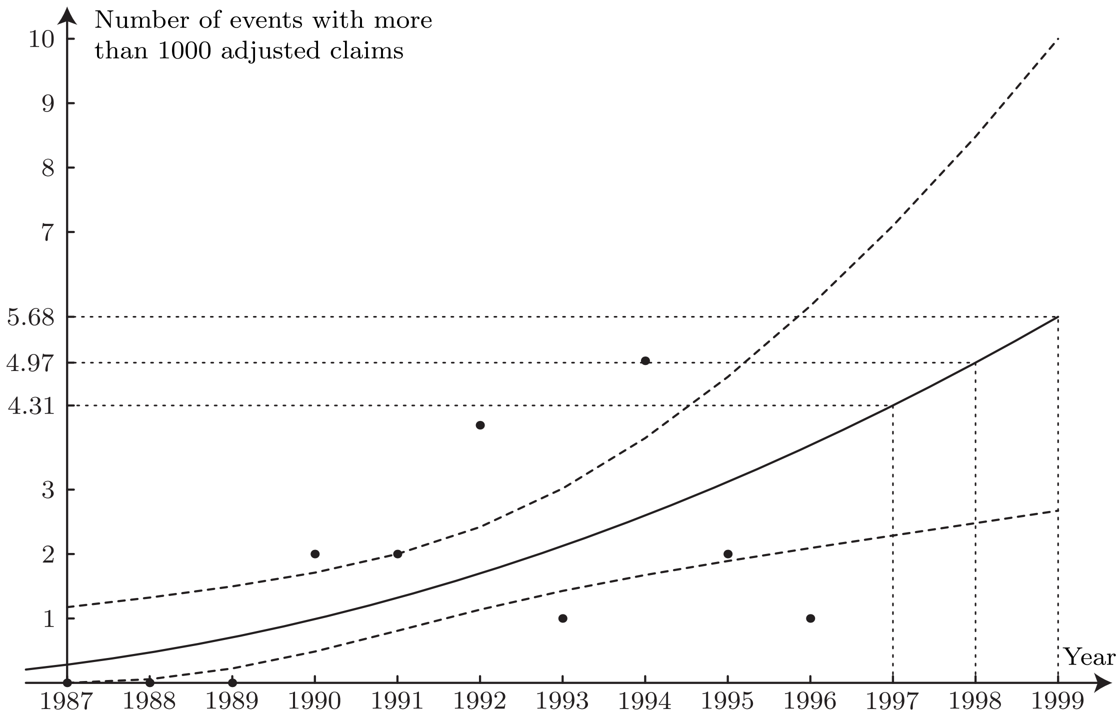

FIGURE 5.5. Extrapolated values of the Poisson parameter in the case of a square-root linear dependence $\lambda_{\alpha,\beta}(y) = (\alpha + \beta(y - 1992))^2$.

way. This log-linear model is already a very pessimistic one with respect to the future development of the event frequency. Due to the exponential amplification of the estimator uncertainty, the log-linear model is certainly not the favourite one for calculating a conservative estimate for the value of the WINCAT coupons.

5.3. **Square-root linear trend of the parameter.** To avoid the possibly negative Poisson parameters of the linear model from Subsection 5.1 and the very pessimistic perspective of the future event frequency in the log-linear model from Subsection 5.2, we want to consider the usual root-linear model, which means that we consider the dependence

$$\lambda_{\alpha,\beta}(y) = (\alpha + \beta(y - 1992))^2, \qquad \alpha, \beta, y \in \mathbb{R},$$

Year y	$\lambda_{\hat\alpha,\hat\beta}(y)$	$\hat p_{6000}$	$P_{\text{CAT}}(y)$	Coupon value
1997	4.31	0.0757	27.8%	CHF 77.84
1998	4.97	0.0757	31.4%	CHF 69.30
1999	5.68	0.0757	35.0%	CHF 63.72
Discounted value of the three WINCAT coupons:				CHF 210.86

TABLE 5.3. Calculation of the discounted value of the three WINCAT coupons (similar to Table 5.2) in the case of a square-root linear dependence $\lambda_{\alpha,\beta}(y) = (\alpha + \beta(y - 1992))^2$ of the Poisson parameter. The maximum likelihood estimates $\hat\alpha \approx 1.30$ and $\hat\beta \approx 0.154$ are used. The extrapolated values $\lambda_{\hat\alpha,\hat\beta}(y)$ are taken from Figure 5.5.

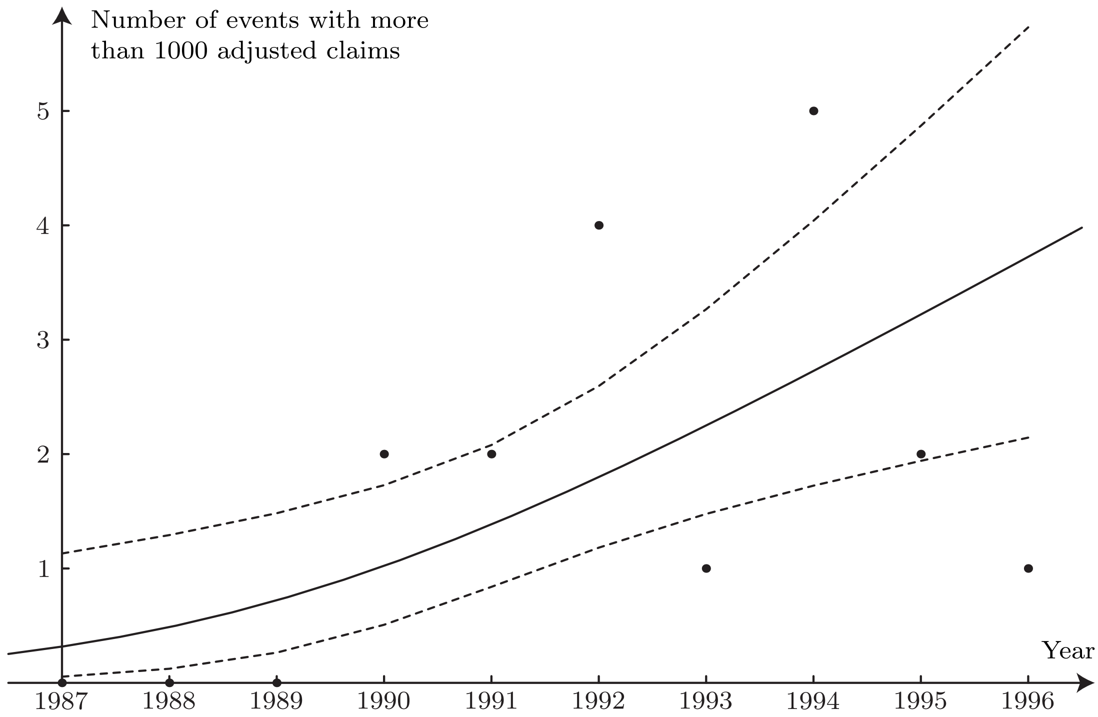

FIGURE 5.6. Observed number of events with more than 1 000 adjusted claims together with a modified-linear fit of the Poisson parameter for such events using $\lambda_{\alpha,\beta}(y) = \log(1 + \exp(\alpha + \beta(y - 1992)))$. The maximum likelihood method leads to $\hat{\alpha} \approx 1.62$ and $\hat{\beta} \approx 0.521$. The dashed piecewise linear curves indicate the estimated 68%-confidence interval for $\lambda_{\hat{\alpha},\hat{\beta}}(y)$ as derived from the log-likelihood ratio statistic.

instead of the log-linear dependence from (5.3). Along the lines of the previous subsection, we obtain Figures 5.4 and 5.5 as well as Table 5.3, which gives a discounted value of CHF 210.86 for the three WINCAT coupons.

In this square-root linear model, the estimated Poisson parameter drops to zero between 1983 and 1984 and increases in the more distant past. This model deficiency, however, is not of great importance for the extrapolation into the future. A more important problem is, as in the log-linear model of Subsection 5.2, the built-in slightly pessimistic perspective of a future quadratic growth of the event frequency. In the last subsection we shall present a model which is better tailored for the extrapolation of the estimated Poisson parameter in our case.

5.4. **Modified-linear trend of the parameter.** To avoid the problems of negative Poisson parameters and too pessimistic perspectives of the future event frequency, we want to consider the dependence

$$\lambda_{\alpha,\beta}(y) = \log\bigl(1 + \exp(\alpha + \beta(y - 1992))\bigr), \qquad \alpha, \beta, y \in \mathbb{R}. \tag{5.5}$$

We shall call it *modified linear,* because $y \mapsto \lambda_{\alpha,\beta}(y)$ is approximately linear for $\alpha + \beta(y - 1992) \gg 0$ and the graph bends smoothly for $\alpha + \beta(y - 1992) \approx 0$ to avoid negative values. The number one in (5.5) arises from this restriction; another value, let's say γ, would lead to the asymptotic value $\log \gamma$ when $\alpha + \beta(y - 1992) \ll 0$. Maximum likelihood estimators and 68%-confidence intervals

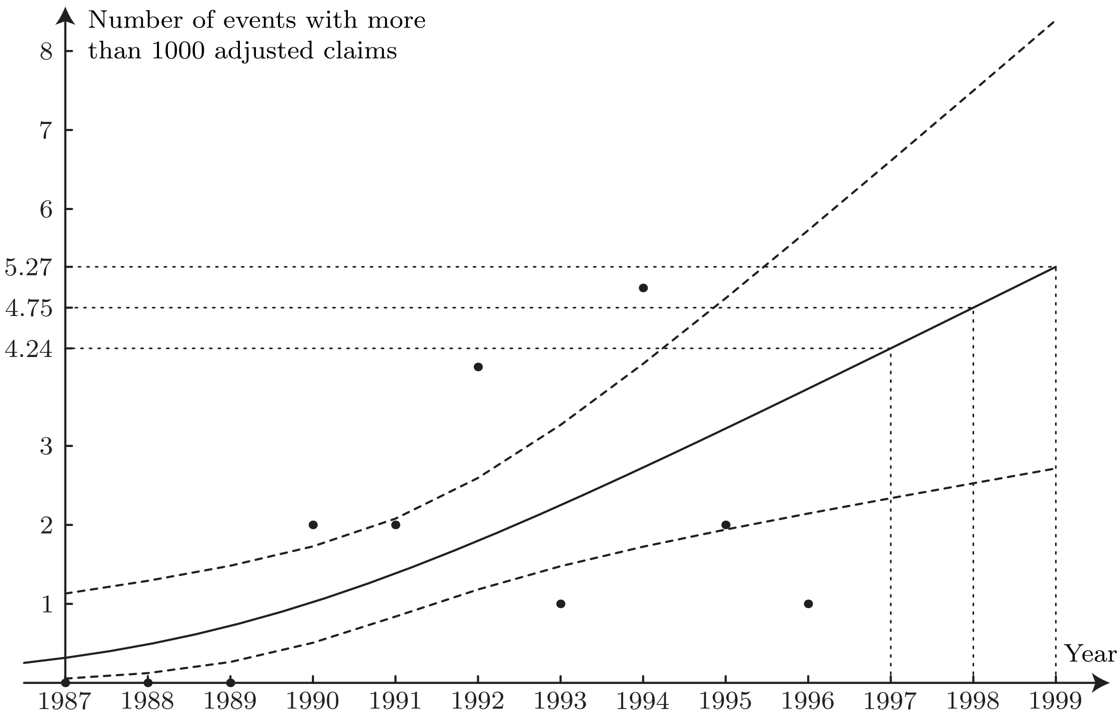

FIGURE 5.7. The estimated extrapolated values of the Poisson parameter using a modified-linear fit of the Poisson parameter. The uncertainty of the estimates grows only linearly.

are calculated with the procedures outlined in Subsections 5.1 and 5.2. The results are shown in Figure 5.6, the maximum likelihood estimators are $\hat{\alpha} \approx 1.61$ and $\hat{\beta} \approx 0.521$. Figure 5.7 shows that the estimator uncertainty is amplified only in a linear way. According to Table 5.4, the discounted value of the three WINCAT coupons is CHF 214.44.

Note that the above value is only CHF 0.07 above the discounted value obtained with the log-linear model in Table 5.2. Furthermore, the value is CHF 3.58 above the discounted value obtained with the square-root linear model, hence all three models give approximately the same result as long as uncertainty of the prediction is not an issue. The discounted value of CHF 223.88 from Table 5.1, obtained in

Year y	$\lambda_{\hat{\alpha},\hat{\beta}}(y)$	$\hat{p}_{6000}$	$P_{\text{CAT}}(y)$	Coupon value
1997	4.24	0.0757	27.4%	CHF 78.21
1998	4.75	0.0757	30.2%	CHF 70.47
1999	5.27	0.0757	32.9%	CHF 65.76
Discounted value of the three WINCAT coupons:				CHF 214.44

TABLE 5.4. Calculation of the discounted value of the three WINCAT coupons (similar to Table 5.2) in the case of a modified-linear dependence $\lambda_{\alpha,\beta}(y) = \log(1+\exp(\alpha+\beta(y-1992)))$ of the Poisson parameter. The estimated values $\hat{\alpha} \approx 1.62$ and $\hat{\beta} \approx 0.521$ are used. The estimated values $\lambda_{\hat{\alpha},\hat{\beta}}(y)$ are taken from Figure 5.7.

Coupon value	Corresponding binomial or composite Poisson model
CHF 244.44	Binomial model of Section 3
	Constant-parameter Poisson model of Section 4 and
CHF 267.48	— *generalized Pareto distribution of Subsection 4.3*
CHF 263.29	— *Pareto distribution of Subsection 4.2*
CHF 247.37	— *Bernoulli distribution of Subsection 4.1*
	Generalized Pareto distribution (4.28) and time-dependent Poisson parameter with the
CHF 223.88	— *linear trend of Subsection 5.1*
CHF 214.44	— *modified-linear trend of Subsection 5.4*
CHF 214.37	— *log-linear trend of Subsection 5.2*
CHF 210.86	— *square-root linear trend of Subsection 5.3*
	Pareto distribution (4.14) and a time-dependent Poisson parameter with the
CHF 215.19	— *linear trend of Subsection 5.1*
CHF 204.96	— *modified-linear trend of Subsection 5.4*
CHF 204.93	— *log-linear trend of Subsection 5.2*
CHF 201.12	— *square-root linear trend of Subsection 5.3*
	Bernoulli distribution and a time-dependent Poisson parameter with the
CHF 189.56	— *linear trend of Subsection 5.1*
CHF 177.44	— *log-linear trend of Subsection 5.2*
CHF 177.36	— *modified-linear trend of Subsection 5.4*
CHF 172.87	— *square-root linear trend of Subsection 5.3*

TABLE 5.5. Comparison of the sum of the three discounted WINCAT coupon values arising from the binomial model and the various composite Poisson models. No risk premium is included. The last eight values are given for comparison only, they arise from additional combinations of the model assumptions discussed in this note.

the case of the linear trend (5.1), is remarkably higher, because the slope in the linear-dependence case is restricted from above by the condition $\lambda_{\alpha,\beta}(1987) \geq 0$ for the Poisson parameter. As expected, all the discounted values of this section are substantially below the value CHF 263.29, which was derived via (4.25) using a constant Poisson parameter and a Pareto fit for the adjusted number of claims per event. What is more important: All the discounted values derived from the models admitting a trend in the Poisson parameter (see Tables 5.1–5.4) are well below the value CHF 229.78 from [2] (see the discussion below (4.26)). This value of CHF 229.78, however, was supposed to be a conservative estimate including a risk premium for the investor. As the present note shows, it is questionable whether this value includes a risk premium at all. The model uncertainty due to the small historic data set is substantially larger than the uncertainty indicated by the estimated standard deviations of the parameters in the time-homogeneous models. Four remarks should be kept in mind when comparing the "conservative" value of CHF 229.78 with the results of this section.

- No explicit risk premium is included in the discounted values calculated in this section.
- Time homogeneity is possible with the models of this section by choosing $\beta = 0$. It is the historic data set that leads to the positive estimates for β.
- The extrapolated estimated Poisson parameters are in the region from 3.78 in Figure 5.1 up to 5.68 in Figure 5.5. In the 10-year historic data set, two observation periods with four and five events, respectively, are recorded, hence the extrapolated parameters are not unreasonable if one accepts the possibility of a trend.
- The "conservative" value of CHF 229.78 was calculated with the estimate $\hat{p}_{6000} \approx 0.086$ from (4.23) obtained by the Pareto fit of the adjusted claim numbers. The discounted values in this section are calculated with the lower estimate $\hat{p}_{6000} \approx 0.0757$ from (4.29) obtained by the generalized-Pareto fit of the adjusted claim numbers. Using the value $\hat{p}_{6000} \approx 0.086$ from the Pareto fit of Subsection 4.2 or the value $\hat{p}_{6000} = 2/17 \approx 0.118$ from the Bernoulli distribution of Subsection 4.1, would lead to even lower estimated values for the WINCAT coupons, see Table 5.5.

Remark. An updated Postscript version of this note will be available from my home page `http://www.math.ethz.ch/~schmock`.

Acknowledgements. I would like to thank Prof. Dr. F. Delbaen (Mathematical Finance, Department of Mathematics, ETH Zürich) and Dr. A. McNeil (Swiss Re Research Fellow at the ETH Zürich) for fruitful discussions.

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