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GENERALIZATION OF THE DYBVIG–INGERSOLL–ROSS THEOREM AND ASYMPTOTIC MINIMALITY

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ABSTRACT. The long-term limit of zero-coupon rates with respect to the maturity does not always exist. In this case we use the limit superior and prove corresponding versions of the Dybvig–Ingersoll–Ross theorem, which says that long-term spot and forward rates can never fall in an arbitrage-free model. Extensions of popular interest rate models needing this generalization are presented. In addition, we discuss several definitions of arbitrage, prove asymptotic minimality of the limit superior of the spot rates, and illustrate our results by several continuous-time short-rate models.

1. INTRODUCTION

To price long-term contracts, like life insurance policies, practitioners model zero-coupon bond prices with long-term maturities to find reasonable discount factors. Empirical investigations of these prices are difficult, since there are only zero-coupon bonds traded with maturity of up to 30 years, and for a life annuity, for example, discount factors for up to 100 years are needed, see e. g. Carriere (1999). To construct reasonable models, we need to know how the long-term zero-coupon rates behave.

Dybvig, Ingersoll and Ross (1996) showed that long-term zero-coupon rates can never fall in an arbitrage-free market. Therefore, if the rates in a model decrease, it is not arbitrage-free. This fundamental theorem is part of textbooks, see e. g. Cairns (2004), and can be used to constrain the parameters of factor models to avoid arbitrage. Yao (1999) and El Karoui, Frachot and Geman (1998) discussed the long-term rates for several well-known models and used the theorem in this context.

In the literature there are two approaches to prove the Dybvig–Ingersoll–Ross theorem. The first approach constructs an arbitrage strategy, if long-term rates fall. Dybvig et al. provide an arbitrage strategy for a general infinite state space in the appendix of their paper. In the case of finitely many states they construct a second arbitrage strategy, which was made rigorous by McCulloch (2000). Recently, Schulze (2007) showed a further arbitrage strategy using another definition of arbitrage than Dybvig et al. The second approach to prove the Dybvig–Ingersoll–Ross theorem is to assume the existence of an equivalent martingale measure. Hubalek, Klein and Teichmann (2002) gave a general proof in this setting.

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In this paper we present a version of the Dybvig–Ingersoll–Ross theorem, which is more general, because Dybvig et al. as well as Hubalek et al. require the existence of the long-term limit of the zero-coupon rates. We show in two different ways that the limit superior of the zero-coupon rates and the forward rates never fall, which is called asymptotic monotonicity. For the first approach, we assume the existence of an equivalent martingale measure. This proof is inspired by the proof of Hubalek et al. For the second approach, we assume that there is no arbitrage opportunity in the limit with vanishing risk, and show again that asymptotic monotonicity holds.

Besides the main theorem, Dybvig et al. showed that the long-term zero-coupon rate equals its minimum future value, if the state space is finite. Using a stricter definition of no-arbitrage, Schulze extended this result to infinite state spaces. Again, the authors assume the existence of the long-term limit. Here we state conditions for asymptotic minimality of the limit superior of the zero-coupon rates. That means, the limit superior of the long-term limit of the zero-coupon rates is the largest random variable, which is known at this time and dominated by the future limit superior of the long-term limit.

The outline of the paper is the following. In Section 2 we give the general notation, state the main theorem about asymptotic monotonicity, and justify the use of the limit superior from the investor's point of view. Furthermore, we specify conditions for asymptotic minimality and define two notions of an *arbitrage opportunity in the limit*. In Section 3, we provide several interest rate models, where the long-term limit of the zero-coupon rates does not exist, to show that our generalization of asymptotic monotonicity is useful. Further examples illustrate the conditions for asymptotic minimality. In Section 4 we prove two auxiliary lemmas. Section 5 contains the proof for asymptotic monotonicity and minimality using an equivalent martingale measure. The proofs using arbitrage arguments are given in Section 6.

2. Statement of the generalized Dybvig–Ingersoll–Ross theorem and asymptotic minimality

2.1. **Notation.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ a filtration of $\mathcal{F}$ with a discrete-time parameter $t \in \mathbb{N}_0$ or a continuous-time parameter $t \in [0,\infty)$. For every maturity $T \in \mathbb{N}$ or $T \in (0,\infty)$, respectively, we assume that the corresponding zero-coupon bond price process $P(t,T)$ with $t \in \{0,1,\dots,T\}$ or $t \in [0,T]$, respectively, is strictly positive and $\mathbb{F}$-adapted with normalization $P(T,T) = 1$.

Define the zero-coupon rate for maturity $T > 0$ in the discrete-time case by

$$R(t,T) := P(t,T)^{-1/(T-t)} - 1, \qquad t \in \{0,1,\dots,T-1\}, \tag{2.1}$$

and in the continuous-time case by

$$R(t,T) := -\frac{\log P(t,T)}{T-t}\,, \qquad t \in [0,T). \tag{2.2}$$

The arbitrage-free forward rate $F(s,t,T)$ for a loan over the future time period $[t,T]$, contracted at time s, is in the discrete-time case defined by

$$F(s,t,T) := \left(\frac{P(s,t)}{P(s,T)}\right)^{1/(T-t)} - 1, \qquad s,t \in \{0,1,\dots,T-1\},\ s \leq t, \tag{2.3}$$

and in the continuous-time case by

$$F(s,t,T) := \frac{1}{T-t}\log\frac{P(s,t)}{P(s,T)}, \qquad s,t\in[0,T),\ s\le t. \tag{2.4}$$

For both time scales we define the long-term spot rate process by

$$l(t) := \limsup_{T\to\infty} R(t,T) = \lim_{n\to\infty} \operatorname*{ess\,sup}_{T>n\vee t} R(t,T), \qquad t\ge 0, \tag{2.5}$$

and the long-term forward rate process by

$$l_F(s,t) := \limsup_{T\to\infty} F(s,t,T) = \lim_{n\to\infty} \operatorname*{ess\,sup}_{T>n\vee t} F(s,t,T), \qquad 0\le s\le t, \tag{2.6}$$

Remark 2.7. For clarity we want to point out that for each $t\ge 0$ the limit superior $l(t)$ of the zero-coupon rates is the pointwise infimum of $\{R_n^*(t)\}_{n\in\mathbb{N}}$, where each $R_n^*(t)$ denotes the essential supremum of $\{R(t,T)\}_{T>n\vee t}$. The essential supremum is the smallest $\mathcal{F}_t$-measurable upper bound. That means, $R_n^*(t)$ is an $\mathcal{F}_t$-measurable random variable, $\mathbb{P}(R_n^*(t)\ge R(t,T))=1$ for all $T>n\vee t$, and every other random variable X dominating a. s. these zero-coupon rates satisfies $\mathbb{P}(X\ge R_n^*(t))=1$. In particular, the essential supremum is uniquely determined up to a set of $\mathbb{P}$-measure zero. The existence of the essential supremum for a collection of random variables is proved, for example, in [7, Appendix A.5]. Note that $\mathbb{P}(R_m^*(t)\ge R_n^*(t))=1$ for all $m\le n$, hence the infimum of $\{R_n^*(t)\}_{n\in\mathbb{N}}$ is $\mathbb{P}$-almost surely equal to the almost surely existing pointwise limit. The limit superior of the forward rates is to be understood in an analogue manner.

In comparison to Dybvig et al. and Hubalek et al., we do not assume that the long-term limits of the zero-coupon rates or the forward rates exist. In Subsection 3.1 we present (extensions of) popular interest rate models, which need this generalization.

From the investor's point of view, the limit superior of the zero-coupon rates is the natural definition, because he/she prefers for the long-term investment those zero-coupon bonds, which give a high investment return based on the information at time t. The following lemmas (proved in Section 4) show that the long-term spot rate $l(t)$ can indeed be approximated by investing in a zero-coupon bond with a suitable maturity, which is chosen based on the information available at time t. Furthermore, $l(t)$ agrees with the long-term forward rates, so it suffices to investigate the behaviour of the long-term spot rates.

Lemma 2.8. *Given $t\ge 0$, there exists a sequence of $\mathcal{F}_t$-measurable random maturities*[1] $T_n\colon \Omega\to(n\vee t,\infty)$, *each one taking only a finite number of values, such that*

$$l(t) \overset{a.s.}{=} \lim_{n\to\infty} R(t,T_n).$$

Lemma 2.9. *The long-term forward and spot rates are almost surely equal, meaning that $l_F(s,t)\overset{a.s.}{=} l(s)$ for all $0\le s\le t$.*

[1] In the discrete-time setting, the random maturities have to be integer-valued. This also applies to Remark 2.27, Definition 2.29, Theorem 2.34 and its corollaries. Since T_n attains only a finite number of values, $R(t,T_n)$ is $\mathcal{F}_t$-measurable.

2.2. **Results using the existence of a forward risk neutral probability measure.** Part of our main results, namely asymptotic monotonicity in Theorem 2.17 and asymptotic minimality in Theorem 2.21 are based on the following two conditions:

Condition 2.10. *We say that this condition holds for times s and t with $0 \le s < t$, if there exist a probability measure $\mathbb{Q}_{s,t}$ on $(\Omega, \mathcal{F}_t)$, which is equivalent to $\mathbb{P}|_{\mathcal{F}_t}$, and a $T_0 > t$ such that, for all $T \ge T_0$,*

$$P(s,T) \ge P(s,t)\,\mathbb{E}_{\mathbb{Q}_{s,t}}[P(t,T)\,|\,\mathcal{F}_s] \quad a.\,s. \tag{2.11}$$

This condition is sufficient for asymptotic monotonicity. For asymptotic minimality in Theorem 2.21 we need the stronger condition:

Condition 2.12 (Existence of forward (time s) risk neutral probability measure)**.** *We say that this condition holds for times s and t with $0 \le s < t$, if there exists a probability measure $\mathbb{Q}_{s,t}$ on $(\Omega, \mathcal{F}_t)$, which is equivalent to $\mathbb{P}|_{\mathcal{F}_t}$ such that, for all $T > t$,*

$$P(s,T) \overset{a.s.}{=} P(s,t)\,\mathbb{E}_{\mathbb{Q}_{s,t}}[P(t,T)\,|\,\mathcal{F}_s]\,. \tag{2.13}$$

We call $\mathbb{Q}_{s,t}$ the forward (time s) risk neutral probability measure for maturity t.

Condition 2.12 says that, simultaneously for all maturities $T > t$, the arbitrage-free forward price $P(s,T)/P(s,t)$, contracted at time s for the T-maturity zero-coupon bond at time t, can be expressed as the $\mathcal{F}_s$-conditional expectation of the price $P(t,T)$ at time t with respect to the measure $\mathbb{Q}_{s,t}$.

Remark 2.14. Suppose a money market account B_t with $t \in \mathbb{N}_0$ or $t \in [0,\infty)$ is given by a strictly positive and $\mathbb{F}$-adapted process. Then the following construction yields a model, where a forward risk neutral probability measure $\mathbb{Q}_{s,t}$ exists simultaneously for all times s and t with $0 \le s < t$. If $\mathbb{Q}$ is a probability measure such that B_0/B_T is $\mathbb{Q}$-integrable for every $T > 0$, then we can define zero-coupon bond prices by

$$P(t,T) = \mathbb{E}_{\mathbb{Q}}\Big[\frac{B_t}{B_T}\,\Big|\,\mathcal{F}_t\Big] = \frac{B_t}{B_0}\,\mathbb{E}_{\mathbb{Q}}\Big[\frac{B_0}{B_T}\,\Big|\,\mathcal{F}_t\Big], \qquad t \in [0,T], \tag{2.15}$$

and the forward (time s) risk neutral probability measure $\mathbb{Q}_{s,t}$ on $(\Omega, \mathcal{F}_t)$ by

$$\frac{d\mathbb{Q}_{s,t}}{d\mathbb{Q}} = \frac{B_s}{P(s,t)B_t}$$

for every $s \in [0,t)$. Since

$$\mathbb{E}_{\mathbb{Q}}\Big[\frac{B_s}{P(s,t)B_t}\,\Big|\,\mathcal{F}_s\Big] \overset{\text{a.s.}}{=} 1,$$

it follows by using Bayes' formula and the tower property, that

$$\mathbb{E}_{\mathbb{Q}_{s,t}}[P(t,T)\,|\,\mathcal{F}_s] \overset{\text{a.s.}}{=} \mathbb{E}_{\mathbb{Q}}\bigg[\frac{B_s}{P(s,t)B_t}\,\mathbb{E}_{\mathbb{Q}}\Big[\frac{B_t}{B_T}\,\Big|\,\mathcal{F}_t\Big]\,\bigg|\,\mathcal{F}_s\bigg] \overset{\text{a.s.}}{=} \frac{P(s,T)}{P(s,t)},$$

hence Condition 2.12 holds for all times s and t with $0 \le s < t$. We will use this construction for the examples in Section 3.

Example 2.16. In the discrete-time case, let $\{r_t\}_{t\in\mathbb{N}}$ be an interest rate process, which is $\mathbb{F}$-adapted and $(-1,\infty)$-valued. We define the money market account by

$$B_t = B_0 \prod_{i=1}^{t}(1+r_i), \qquad t \in \mathbb{N}_0,$$

where B_0 is strictly positive and $\mathcal{F}_0$-measurable. For a probability measure $\mathbb{Q}$ such that B_0/B_T is $\mathbb{Q}$-integrable for every $T \in \mathbb{N}$, we define the corresponding zero-coupon bond prices by (2.15), which means

$$P(t,T) = \mathbb{E}_{\mathbb{Q}}\biggl[\prod_{i=t+1}^{T}\frac{1}{1+r_i}\,\bigg|\,\mathcal{F}_t\biggr], \qquad t \in \{0,1,\dots,T\}.$$

By Remark 2.14, a forward risk neutral probability measure exists in this model simultaneously for all times $s,t \in \mathbb{N}_0$ with $s<t$.

The following result, which we prove in Section 5, states that the long-term spot and forward rates, given by (2.5) and (2.6), respectively, almost surely never fall. This is also called *asymptotic monotonicity.* Under the assumption, that the long-term limits of the spot and forward rates exist, this is the so-called Dybvig–Ingersoll–Ross theorem. Economically, from time s to a later time t, the available information increases, so a more informed decision concerning the best zero-coupon bonds for long-term investments can be made. However, to take advantage of this additional information, the gains during $[s,t]$ on zero-coupon bonds with a large maturity T should be negligible compared to the total gains until T, at least in the limit $T\to\infty$, see Example 2.18 for a counterexample. Therefore, in a reasonable economic environment as specified by Condition 2.10, the long-term spot rate at time t should be greater than the long-term spot rate at time s.

Theorem 2.17. *If Condition 2.10 holds for times s and t with $0 \le s < t$, then*

(a) $l(s) \le l(t)$ *a. s. and*
(b) $l_F(s,s') \le l_F(t,t')$ *a. s. for all $s' \ge s$ and $t' \ge t$.*

Examples 3.16, 3.20 and 3.22 show that the inequalities can be strict everywhere on Ω.

Example 2.18. Given $0 \le s < t$, the deterministic, continuous-time example with $P(s,T) = e^{-(T-s)}$ for all $T \ge s$ and $P(t,T) = 1$ for all $T \ge t$ shows, that $l(s) = 1 > l(t) = 0$ can happen, if there is arbitrage by investing in the zero-coupon bonds with maturity $T > t$. To exploit the arbitrage in this example, sell at time s one t-maturity bond and buy e^{T-t} zero-coupon bonds of maturity T with $T>t$.

Asymptotic monotonicity raises the question, whether $l(s)$ is the largest $\mathcal{F}_s$-measurable random variable, which is almost surely dominated by $l(t)$. To discuss this question, we need the following definition.

Definition 2.19. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$. For an $\overline{\mathbb{R}}$-valued random variable X, we define the upper $\mathcal{G}$-measurable envelope $X^{\mathcal{G}}$ as the essential infimum of all $\overline{\mathbb{R}}$-valued, $\mathcal{G}$-measurable random variables Z with $Z \ge X$ a. s. Similarly, we define the lower $\mathcal{G}$-measurable envelope $X_{\mathcal{G}}$ as the essential supremum of all $\overline{\mathbb{R}}$-valued, $\mathcal{G}$-measurable Z with $Z \le X$ a. s.

Observe that $X_{\mathcal{G}} \le X \le X^{\mathcal{G}}$ a. s., and asymptotic monotonicity implies $l(s) \le l(t)_{\mathcal{F}_s}$ a. s. Note that even in case of convergence of the zero-coupon rates $R(t,T)$

to $l(t)$ as $T \to \infty$, the existence of a forward risk neutral probability measure does not imply asymptotic minimality in the sense that $l(s) \overset{\text{a.s.}}{=} l(t)_{\mathcal{F}_s}$, as Example 3.22 shows. In this example of a stochastic interest rate model, the long-term spot rate l jumps up from 0 to 1 at time $t = 1$ with probability 1. The following, purely analytical condition is a convenient additional assumption for proving asymptotic minimality in Theorem 2.21 below.

Definition 2.20. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$. An $\overline{\mathbb{R}}$-valued random variable X is said to dominate the random variables $\{X_t\}_{t>0}$ in the $(\mathcal{G}, \mathbb{P})$-superexponential sense along a $\mathcal{G}$-measurable subsequence, if[2]

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}\big[(\max\{X_t - X, 0\})^t \,|\, \mathcal{G}\big] \overset{\text{a.s.}}{=} -\infty.$$

Theorem 2.21 (Asymptotic minimality)**.** *Let Condition 2.12 be satisfied for times s and t with $0 \le s < t$. Assume in addition that the upper $\mathcal{F}_s$-measurable envelope $V_t^{\mathcal{F}_s}$ of the limiting annual discount factor at time t given by*[3]

$$V_t = \begin{cases} 1/(l(t)+1) & \text{in the discrete-time case,} \\ \exp(-l(t)) & \text{in the continuous-time case,} \end{cases} \tag{2.22}$$

dominates $\{P(t, t+u)^{1/u}\}_{u>0}$ in the $(\mathcal{F}_s, \mathbb{Q})$-superexponential sense along an $\mathcal{F}_s$-measurable subsequence, which means that

$$\liminf_{T\to\infty} \big(\mathbb{E}_{\mathbb{Q}}\big[\max\{P(t,T) - (V_t^{\mathcal{F}_s})^{T-t}, 0\} \,\big|\, \mathcal{F}_s\big]\big)^{1/(T-t)} \overset{\text{a.s.}}{=} 0. \tag{2.23}$$

Then $l(s) \overset{\text{a.s.}}{=} l(t)_{\mathcal{F}_s}$ and $l_F(s, s') = l_F(t, t')_{\mathcal{F}_s}$ a. s. for all $s' \ge s$ and $t' \ge t$.

Remark 2.24. For asymptotic minimality we cannot weaken the requirements for the probability measure, because we use the probability measure in Condition 2.10 to show asymptotic monotonicity, but we need also the reversed inequality

$$P(s,T) \le P(s,t)\, \mathbb{E}_{\mathbb{Q}_{s,t}}[P(t,T) \,|\, \mathcal{F}_s] \qquad \text{a. s.},$$

for the estimate in (5.15) in the proof of Theorem 2.21.

Remark 2.25. Note that $V_t^{\mathcal{F}_s} \overset{\text{a.s.}}{=} 1/(l(t)_{\mathcal{F}_s} + 1)$ and $V_t^{\mathcal{F}_s} \overset{\text{a.s.}}{=} \exp(-l(t)_{\mathcal{F}_s})$, respectively, and since

$$V_t \overset{\text{a.s.}}{=} \begin{cases} \liminf_{T\to\infty} \frac{1}{R(t,T)+1} & \text{in the discrete-time case,} \\ \liminf_{T\to\infty} \exp(-R(t,T)) & \text{in the continuous-time case,} \end{cases}$$

we obtain

$$V_t \overset{\text{a.s.}}{=} \liminf_{T\to\infty} P(t,T)^{1/(T-t)}. \tag{2.26}$$

Remark 2.27. If there exists a sequence $\{T_n\}_{n\in\mathbb{N}}$ of $\mathcal{F}_s$-measurable random times, taking at most countable many values in (t, ∞) and tending to infinity as $n \to \infty$, such that for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ satisfying

$$P(t, T_n)^{1/(T_n - t)} \le V_t^{\mathcal{F}_s} + \varepsilon \qquad \text{a. s.} \tag{2.28}$$

for all $n \ge n_\varepsilon$, then (2.23) holds. Due to (2.26) and $V_t \le V_t^{\mathcal{F}_s}$, this uniformity certainly holds for all $s \in [0, t]$ simultaneously when $\mathcal{F}_t$ is finite and the limit

[2]We use here the convention $\log 0 = -\infty$. Analogously to (2.5) and (2.6), the limit inferior is the limit as $n \to \infty$ of the essential infima over all $t > n$.

[3]We use here the conventions $1/0 = \infty$, $1/\infty = 0$, $\exp(\infty) = \infty$, and $\exp(-\infty) = 0$.

inferior in (2.26) is attained along a deterministic sequence $\{T_n\}_{n\in\mathbb{N}}$. The latter condition in turn is satisfied when $l(t) = \lim_{n\to\infty} R(t, T_n)$ a. s.

2.3. **Results using different notions for absence of arbitrage.** In the last section we assumed the existence of a forward risk neutral probability measure, resp. that Condition 2.10 holds. The second approach uses no-arbitrage arguments to show asymptotic monotonicity and minimality. The next definition gives two different notions of arbitrage and applies to discrete as well as continuous time. It is inspired by the definition of *arbitrage in the limit,* which is used by Schulze, and the definition of arbitrage used in Dybvig et al.

Definition 2.29. Given times $0 \le s < t$, the zero-coupon bonds with maturity $T \ge t$ provide an *arbitrage opportunity in the limit* for times s and t, if there exist a sequence $\{(\varphi_n, \psi_n)\}_{n\in\mathbb{N}}$ of $\mathcal{F}_s$-measurable, $\mathbb{R}^2$-valued portfolio compositions and a sequence $\{T_n\}_{n\in\mathbb{N}}$ of $\mathcal{F}_s$-measurable random maturities $T_n\colon \Omega \to (n \vee t, \infty)$, each one taking only a finite number of values, such that

(a) $V_n(s) := \varphi_n P(s, T_n) + \psi_n P(s, t) \overset{\text{a.s.}}{=} 0$ for all $n \in \mathbb{N}$,
(b) $\mathbb{P}(\liminf_{n\to\infty} V_n(t) > 0) > 0$, where $V_n(t) := \varphi_n P(t, T_n) + \psi_n$, and
(c) $\liminf_{n\to\infty} V_n(t) \ge 0$ a. s.

We say that the zero-coupon bonds provide an *arbitrage opportunity in the limit with vanishing risk* for times s and t, if (c) is replaced by

(d) for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $V_n(t) \ge -\varepsilon$ a. s. for all $n \ge n_\varepsilon$.

Remark 2.30. Part (a) in Definition 2.29 always holds if $\psi_n := -\varphi_n P(s, T_n)/P(s, t)$ for all $n \in \mathbb{N}$.

Remark 2.31. Since (d) implies (c), the assumption of *no arbitrage opportunity in the limit* is stronger than *no arbitrage opportunity in the limit with vanishing risk.* If $\mathcal{F}_t$ is finite, then pointwise implies uniform convergence, hence (c) implies (d) and both notions of arbitrage are equivalent. Example 3.26 below shows that even the stronger assumption of *no arbitrage opportunity in the limit* does not imply the existence of a forward risk neutral probability measure in Condition 2.12. Even the weaker Condition 2.10 does not hold in this example.

Lemma 2.32 below shows that Condition 2.12 implies the weaker no-arbitrage condition, which by Theorem 2.33 is sufficient for asymptotic monotonicity. Actually, the no-arbitrage condition can be further weakened by excluding only arbitrage due to a positive investment in the long-term zero-coupon bonds. The lemma and the following theorems are proved in Section 6.

Lemma 2.32. *If there exists a forward time s risk neutral probability measure for maturity t as in Condition 2.12 with $0 \le s < t$, then there is no arbitrage opportunity in the limit with vanishing risk for times s and t.*

Theorem 2.33. *Consider times $0 \le s < t$. Assume that there is no arbitrage opportunity in the limit with vanishing risk for times s and t in the sense of Definition 2.29 by investing in the long-term zero-coupon bonds (with $\varphi_n \ge 0$ for all $n \in \mathbb{N}$). Then $l(s) \le l(t)$ a. s. and $l_F(s, s') \le l_F(t, t')$ a. s. for all $s' \ge s$ and $t' \ge t$.*

For the remaining results, we need the stronger assumption of *no arbitrage opportunity in the limit*, however, for Theorem 2.34 below we only have to exclude this limiting arbitrage by short-selling of the long-term zero-coupon bonds. The heuristic justification of the following theorem is as follows: If, with strictly positive probability, the worst long-term spot rate, which we will incur by placing our investment orders for time t already at an earlier time s based on the information available at s, is strictly larger than the best long-term spot rate we can earn by investing already at time s, then the prices of the long-term zero-coupon bonds must fall substantially during $[s,t]$, offering an arbitrage possibility in the limit by short-selling these bonds.

Theorem 2.34. *Consider times $0 \le s < t$. Assume that there is no arbitrage opportunity in the limit for times s and t in the sense of Definition 2.29 by short-selling the long-term zero-coupon bonds (with $\varphi_n \le 0$ for all $n \in \mathbb{N}$). Then for every sequence $\{T_n\}_{n\in\mathbb{N}}$ of $\mathcal{F}_s$-measurable random maturities $T_n\colon \Omega \to (n \vee t, \infty)$, each one taking only finitely many values,*

$$\Big(\liminf_{n\to\infty} R(t,T_n)\Big)_{\mathcal{F}_s} \le l(s) \quad a.\,s. \tag{2.35}$$

Examples 3.3 and 3.20 show that, for certain sequences of random (or even deterministic) maturities, the inequality in (2.35) can be strict everywhere on Ω. In these models, the limit of $R(t,T_n)$ as $n \to \infty$ does not exist. Note that the deterministic model of Example 2.18, which admits an arbitrage possibility by investing in the zero-coupon bonds with maturity $T > t$, satisfies the assumptions of Theorem 2.34.

Using the definition of the long-term zero-coupon rate $l(t)$ from (2.5), the $\mathcal{F}_s$-measurability of $l(s)$ and the definition of the lower $\mathcal{F}_s$-measurable envelope in Definition 2.19, we obtain from Theorems 2.33 and 2.34:

Corollary 2.36 (Asymptotic minimality)**.** *Consider $0 \le s < t$. If there is no arbitrage opportunity in the limit for s and t in the sense of Definition 2.29, then*

$$\Big(\liminf_{T\to\infty} R(t,T)\Big)_{\mathcal{F}_s} \le l(s) \le \Big(\limsup_{T\to\infty} R(t,T)\Big)_{\mathcal{F}_s} \quad a.\,s. \tag{2.37}$$

In particular, if $\lim_{T\to\infty} R(t,T)$ exists a. s., then $l(s) \overset{a.s.}{=} l(t)_{\mathcal{F}_s}$.

If the limit of $R(t,T)$ as $T \to \infty$ does not exist a. s., we might still get asymptotic minimality. Using asymptotic monotonicity and Theorem 2.34, each sequence $\{T_n\}_{n\in\mathbb{N}}$ of $\mathcal{F}_s$-measurable random maturities, each one taking only finitely many values, satisfies

$$\Big(\liminf_{n\to\infty} R(t,T_n)\Big)_{\mathcal{F}_s} \le l(s) \le l(t)_{\mathcal{F}_s}, \quad \text{a.\,s.},$$

if there is no arbitrage opportunity in the limit. If a special sequence of maturities satisfies additionally the reversed inequality, we have asymptotic minimality. Note that the sequence from Lemma 2.8 cannot be used in general, because these maturities are only $\mathcal{F}_t$-measurable.

Corollary 2.38 (Asymptotic minimality)**.** *Consider $0 \le s < t$. If there is no arbitrage opportunity in the limit for times s and t in the sense of Definition 2.29 and if there exists a sequence $\{T_n\}_{n\in\mathbb{N}}$ of $\mathcal{F}_s$-measurable random maturities T_n: $\Omega \to (n \vee t, \infty)$, each one taking only finitely many values, such that*

$$l(t)_{\mathcal{F}_s} \le \liminf_{n\to\infty} R(t,T_n) \quad a.\,s., \tag{2.39}$$

then $l(s) \overset{a.s.}{=} l(t)_{\mathcal{F}_s}$.

Remark 2.40. In Corollaries 2.36 and 2.38, it is actually sufficient to assume that there is no arbitrage opportunity in the limit with vanishing risk for times s and t by investing in the long-term zero-coupon bonds (with $\varphi_n \geq 0$ for all $n \in \mathbb{N}$) and that there is no arbitrage opportunity in the limit by short-selling the long-term zero-coupon bonds (with $\varphi_n \leq 0$ for all $n \in \mathbb{N}$).

Remark 2.41. Using the almost sure equivalence of the long-term spot and forward rates in Lemma 2.9, we can also transfer Theorem 2.34 and its corollaries to the long-term forward rates. We refrain from spelling out the details.

Remark 2.42. We could relax Definition 2.29(d) to

(e) there exists $n_0 \in \mathbb{N}$, such that the negative parts $V_n^-(t) := \max\{0, -V_n(t)\}$ for all $n \geq n_0$ are uniformly integrable and $\liminf_{n\to\infty} V_n(t) \geq 0$ a. s.

and get more limiting arbitrage opportunities in this way. This would strengthen the no-arbitrage assumption. However, using a more general version of Fatou's lemma for conditional expectations[4], the proof of Lemma 2.32 carries over, where the existence of a forward risk neutral probability measure for times s and t in Condition 2.12 implies no arbitrage in the limit with vanishing risk. Hence this Condition is still stronger. Therefore, this stronger no-arbitrage assumption would not be strong enough to imply asymptotic minimality, as Example 3.22 illustrates. In particular, the limiting arbitrage strategies given there cannot satisfy (e).

Remark 2.43. The proofs of the above theorems, corollaries and lemmas do not use path properties of the processes $\{P(t,T)\}_{0\leq t\leq T}$ (like being càdlàg or a semimartingale), and we also do not need a bank account process or additional assumptions on the filtration $\mathbb{F}$ (like containing all null sets of $\mathcal{F}$ or being right-continuous). Furthermore, we allow for $\mathbb{P}(P(t,T) > 1) > 0$, which can happen for models with negative interest rates like the Vasiček model or the Heath–Jarrow–Morton model.

3. Examples

In this section we show with illustrative examples first, that the long-term zero-coupon rates do not always exist and our generalization of the Dybvig–Ingersoll–Ross theorem is therefore useful. In these examples asymptotic monotonicity holds for the limit superior of the zero-coupon rates, resp. the forward rates. Four further examples illustrate the asymptotic minimality conditions as explained in Section 2. In Example 3.20 we describe a very simple stochastic interest rate model with $\Omega = \{0,1\}$. Although this model provides no arbitrage opportunity in the limit and a forward risk neutral probability measure exists, asymptotic minimality does not hold. There *does not* exist a deterministic sequence $\{T_n\}_{n\in\mathbb{N}}$ with $T_n \to \infty$ such that (2.39) holds. Therefore, the absence of arbitrage opportunities in the limit for times s and t with $0 \leq s < t$ or the existence of a forward risk neutral probability measure is not sufficient for asymptotic minimality in the sense of $l(s) \overset{\text{a.s.}}{=} l(t)_{\mathcal{F}_s}$. These conditions are not even necessary, see Example 3.23. Example 3.24 shows that asymptotic minimality is not an interval property, meaning that for times $0 < s < t < u$ the property $l(s) \overset{\text{a.s.}}{=} l(u)_{\mathcal{F}_s}$ *does not* imply $l(t) \overset{\text{a.s.}}{=} l(u)_{\mathcal{F}_t}$. Furthermore, the example shows that even if there is no arbitrage opportunity in the limit for times s and u, it is possible to have an arbitrage opportunity for times t and u.

[4]See *Fatou's lemma* at en.wikipedia.org/wiki/, version of October 11, 2008.

All these examples are continuous-time short-rate models, and there exists a forward risk neutral probability measure for all times defined in Condition 2.12 by construction as pointed out in Remark 2.14. Hence by Lemma 2.32, these models do not provide an arbitrage opportunity in the limit with vanishing risk. For a model not satisfying Condition 2.12, see Example 3.26.

The general set-up of these models (with the exception of the last one) is given as follows. For a given $\mathbb{F}$-progressive interest rate intensity process $\{r_t\}_{t\geq 0}$ with locally integrable paths, we define the money market account by

$$B_t = \exp\Bigl(\int_0^t r_u \, du\Bigr), \qquad t \in [0, \infty).$$

Assume that $1/B_t$ is $\mathbb{Q}$-integrable for every $t > 0$. Using (2.15), the zero-coupon bond prices are given by

$$P(t,T) = \mathbb{E}_{\mathbb{Q}}\Bigl[\exp\Bigl(-\int_t^T r_u \, du\Bigr)\Bigm|\mathcal{F}_t\Bigr], \qquad 0 \leq t \leq T. \tag{3.1}$$

Therefore, the definition of $R(t,T)$ in (2.2) implies

$$R(t,T) = -\frac{1}{T-t}\log \mathbb{E}_{\mathbb{Q}}\Bigl[\exp\Bigl(-\int_t^T r_u \, du\Bigr)\Bigm|\mathcal{F}_t\Bigr], \qquad 0 \leq t < T. \tag{3.2}$$

3.1. **Models where the limits of the zero-coupon rates do not exist.** In the following examples we discuss models, where the limits of the zero-coupon rates $R(t,T)$ for $T \to \infty$ do not exist. The idea is to vary the behaviour of the short rate on longer and longer time periods to get oscillating means. We illustrate this with a simple deterministic model and then with two short-rate models having an (exponentially) affine term structure. More specifically, we consider a variant of the familiar Vasiček model with time-dependent coefficients, which was proposed by Vasiček (1977) and Hull and White (1990). Secondly, we study the behaviour of the long-term spot rate in the model of Cox, Ingersoll and Ross (1985) with time-dependent coefficients (but constant dimension).

In both examples the mean level or the volatility of the short rate changes cyclically but decelerates over time. An economical justification for this behaviour can be the dependence on the business cycles, which become longer and longer. So, if the lengths of the business cycles increase exponentially, then the limits of the zero-coupon rates might not exist, as our examples show.

In our last example, we use the well-known Heath–Jarrow–Morton framework, proposed in [9], and choose an oscillating but decaying volatility function for the forward rates such that the limits of the zero-coupon rates do not exist, see Example 3.16 below. Since we specialize to a deterministic volatility function in product form, this example is related to the extended Vasiček model, cf. [14, Section 10.2].

Example 3.3 (Deterministic model)**.** Define the set

$$A = \bigl[\tfrac{1}{3}, 1\bigr) \cup \bigcup_{k=0}^{\infty} \bigl[2^{2k+1}, 2^{2k+2}\bigr), \tag{3.4}$$

the càdlàg interest rate intensity $r_t = 1_A(t)$ for $t \geq 0$, and the continuous function

$$R_A(t,T) = \frac{1}{T-t}\int_t^T 1_A(u)\, du = \frac{\lambda(A \cap [t,T])}{T-t}, \qquad 0 \leq t < T, \tag{3.5}$$

where λ denotes the Lebesgue measure. Since $\{r_t\}_{t\geq 0}$ is deterministic, (3.2) implies $R(t,T) = R_A(t,T)$ for all $0 \leq t < T$. Note that $T \geq 1$ is a local minimum of $R_A(0,\cdot)$ if and only if there exists $n \in \mathbb{N}_0$ with $T = 2^{2n+1}$. Since

$$\lambda(A \cap [0, 2^{2n+1}]) = \frac{2}{3} + \sum_{k=0}^{n-1} 2^{2k+1} = \frac{2}{3} + 2\frac{4^n - 1}{3} = \frac{2^{2n+1}}{3}, \qquad n \in \mathbb{N}_0,$$

we have $R_A(0, 2^{2n+1}) = 1/3$. Furthermore, $T \geq 2$ is a local maximum of $R_A(0,\cdot)$ if and only if there exists $n \in \mathbb{N}_0$ with $T = 2^{2n+2}$. Since $\lambda(A \cap [0, 2^{2n+2}]) = \lambda(A \cap [0, 2^{2n+1}]) + 2^{2n+1} = 2^{2n+3}/3$, we get $R_A(0, 2^{2n+2}) = 2/3$. Hence, we have $R_A(0,T) \in [1/3, 2/3]$ for all $T \geq 1$, and the interval $[1/3, 2/3]$ is also the set of all accumulation points of $\{R_A(0,T)\}_{T>0}$. Since $|R_A(t,T) - R_A(0,T)| \leq 2t/T$ for $0 \leq t < T$, the latter is also true for $\{R_A(t,T)\}_{T>t}$, in particular the limit of $R(t,T)$ as $T \to \infty$ does not exist. Since $l(t) = \limsup_{T\to\infty} R_A(t,T) = 2/3$ for all $t \in [0,\infty)$, asymptotic minimality holds. This can also be shown by verifying the assumptions of Corollary 2.38. For $t \in [0,\infty)$ and $T_n := 2^{2n+2}$ with $n \in \mathbb{N}_0$ such that $T_n > t$,

$$|R_A(t,T_n) - l(t)| = |R_A(t,T_n) - R_A(0,T_n)| \leq \frac{2t}{2^{2n+2}} \overset{n\to\infty}{\longrightarrow} 0,$$

hence (2.39) is satisfied. Since the model is deterministic, the σ-algebra $\mathcal{F}_t$ is finite. Remark 2.31 implies that no arbitrage opportunity in the limit is equivalent to no arbitrage opportunity in the limit with vanishing risk. Furthermore, the example illustrates that the inequality (2.35) in Theorem 2.34 can be strict, because $l(0) = 2/3$ but $\liminf_{n\to\infty} R(0,T_n) = 1/3$ for $T_n := 2^{2n+1}$ with $n \in \mathbb{N}$. Note that this example can be generalized to an interest intensity process $r_t = a + b1_A(t)$ for $t \geq 0$, where $a, b \in \mathbb{R}$ and $b \neq 0$.

A broad class of interest rate models have an (exponentially) affine term structure, i. e., the price process of a zero-coupon bond with maturity $T > 0$ admits the representation

$$P(t,T) = \exp\bigl(A(t,T) + B(t,T)r_t\bigr), \qquad t \in [0,T),$$

with deterministic real-valued functions A and B, cf. [2, Chapter 22.3]. Hence, the zero-coupon rate process for $T > 0$ is given by

$$R(t,T) = -\frac{A(t,T) + B(t,T)r_t}{T-t}, \qquad t \in [0,T). \tag{3.6}$$

Therefore, if for $t \geq 0$ the short rate r_t is not deterministic, then the limit of $\{R(t,T)\}_{T>t}$ exists a. s. if and only if the limits of $A(t,T)/T$ and $B(t,T)/T$ for $T \to \infty$ exist. In the following we consider generalizations of the familiar Vasiček and Cox–Ingersoll–Ross models, which both belong to the (exponentially) affine term structure models. In these generalized models we show that, with appropriate choices of time-dependent coefficients, the limit of $A(t,T)/T$ as $T \to \infty$ does not exist.

Example 3.7 (Vasiček model with time-dependent coefficients)**.** Let $\alpha > 0$ be a parameter for the mean reverting strength. Suppose the mean level $\mu\colon [0,\infty) \to \mathbb{R}$ is a locally integrable function and the volatility $\sigma\colon [0,\infty) \to \mathbb{R}$ is a locally square-integrable function. Let $\{W_t\}_{t\geq 0}$ be a standard Brownian motion under $\mathbb{Q}$, and let the initial value r_0 be normally distributed (possibly with zero variance) and independent of the Brownian motion. Define the interest rate intensity process by

$$r_t = e^{-\alpha t}\Bigl(r_0 + \alpha\int_0^t e^{\alpha s}\mu_s\,ds + \int_0^t e^{\alpha s}\sigma_s\,dW_s\Bigr), \qquad t \ge 0. \tag{3.8}$$

Using Itô's formula, it follows that $\{r_t\}_{t\ge0}$ is a strong solution of the stochastic differential equation

$$dr_t = \alpha(\mu_t - r_t)\,dt + \sigma_t\,dW_t\,, \qquad t \ge 0,$$

with initial value r_0. Note that $\{r_t\}_{t\ge0}$ is a Gaussian process with continuous paths, see e. g. [1, Chapter 8]. It follows from (3.8) that

$$r_u = e^{-\alpha(u-t)}r_t + \alpha\int_t^u e^{-\alpha(u-s)}\mu_s\,ds + \int_t^u e^{-\alpha(u-s)}\sigma_s\,dW_s, \qquad 0 \le t \le u,$$

hence the conditional distribution of the integral $I_{t,T} = \int_t^T r_u\,du$ given r_t is a normal one. In particular, the process $\{r_t\}_{t\ge0}$ is Markovian and (3.2) simplifies to

$$R(t,T) = -\frac{1}{T-t}\log\mathbb{E}_{\mathbb{Q}}\bigl[\exp\bigl(-I_{t,T}\bigr)\,\big|\,r_t\bigr], \qquad 0 \le t < T. \tag{3.9}$$

Using the stochastic Fubini theorem, see e. g. Protter (2004), we obtain

$$\begin{aligned} I_{t,T} - r_t\int_t^T e^{-\alpha(u-t)}\,du - \alpha\int_t^T \mu_s\int_s^T e^{-\alpha(u-s)}\,du\,ds \\ = \int_t^T \sigma_s\underbrace{\int_s^T e^{-\alpha(u-s)}\,du}_{=\,(1-e^{-\alpha(T-s)})/\alpha}\,dW_s, \qquad 0 \le t \le T. \end{aligned}$$

Since the stochastic integral on the right-hand side is independent of r_t with zero expectation, it follows that

$$\mathbb{E}_{\mathbb{Q}}\bigl[I_{t,T}\,\big|\,r_t\bigr] = r_t\,\frac{1-e^{-\alpha(T-t)}}{\alpha} + \int_t^T \bigl(1-e^{-\alpha(T-s)}\bigr)\mu_s\,ds, \qquad 0 \le t \le T,$$

and, using the Itô isometry,

$$\mathrm{Var}_{\mathbb{Q}}(I_{t,T}\,|\,r_t) = \frac{1}{\alpha^2}\int_t^T \bigl(1-e^{-\alpha(T-s)}\bigr)^2\sigma_s^2\,ds, \qquad 0 \le t \le T.$$

If X has a normal distribution, then $\log\mathbb{E}\bigl[e^{-X}\bigr] = -\mathbb{E}[X] + \frac{1}{2}\mathrm{Var}(X)$. Applying these results to (3.9) leads to

$$\begin{aligned} R(t,T) = r_t\,\frac{1-e^{-\alpha(T-t)}}{\alpha(T-t)} + \frac{1}{T-t}\int_t^T \bigl(1-e^{-\alpha(T-s)}\bigr)\mu_s\,ds \\ - \frac{1}{2\alpha^2(T-t)}\int_t^T \bigl(1-e^{-\alpha(T-s)}\bigr)^2\sigma_s^2\,ds, \qquad 0 \le t < T. \end{aligned} \tag{3.10}$$

Given $t \ge 0$, the limit of the zero-coupon rates $\{R(t,T)\}_{T>t}$ exists in $\mathbb{R}$ if and only if the limit of the difference of the last two terms in (3.10) exists in $\mathbb{R}$ as $T \to \infty$. It remains to choose suitable time-dependent functions for the mean level μ or the volatility σ such that this is not the case. Let us discuss three specific choices.

If μ is bounded and $\lim_{s\to\infty}\sigma_s^2=\infty$, then, for every $n\in\mathbb{N}$, there exists $T_n\ge t$ such that $\sigma_s^2\ge n$ for all $s\ge T_n$. Since $1-e^{-\alpha(T-s)}\ge 1/2$ for $s\le T-1/\alpha$, we get for all $T\ge T_n+1/\alpha$,

$$\frac{4}{T-t}\int_t^T\bigl(1-e^{-\alpha(T-s)}\bigr)^2\sigma_s^2\,ds\ge\frac{1}{T-t}\int_{T_n}^{T-1/\alpha}\sigma_s^2\,ds\ge n\frac{T-T_n-1/\alpha}{T-t}\overset{T\to\infty}{\longrightarrow}n.$$

Hence $l(t)=\limsup_{T\to\infty}R(t,T)=-\infty$ by (3.10) and, in particular, asymptotic minimality holds. Note, that (2.39) is satisfied. A similar argumentation shows that $l(t)=\pm\infty$ if σ is bounded and $\lim_{s\to\infty}\mu_s=\pm\infty$, respectively.

We now discuss cases where the mean level μ and the volatility σ remain bounded. Note that, for every $a>0$ and bounded measurable function $g\colon[0,\infty)\to\mathbb{R}$,

$$\left|\int_t^T e^{-a(T-s)}g(s)\,ds\right|\le\frac{1-e^{-a(T-t)}}{a}\|g\|_\infty\le\frac{\|g\|_\infty}{a},\qquad 0\le t\le T.$$

Therefore, if follows from (3.10) that, for every $t\ge 0$,

$$R(t,T)=\frac{1}{T-t}\int_t^T\mu_s\,ds-\frac{1}{2\alpha^2(T-t)}\int_t^T\sigma_s^2\,ds+O\Bigl(\frac{1}{T}\Bigr)\tag{3.11}$$

as $T>t$ tends to infinity.

We first consider a constant volatility $\sigma\in\mathbb{R}$ and a time-dependent mean level $\mu_s:=a+b1_A(s)$ for $s\ge 0$ with A given by (3.4), $a\in\mathbb{R}$ and $b>0$. Using (3.11) we obtain, for every $t\ge 0$,

$$R(t,T)=a+bR_A(t,T)-\frac{\sigma^2}{2\alpha^2}+O\Bigl(\frac{1}{T}\Bigr)\qquad\text{as }T\to\infty,$$

with $R_A(t,T)$ given by (3.5). It is shown in Example 3.3 that the limit of $R_A(t,T)$ as $T\to\infty$ does not exist, hence the limit of $\{R(t,T)\}_{T>t}$ does not exist either. Since $\limsup_{T\to\infty}R_A(t,T)=2/3$ by the results from Example 3.3, we see that

$$l(t)=\limsup_{T\to\infty}R(t,T)=a+\frac{2b}{3}-\frac{\sigma^2}{2\alpha^2},\qquad t\ge 0,$$

hence asymptotic minimality holds for all $0\le s<t$. A similar result can be obtained, if we choose a constant mean level $\mu\in\mathbb{R}$ and a time-dependent volatility $\sigma_s:=a+b1_A(s)$ with $a,b\in\mathbb{R}$ satisfying $2ab+b^2\ne 0$.

To illustrate explicitly that a bounded, continuously varying volatility function σ can also lead to oscillating zero-coupon rates, we consider a constant mean level $\mu\in\mathbb{R}$ and a volatility function of the form

$$\sigma_t=\sqrt{a+b\sin(\log(t+1))+b\cos(\log(t+1))},\qquad t\ge 0,$$

with $a,b\in(0,\infty)$ satisfying $a\ge\sqrt{2}\,b$. Then $0\le\sigma_t\le\sqrt{2a}$ for all $t\ge 0$. Furthermore, for all $T>0$ and $t\in[0,T)$,

$$\frac{1}{T-t}\int_t^T\sigma_s^2\,ds=a+b\,\frac{(T+1)\sin(\log(T+1))-(t+1)\sin(\log(t+1))}{T-t}.\tag{3.12}$$

Together with (3.11) we obtain for the long-term spot rate process

$$l(t)=\limsup_{T\to\infty}R(t,T)=\mu-\frac{a-b}{2\alpha^2},\qquad t\ge 0,\tag{3.13}$$

but for the limes inferior of the zero coupon rates

$$\liminf_{T\to\infty}R(t,T)=\mu-\frac{a+b}{2\alpha^2},\qquad t\ge 0.$$

Hence, the limit of $\{R(t,T)\}_{T>t}$ as $T \to \infty$ does not exist. Since the long-term spot-rate process given by (3.13) is a deterministic constant, asymptotic minimality holds for all times $0 \le s < t$.

Example 3.14 (Cox–Ingersoll–Ross model with time-dependent coefficients)**.** Let α: $[0,\infty) \to (0,\infty)$ and β: $[0,\infty) \to (-\infty,0)$ be two continuously differentiable functions and let $\{W_t\}_{t\ge0}$ be a standard Brownian motion under $\mathbb{Q}$. Analogously to [14, Sections 10.3.2 and 10.3.3], by considering a squared Bessel process of dimension $\delta \in (0,\infty)$ with respect to some probability measure $\mathbb{P}$, applying a suitable measure change to $\mathbb{Q}$ using Girsanov's theorem, and rescaling the state space by the function α, we can construct an interest rate intensity process $\{r_t\}_{t\ge0}$ which solves the stochastic differential equation

$$dr_t = \Bigl(\delta\alpha(t) + \Bigl(2\beta(t) + \frac{\alpha'(t)}{\alpha(t)}\Bigr) r_t\Bigr)\,dt + 2\sqrt{\alpha(t) r_t}\,dW_t, \qquad t \ge 0,$$

with deterministic initial value $r_0 \ge 0$. If for given $0 \le t < T$ there is a solution $F_T : [t,T] \to \mathbb{R}$ to the Riccati equation

$$F_T^2(u) + F_T'(u) = 2\alpha(u) + \beta^2(u) + \beta'(u), \qquad u \in [t,T],$$

with the terminal condition $F_T(T) = \beta(T)$, then it follows as in [14, Sections 10.3.3 and 10.3.4] that the corresponding zero-coupon rate is given by

$$R(t,T) = -\frac{1}{2(T-t)}\biggl(\frac{F_T(t)-\beta(t)}{\alpha(t)} r_t + \delta \int_t^T \bigl(F_T(u) - \beta(u)\bigr)\,du\biggr),$$

which corresponds to (3.6) resulting from an (exponentially) affine term structure.

We now make specific choices for α and β. For $b > 0$ and $a > \sqrt{2}b$ define the function

$$\beta(t) = -a + b\sin(\log(t+1)) + b\cos(\log(t+1)), \qquad t \ge 0.$$

Note that β is continuously differentiable and that $-a - \sqrt{2}b \le \beta(t) < 0$ for all $t \ge 0$. Furthermore, for $c > 0$ with $c^2 > (a+\sqrt{2}b)^2 + \sqrt{2}b$, define the function

$$\alpha(t) = \frac{1}{2}\bigl(c^2 - \beta^2(t) - \beta'(t)\bigr), \qquad t \ge 0.$$

Since $\beta^2(t) \le (a+\sqrt{2}b)^2$ and $\beta'(t) \le \sqrt{2}b$, it follows that $\alpha(t) > 0$ for all $t \ge 0$. For these functions α and β, the Riccati equation simplifies, for each $T > 0$, to

$$F_T^2(u) + F_T'(u) = c^2, \qquad u \in [0,T].$$

The solution for the terminal condition $F_T(T) = \beta(T)$ is

$$F_T(u) = c\tanh(cu + g_T), \qquad u \in [0,T],$$

where $g_T := \operatorname{artanh}(\beta(T)/c) - cT$. Since $|\beta(T)| \le a + \sqrt{2}b < |c|$ for all $T > 0$, the area tangents hyperbolicus of $\beta(T)/c$ is well-defined and bounded with respect to T. Note that $\frac{d}{dx}\log(\cosh(cx + g_T)) = c\tanh(cx + g_T)$ for all $x \in \mathbb{R}$. Therefore,

$$\int_t^T F_T(u)\,du = \log\frac{\cosh\bigl(\operatorname{artanh}\frac{\beta(T)}{c}\bigr)}{\cosh\bigl(\operatorname{artanh}\bigl(\frac{\beta(T)}{c}\bigr) - c(T-t)\bigr)}, \qquad 0 \le t \le T.$$

Using $\cosh x = (e^x + e^{-x})/2$ for $x \in \mathbb{R}$ and the boundedness of $\operatorname{artanh}(\beta(T)/c)$, it follows that

$$\lim_{T\to\infty} \frac{1}{T-t}\int_t^T F_T(u)\,du = -c, \qquad t \ge 0.$$

Finally, integration of β, cf. (3.12), yields

$$l(t) = \limsup_{T\to\infty} R(t,T) = \frac{\delta}{2}(c-a+b), \qquad t \geq 0 \tag{3.15}$$

but

$$\liminf_{T\to\infty} R(t,T) = \frac{\delta}{2}(c-a-b), \qquad t \geq 0.$$

Since l in (3.15) is a deterministic constant, asymptotic minimality holds.

Our next example is the well-known Gaussian Heath–Jarrow–Morton model, cf. [14, Chapter 11], with deterministic but time-dependent volatility of the forward rates. For this volatility we choose a non-negative function, which fluctuates over time but converges to zero when the maturities tend to infinity. Again, we assume, that the volatility varies with the business cycles of exponentially increasing lengths.

Example 3.16 (Gaussian Heath–Jarrow–Morton model)**.** Let $\sigma_1, \sigma_2\colon [0,\infty) \to \mathbb{R}$ denote bounded measurable functions. Define the volatility $\sigma\colon [0,\infty)^2 \to \mathbb{R}$ of the forward rates by $\sigma(u,v) = \sigma_1(u)\sigma_2(v)$ for all $u, v \geq 0$. Suppose that the deterministic forward rate curve $f(0,\cdot)\colon [0,\infty) \to \mathbb{R}$ at time zero is locally integrable. We set up the model directly using the spot martingale measure $\mathbb{Q}$, under which zero-coupon bond prices, discounted by the bank account process, are martingales. Therefore, let $\{W_t\}_{t\geq 0}$ be a standard Brownian motion under $\mathbb{Q}$ and let the forward rates satisfy

$$f(t,T) = f(0,T) + \int_0^t \sigma(u,T)\sigma^*(u,T)\,du + \int_0^t \sigma(u,T)\,dW_u, \qquad 0 \leq t \leq T,$$

with integrated volatility $\sigma^*(u,T) := \int_u^T \sigma(u,v)\,dv$ so that they obey the Heath–Jarrow–Morton drift condition. The short-term interest rate intensity process is given by $r_t = f(t,t)$ for $t \geq 0$. Then, for each maturity $T > 0$ and time $t \in [0,T)$, the zero-coupon rate is given by

$$R(t,T) = \frac{1}{T-t}\biggl(\int_0^T f(0,u)\,du - \int_0^t \Bigl(r_u - \frac{1}{2}(\sigma^*(u,T))^2\Bigr)\,du + \int_0^t \sigma^*(u,T)\,dW_u\biggr),$$

see e. g. [14, Chapter 11, pp. 388–389].

Given $t \geq 0$, the limit of the zero-coupon rates $\{R(t,T)\}_{T>t}$ as $T \to \infty$ might not exist, if the averages of the initial forward rates $\{f(0,u)\}_{u\in[t,T]}$ do not converge, see Examples 3.3 and 3.7 for such functions. In the following, we therefore assume the existence of the limit of these averages so that we can define

$$f^* = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(0,u)\,du.$$

To further discuss the limiting behaviour of the zero-coupon rates, we first consider their stochastic component. Substituting the stochastic integral from the short-rate r_v into the formula for $R(t,T)$ and using the stochastic Fubini theorem, we obtain

$$\begin{aligned}\frac{1}{T-t}\biggl(\int_0^t \sigma^*(u,T)\,dW_u - \int_0^t\int_0^v \sigma(u,v)\,dW_u\,dv\biggr)\\ = \frac{1}{T-t}\int_t^T \sigma_2(v)\,dv \int_0^t \sigma_1(u)\,dW_u, \qquad 0 \leq t < T,\end{aligned} \tag{3.17}$$

which converges to zero as $T \to \infty$ whenever the averages of $\{\sigma_2(v)\}_{v\in[t,T]}$ do. To obtain the oscillating behaviour of the zero-coupon rates, define

$$\sigma_2(v) = \frac{1}{2\sqrt{v+1}}\big(a + \sin(b\log(v+1)) + 2b\cos(b\log(v+1))\big), \qquad v \geq 0,$$

with parameters[5] $a, b \in \mathbb{R}$. Then $\lim_{v\to\infty}\sigma_2(v) = 0$, hence the stochastic part given in (3.17) tends to zero as $T \to \infty$. For all $T > 0$ and $u \in [0, T]$,

$$\int_u^T \sigma_2(v)\,dv = \sqrt{T+1}\big(a + \sin(b\log(T+1))\big) - \sqrt{u+1}\big(a + \sin(b\log(u+1))\big).$$

Hence, using the above expression for $R(t,T)$, the long-term spot rate is given by

$$\begin{aligned} l(t) &= \limsup_{T\to\infty} R(t,T) \\ &= f^* + \limsup_{T\to\infty} \frac{1}{2(T-t)} \int_0^t \sigma_1^2(u)\Big(\int_u^T \sigma_2(v)\,dv\Big)^2 du \\ &= f^* + \frac{1}{2}\limsup_{T\to\infty}\big(a + \sin(b\log(T+1))\big)^2 \int_0^t \sigma_1^2(u)\,du \\ &= f^* + \frac{1}{2}(|a| + 1_{b\neq 0})^2 \int_0^t \sigma_1^2(u)\,du, \qquad t \geq 0, \end{aligned} \tag{3.18}$$

where $1_{b\neq0}$ equals 1 if $b \neq 0$ and 0 otherwise. In particular, if a and b are not both zero, then, for all times $0 \leq s < t$ with $\int_s^t \sigma_1^2(u)\,du > 0$, asymptotic monotonicity in the sense $l(s) = l(t)_{\mathcal{F}_s}$ does not hold. The same reasoning as above yields

$$\liminf_{T\to\infty} R(t,T) = f^* + \frac{1}{2}\big(\max\{|a| - 1_{b\neq0}, 0\}\big)^2 \int_0^t \sigma_1^2(u)\,du, \qquad t \geq 0, \tag{3.19}$$

hence, for $t \geq 0$, the limit of the zero-coupon rates does not exist if $\int_0^t \sigma_1^2(u)\,du > 0$ and $b \neq 0$. Furthermore, if $|a| > 1_{b\neq0}$ and σ_1 is not the zero function, then this model provides arbitrage opportunities in the limit for all times $0 \leq s < t$ satisfying

$$(|a| + 1_{b\neq0})^2 \int_0^s \sigma_1^2(u)\,du < (|a| - 1_{b\neq0})^2 \int_0^t \sigma_1^2(u)\,du$$

by short-selling the long-term zero-coupon bonds: For every deterministic sequence $\{T_n\}_{n\in\mathbb{N}}$ tending to infinity, we have $l(s) < \liminf_{n\to\infty} R(t, T_n)$ by (3.18) and (3.19), hence the assumptions of Theorem 2.34 have to be violated.

3.2. **Models violating the asymptotic minimality.** We now present four short-rate models in continuous time, which illustrate the link between asymptotic minimality and the existence of a forward risk neutral probability measure in Condition 2.12, no arbitrage in the limit and convergence of the spot rate.

Example 3.20. On $\Omega = \{0, 1\}$ consider $X(\omega) = \omega$ for $\omega \in \Omega$, let $\mathbb{Q}$ denote the uniform distribution, $\mathcal{F}_t = \{\varnothing, \Omega\}$ for $t \in [0, 1/3)$ and $\mathcal{F}_t$ equal to the power set of Ω for $t \geq 1/3$. With A given by (3.4), define the interest rate intensity process by

$$r_t = X 1_A(t) + (1 - X) 1_{A^c\cap[1/3,\infty)}(t), \qquad t \in [0, \infty).$$

[5] If we choose $a \geq \sqrt{1_{b\neq0} + 4b^2}$, then $\sigma_2(v) \geq 0$ for all $v \geq 0$.

Note that $\{r_t\}_{t\geq 0}$ is adapted and càdlàg. Using (3.2) and Jensen's inequality, we get for all $t \in [0, 1/3)$ and $T > 1/3$,

$$\begin{aligned} R(t,T) &= -\frac{1}{T-t}\log\frac{\exp(-\lambda(A\cap[t,T])) + \exp(-\lambda(A^{\mathrm{c}}\cap[1/3,T]))}{2} \\ &\leq \frac{\lambda(A\cap[t,T]) + \lambda(A^{\mathrm{c}}\cap[1/3,T])}{2(T-t)} = \frac{T-1/3}{2(T-t)} \leq \frac{1}{2}, \end{aligned}$$

hence $l(t) \leq 1/2$. For $t \geq 1/3$, X is $\mathcal{F}_t$-measurable and we get from (3.2)

$$\begin{aligned} R(t,T) &= \frac{X\lambda(A\cap[t,T]) + (1-X)\lambda(A^{\mathrm{c}}\cap[t,T])}{T-t} \\ &= XR_A(t,T) + (1-X)(1-R_A(t,T)), \qquad T > t, \end{aligned}$$

with $R_A(t,T)$ given by (3.5). Therefore, $l(t) = \limsup_{T\to\infty} R(t,T) = 2/3$ for all $t \geq 1/3$, because the points in $[1/3, 2/3]$ are the accumulation points of $R_A(t,T)$ as $T \to \infty$, see Example 3.3.

In this example asymptotic minimality fails for all times $s \in [0, 1/3)$ and $t \in [1/3, T]$. By construction there exists a forward risk neutral probability measure, which implies with Lemma 2.32, that there is no arbitrage opportunity in the limit with vanishing risk. Since $\mathcal{F}_t$ is finite for each $t \geq 0$, the model provides also no arbitrage opportunity in the limit with Remark 2.31. Therefore Condition 2.12, resp. the weaker Condition 2.10, and the two different notions of no-arbitrage are not sufficient for asymptotic minimality.

Indeed, the inequality (2.39) fails, which is sufficient for asymptotic minimality in combination with no arbitrage opportunity in the limit. Consider an arbitrary deterministic sequence $\{T_n\}_{n\in\mathbb{N}}$ tending to infinity. Then

$$\liminf_{n\to\infty} R(t,T_n) = X \liminf_{n\to\infty} R_A(t,T_n) + (1-X)\Big(1 - \limsup_{n\to\infty} R_A(t,T_n)\Big). \tag{3.21}$$

Assume (2.39) holds for $\omega = 1$, then $\liminf_{n\to\infty} R_A(t,T_n) \geq l(t)_{\mathcal{F}_s} = 2/3$. Therefore, $\limsup_{n\to\infty} R_A(t,T_n) \geq 2/3$. With (3.21) follows that the inequality (2.39) fails for $\omega = 0$.

Finally, suppose $T_n := 2^n$ for $n \in \mathbb{N}$. We have seen in Example 3.3, that $\liminf_{n\to\infty} R_A(t,T_n) = 1/3$ and $\limsup_{n\to\infty} R_A(t,T_n) = 2/3$ for all $t \geq 0$. Hence, for $1/3 \leq s \leq t$ the inequality (2.35) is strict on Ω, i. e.

$$\liminf_{n\to\infty} R(t,T_n) = 1/3 < 2/3 = l(s).$$

Even, if the limit of the zero-coupon bonds and a forward risk neutral probability measure exist, this is not sufficient for asymptotic minimality, which is shown by the following example.

Example 3.22. Consider $\Omega = (0,1]$ with Lebesgue measure $\mathbb{Q}$, define $\mathcal{F}_t = \{\varnothing, \Omega\}$ for $t \in [0,1)$ and let $\mathcal{F}_t$ denote the Borel σ-algebra of $(0,1]$ for $t \geq 1$. Let $\tau(\omega) = 1/\omega$ for $\omega \in \Omega$ denote the random time, when the interest rate intensity jumps to 1, i. e., we define the interest rate intensity process by $r_t = 1_{[\tau,\infty)}(t)$ for $t \geq 0$. Then τ is $\mathcal{F}_1$-measurable and (3.2) implies for $T > 1$

$$R(1,T) = \frac{1}{T-1}\int_1^T r_u\,du = \frac{T-(T\wedge\tau)}{T-1} \overset{T\to\infty}{\longrightarrow} 1$$

everywhere on Ω, hence $l(1) = 1$. For $t \in [0, 1)$ and $T \geq 1$, (3.2) implies that

$$R(t,T) = -\frac{1}{T-t} \log \mathbb{E}_{\mathbb{Q}}\Bigg[\underbrace{\exp\Big(-\int_1^T r_u\,du\Big)}_{\geq 1_{\{\tau\geq T\}}}\Bigg] \leq -\frac{1}{T-t}\log\frac{1}{T} \overset{T\to\infty}{\longrightarrow} 0,$$

hence $l(t) = 0$ due to non-negative interest rates. Therefore, asymptotic minimality does not hold. This does not contradict Corollary 2.36, because this model provides an arbitrage opportunity in the limit for the times $s \in [0, 1)$ and $t = 1$ by short-selling long-term zero-coupon bonds. Choose a $(1, \infty)$-valued deterministic sequence $\{T_n\}_{n\in\mathbb{N}}$ tending to infinity, define $\varphi_n = -\exp((T_n - 1)/2)$ for each $n \in \mathbb{N}$ and fix $\{\psi_n\}_{n\in\mathbb{N}}$ according to Remark 2.30 so that Definition 2.29(a) holds. Then

$$V_n(1) = -\exp\Big((T_n-1)\Big(\frac{1}{2} - R(1,T_n)\Big)\Big) + \frac{\exp\big(\frac{1}{2}(T_n-1) - R(s,T_n)(T_n-s)\big)}{P(s,1)},$$

hence $\liminf_{n\to\infty} V_n(1) \overset{\text{a.s.}}{=} \infty$ and parts (b) and (c) of Definition 2.29 hold.

The existence of a forward risk neutral probability measure or the absence of arbitrage opportunities in the limit, is not even necessary for asymptotic minimality.

Example 3.23. Consider $\Omega = (0, 1]$ with Lebesgue measure $\mathbb{Q}$, define $\mathcal{F}_t = \{\varnothing, \Omega\}$ for $t \in [0, 1)$ and let $\mathcal{F}_t$ denote the Borel σ-algebra of $(0, 1]$ for $t \geq 1$. Let $\tau(\omega) = 1/\omega$ for $\omega \in \Omega$ be a random time. Define the interest rate intensity process[6] by $r_t = 1 - \frac{1}{t}1_{[1,\tau)}(t)$ for $t \geq 0$. With (3.1) the zero-coupon bond price for maturity $T \geq 1$ is given by

$$P(t,T) = e^{-(T-t)}\,\mathbb{E}_{\mathbb{Q}}[\tau\wedge T] = e^{-(T-t)}(1+\log T), \qquad t\in[0,1).$$

Using the definition of the zero-coupon rates in (2.2), we obtain for every $t \in [0, 1)$ and $T \geq 1$

$$R(t,T) = 1 - \frac{\log(1+\log T)}{T-t} \overset{T\to\infty}{\longrightarrow} 1.$$

For $t = 1$ the zero-coupon prices for $T \geq 1$ are given by $P(1,T) = e^{-(T-1)}(\tau \wedge T)$ and therefore the zero-coupon rates equal

$$R(1,T) = 1 - \frac{\log(\tau\wedge T)}{T-1} \overset{T\to\infty}{\longrightarrow} 1.$$

Hence asymptotic minimality holds.

On the other hand, we can construct an arbitrage opportunity in the limit for the times $s = 0$ and $t = 1$ according to Definition 2.29 by short-selling long-term zero-coupon bonds. For this define $T_n = n + 1$,

$$\varphi_n = -\frac{1}{e\,P(0,T_n)} = -\frac{e^{T_n-1}}{1+\log T_n},$$

and $\psi_n = 1$ for all $n \in \mathbb{N}$. Then $V_n(0) = 0$ for all $n \in \mathbb{N}$ and

$$V_n(1) = -\frac{(\tau\wedge T_n)}{1+\log T_n} + 1 \overset{n\to\infty}{\longrightarrow} 1 \qquad \text{on } \Omega.$$

[6] This example can be slightly simplified if we omit the 1 and allow negative interest rates.

Example 3.24. Consider $\Omega = \mathbb{N}$, define the filtration

$$\mathcal{F}_t = \begin{cases} \{\varnothing, \Omega\} & \text{for } t \in [0,1), \\ \{\varnothing, \{1\}, \Omega \setminus \{1\}, \Omega\} & \text{for } t \in [1,2), \\ \mathcal{P}(\Omega) & \text{for } t \in [2,\infty), \end{cases}$$

where $\mathcal{P}(\Omega)$ denotes the power set, and the probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{P}(\Omega))$ by $\mathbb{Q}(\{\omega\}) = 1/\omega - 1/(\omega+1)$ for all $\omega \in \Omega$. Let $\tau(\omega) = \omega$ for $\omega \in \Omega$ denote the random time, when the interest rate intensity jumps to $1 - 1/\omega$, i. e., we define the interest rate intensity process by $r_t = (1-1/\tau)1_{[\tau,\infty)}(t)$ for $t \geq 0$. Then τ is $\mathcal{F}_2$-measurable and (3.2) implies for $T > 2$

$$R(2,T) = \frac{1}{T-2}\int_2^T r_u\,du = \Big(1 - \frac{1}{\tau}\Big)\frac{T - (T \wedge (\tau \vee 2))}{T-2} \overset{T\to\infty}{\longrightarrow} 1 - \frac{1}{\tau} \tag{3.25}$$

everywhere on Ω, hence $l(2) = 1 - 1/\tau$. Therefore $l(2)_{\mathcal{F}_0} = 0$ and $l(2)_{\mathcal{F}_1} = \frac{1}{2}1_{\Omega\setminus\{1\}}$.

For $T > 0$, we always have that $R(0,T) \geq 0$ and (3.2) implies that

$$R(0,T) = -\frac{1}{T}\log \mathbb{E}_{\mathbb{Q}}\Big[\underbrace{\exp\Big(-\int_0^T r_u\,du\Big)}_{\geq 1_{\{\tau \geq \lceil T\rceil\}}}\Big] \leq -\frac{1}{T}\log\frac{1}{\lceil T\rceil} \overset{T\to\infty}{\longrightarrow} 0,$$

hence $l(0) = 0$ and asymptotic minimality holds for times 0 and 2.

For $T > 1$, (3.2) implies as in (3.25) that $l(1) = 0$ on $\{1\}$ and that on the complement $\Omega \setminus \{1\}$

$$R(1,T) = -\frac{1}{T-1}\log \mathbb{E}_{\mathbb{Q}}\Big[\underbrace{\exp\Big(-\int_1^T r_u\,du\Big)}_{\geq 1_{\{\tau \geq \lceil T\rceil\}}}\Big|\,\tau \geq 2\Big] \leq -\frac{1}{T-1}\log\frac{2}{\lceil T\rceil} \overset{T\to\infty}{\longrightarrow} 0,$$

hence $l(1) = 0$ on Ω and asymptotic minimality *does not* hold for times 1 and 2. To construct an arbitrage opportunity in the limit for times 1 and 2 according to Definition 2.29 by short-selling the long-term zero-coupon bonds, define for each $n \in \mathbb{N}$ the deterministic maturity $T_n = n + 2$ and the strategy by

$$\varphi_n = -1_{\Omega\setminus\{1\}}\exp\big((T_n - 1)R(1,T_n) - R(1,2)\big)$$

and $\psi_n = 1_{\Omega\setminus\{1\}}$. Then (φ_n, ψ_n) is $\mathcal{F}_1$-measurable and $V_n(1) = 0$ for all $n \in \mathbb{N}$. Furthermore, we obtain for all $n \in \mathbb{N}$

$$\varphi_n P(2,T_n) = -1_{\Omega\setminus\{1\}}\exp\big((T_n - 2)(R(1,T_n) - R(2,T_n)) + R(1,T_n) - R(1,2)\big).$$

Since $l(1) = \lim_{n\to\infty} R(1,T_n) = 0$ on Ω as well as $l(2) = \lim_{n\to\infty} R(2,T_n) = 1 - 1/\tau \geq 1/2$ on $\Omega \setminus \{1\}$ by (3.25), we get $\liminf_{n\to\infty} \varphi_n P(2,T_n) = 0$. Therefore, $\liminf_{n\to\infty} V_n(2) = \psi_n \geq 0$ and with probability $\mathbb{Q}(\Omega\setminus\{1\}) = 1/2$ the limes inferior is strictly greater than zero.

3.3. A model without forward risk neutral probability measure and without limiting arbitrage opportunities.

The following example is inspired by the infinite-horizon model considered in Example 7.2 in Pliska (1997). It shows that in general for a model, which does not provide an arbitrage opportunity in the limit, there must not exist a forward risk neutral probability measure. The other implication is also not true, see Example 3.22.

Example 3.26. Define $\Omega = \mathbb{N}$, $\mathcal{F}_0 = \{\varnothing, \mathbb{N}\}$ and $\mathcal{F}_1 = \mathcal{P}(\mathbb{N})$, the set of all subsets of $\mathbb{N}$. Let $\mathbb{P}$ be any probability measure on $\mathcal{F}_1$ with $\mathbb{P}(\{\omega\}) > 0$ for all $\omega \in \Omega$. Define zero-coupon bond prices by $P(0,n) = 1$ for all $n \in \mathbb{N}$ and

$$P(1,n)(\omega) = \begin{cases} 1 & \text{if } \omega \le n-2, \\ (n^2+1)/2 & \text{if } \omega = n-1, \\ 1/2 & \text{if } \omega \ge n, \end{cases}$$

for all $\omega \in \Omega$ and integers $n \ge 2$. By (2.1) and (2.5), it follows that $l(0) = l(1) = 0$, hence asymptotic monotonicity and minimality hold for times $s = 0$ and $t = 1$.

To verify that there is no arbitrage opportunity in the limit for times $s = 0$ and $t = 1$, consider an $\mathcal{F}_0$-measurable, hence deterministic sequence $\{T_n\}_{n\in\mathbb{N}}$ of maturities with $T_n > n$ for all $n \in \mathbb{N}$ and deterministic portfolios (φ_n, ψ_n) with $\varphi_n = -\psi_n$ for all $n \in \mathbb{N}$ so that Definition 2.29(a) is satisfied. Then $V_n(1)(\omega) = 0$ for all $\omega \in \{1, 2, \dots, n-2\}$, hence $\liminf_{n\to\infty} V_n(1) = 0$ on Ω. Therefore, part (c) of Definition 2.29 is satisfied, but part (b) does not hold.

To show that Condition 2.10 for times $s = 0$ and $t = 1$ is not satisfied (and there does not exist a forward risk neutral measure for times $s = 0$ and $t = 1$ like in Condition 2.12), we argue by contradiction. Assume that there exists an equivalent probability measure $\mathbb{Q} = \mathbb{Q}_{0,1}$ such that $P(0,n) \ge \mathbb{E}_{\mathbb{Q}}[P(1,n)]$ for all integers $n \ge n_0 \ge 2$. This implies

$$\mathbb{Q}(\{n-1, n, \dots\}) \ge \frac{n^2+1}{2}\mathbb{Q}(\{n-1\}) + \frac{1}{2}\mathbb{Q}(\{n, n+1, \dots\}),$$

hence $n^2\mathbb{Q}(\{n-1\}) \le \mathbb{Q}(\{n-1, n, \dots\})$ for all integers $n \ge n_0$. Define the constant $c = (n_0 - 1)\,\mathbb{Q}(\{n_0 - 1, n_0, \dots\})/n_0 > 0$. Then we get by induction

$$\mathbb{Q}(\{n-1, n, \dots\}) \ge \frac{cn}{n-1} \tag{3.27}$$

for all integers $n \ge n_0$, because

$$\begin{aligned} \mathbb{Q}(\{n, n+1, \dots\}) &= \mathbb{Q}(\{n-1, n, \dots\}) - \mathbb{Q}(\{n-1\}) \\ &\ge \Bigl(1 - \frac{1}{n^2}\Bigr)\mathbb{Q}(\{n-1, n, \dots\}) \ge \Bigl(1 - \frac{1}{n^2}\Bigr)\frac{cn}{n-1} = c\frac{n+1}{n} \end{aligned}$$

for all integers $n \ge n_0 + 1$. However, (3.27) for $n \to \infty$ implies $\mathbb{Q}(\varnothing) = c > 0$, which is impossible for a probability measure.

4. Proofs of auxiliary results

Proof of Lemma 2.8. Consider a finite non-empty set $I \subset (n \vee t, \infty)$ of zero-coupon bond maturities, which is required to be also a subset of $\mathbb{N}$ in the discrete-time case. Let $M_I := \max_{u\in I} R(t,u)$ denote the maximal available zero-coupon rate. Define the random maturity $T_I\colon \Omega \to I$ as the first one realizing this maximal rate, i. e.

$$T_I = \sum_{u\in I} u\, 1_{\{R(t,u)=M_I,\, R(t,v)<M_I \text{ for all } v\in I,\, v<u\}}.$$

Note that T_I is $\mathcal{F}_t$-measurable and that $R(t, T_I) = M_I$. By [7, Theorem A.32(b)], there exists, for every $n \in \mathbb{N}$, an increasing sequence $\{I_{k,n}\}_{k\in\mathbb{N}}$ of finite subsets of $(n \vee t, \infty)$, which are also subsets of $\mathbb{N}$ in the discrete-time case, such that

$$S_n := \operatorname*{ess\,sup}_{T > n\vee t} R(t,T) = \lim_{k\to\infty} R(t, T_{I_{k,n}}) \qquad \text{a. s.}$$

Hence, for every $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ such that the essential supremum is nearly reached with high probability, e. g. with the abbreviation $T_n := T_{I_{k_n,n}}$,

$$\mathbb{P}\big(\min\{S_n, n\} - 2^{-n} \le R(t, T_n) \le S_n\big) \ge 1 - 2^{-n}.$$

The a. s. limit of $\{S_n\}_{n\in\mathbb{N}}$ exists due to the monotonicity of the essential suprema. Hence, using the first Borel–Cantelli lemma, the a. s. limit of $\{R(t, T_n)\}_{n\in\mathbb{N}}$ exists and agrees with the one of $\{S_n\}_{n\in\mathbb{N}}$. □

Proof of Lemma 2.9. Fix $0 \le s \le t$. In the continuous-time case, using the definition of the arbitrage-free forward rate in (2.4) and the zero-coupon rate in (2.2),

$$F(s, t, T) = \frac{\log P(s, t)}{T - t} + \frac{T - s}{T - t} R(s, T), \qquad T \in (t, \infty).$$

Since the first summand tends to zero almost surely as $T \to \infty$, it follows that

$$l_F(s, t) = \limsup_{T\to\infty} F(s, t, T) = \limsup_{T\to\infty} \frac{T - s}{T - t}\, R(s, T) = l(s) \qquad \text{a. s.,}$$

by the definition of the long-term forward rate in (2.6) and the long-term zero-coupon rate in (2.5). In the discrete-time case, using the definition (2.3) of the arbitrage-free forward rate and the definition (2.1) of the zero-coupon rate, we see that it is enough to prove

$$\limsup_{T\to\infty} \log\left(\frac{P(s, t)}{P(s, T)}\right)^{1/(T-t)} \overset{\text{a.s.}}{=} \limsup_{T\to\infty} \log P(s, T)^{-1/(T-s)}.$$

However, that is what we just verified for the continuous-time case. □

5. Proofs for asymptotic monotonicity and minimality assuming the existence of a forward risk neutral probability measure

The key observation for our generalization is the following lemma, which uses notation introduced in Definition 2.19.

Lemma 5.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$.*

(a) *For every non-negative random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$, the function $(0, \infty) \ni t \mapsto \mathbb{E}[X^t | \mathcal{G}]^{1/t}$ is non-decreasing a. s. and*

$$X \le \lim_{t\to\infty} \mathbb{E}\big[X^t | \mathcal{G}\big]^{1/t} = X^{\mathcal{G}} \qquad \text{a. s.} \tag{5.2}$$

(b) *Let $\{X_t\}_{t>0}$ be a collection of non-negative random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. For each $n \in \mathbb{N}$ let Y_n denote the essential infimum of $\{X_t\}_{t>n}$. Then*

$$X := \liminf_{t\to\infty} X_t \le \lim_{n\to\infty} \mathbb{E}[Y_n^n | \mathcal{G}]^{1/n} = X^{\mathcal{G}} \le \liminf_{t\to\infty} \mathbb{E}[X_t^t | \mathcal{G}]^{1/t} \quad \text{a. s.} \tag{5.3}$$

(c) *If in (b) the random variable $X^{\mathcal{G}}$ dominates $\{X_t\}_{t>0}$ in the $(\mathcal{G}, \mathbb{P})$-super-exponential sense along a subsequence according to Definition 2.20, then the last inequality in (5.3) is an a. s. equality.*

Remark 5.4. For the trivial case $\mathcal{G} = \{\varnothing, \Omega\}$, Lemma 5.1(a) implies the well-known result $\lim_{p\to\infty} \|X\|_{L^p} = \|X\|_{L^\infty}$.

Remark 5.5. For a non-negative random variable Z with $\mathbb{E}[Z] = \infty$, we define $\mathbb{E}[Z | \mathcal{G}] = \sup_{n\in\mathbb{N}} \mathbb{E}[\min\{Z, n\} | \mathcal{G}]$. For a σ-integrable random variable with respect to $\mathcal{G}$, the generalization of the conditional expectation is given in [8, Chapter 4].

Remark 5.6. Part (b) of Lemma 5.1 was introduced by Hubalek et al. (2002) with the additional assumption that the sequence $\{X_t\}_{t>0}$ converges. A further application of the lemma is to prove that the long volatilities, implied by the Black–Scholes formula, cannot fall, which was done by Rogers and Tehranchi (2006).

Example 5.7. Note that $X < X^{\mathcal{G}}$ is possible in (5.2), even for a bounded X. As an example, consider $\Omega = (0,1)$ with Lebesgue measure and Borel σ-algebra, $\mathcal{G} = \{\varnothing, \Omega\}$ and $X(\omega) = \omega$ for $\omega \in \Omega$. Then $\mathbb{E}[X^n|\mathcal{G}]^{1/n} = (n+1)^{-1/n}$ and $X^{\mathcal{G}} = 1$.

Example 5.8. Note that the last inequality in (5.3) can be strict for a bounded sequence $\{X_n\}_{n\in\mathbb{N}}$ which converges everywhere in a monotone way. In the setting of Example 5.7, consider $X_n = 1_{(0,1/n)}$ for $n \in \mathbb{N}$ with pointwise limit $X = 0$, hence $X^{\mathcal{G}} = 0$. However, $\mathbb{E}[X_n^n|\mathcal{G}]^{1/n} = n^{-1/n} \to 1$ as $n \to \infty$.

Proof of Lemma 5.1. (a) Consider $0 < s < t < \infty$. Jensen's inequality for conditional expectations, applied to the convex function $\varphi(x) = x^{t/s}$ implies

$$\mathbb{E}[X^s|\mathcal{G}]^{1/s} = \bigl(\varphi(\mathbb{E}[X^s|\mathcal{G}])\bigr)^{1/t} \le \mathbb{E}[\varphi(X^s)|\mathcal{G}]^{1/t} = \mathbb{E}\bigl[X^t|\mathcal{G}\bigr]^{1/t} \quad \text{a. s.}$$

Due to this monotonicity the almost sure limit $C := \lim_{n\to\infty} \mathbb{E}[X^{t_n}|\mathcal{G}]^{1/t_n}$ exists along every sequence $t_n \nearrow \infty$ and every other sequence gives a. s. the same limit. Note that C is $\mathcal{G}$-measurable. If Z is a $\mathcal{G}$-measurable random variable satisfying $\mathbb{P}(X \le Z) = 1$, then $\mathbb{E}[X^t|\mathcal{G}]^{1/t} \le \mathbb{E}[Z^t|\mathcal{G}]^{1/t} = Z$ a. s. for all $t > 0$.

It remains to show that $X \le C$ a. s., which we do by contradiction. We assume for the set $A := \{X > C\}$ that $\mathbb{P}(A) > 0$. Since $A \subset \{C < \infty\}$ there exist $k \in \mathbb{N}$ with $\mathbb{P}(A \cap \{C \le k\}) > 0$. Furthermore, there exists $l \in \mathbb{N}$ such that $\mathbb{P}(B) > 0$ for $B := \{X \ge C + 1/l,\, C \le k\}$. We obtain

$$\mathbb{E}[X1_B] \ge \mathbb{E}[C1_B] + \mathbb{P}(B)/l > \mathbb{E}[C1_B]\,, \tag{5.9}$$

because $\mathbb{P}(B) > 0$ and $\mathbb{E}[C1_B] \le k\,\mathbb{P}(B) < \infty$. In the remaining part of the proof, we use the convention $\infty \cdot 0 = 0$ for products. Using the conditional Hölder inequality[7] and the fact that $\mathbb{E}[X^n|\mathcal{G}]^{1/n} \le C$ a. s., it follows for all $n \in \mathbb{N}$ that

$$\mathbb{E}[X1_B|\mathcal{G}] \le \mathbb{E}[X^n|\mathcal{G}]^{1/n}\,\mathbb{E}[1_B|\mathcal{G}]^{1-1/n} \le C\,\mathbb{E}[1_B|\mathcal{G}]^{1-1/n} \qquad \text{a. s.}$$

Passing to the limit $n \to \infty$ and using the $\mathcal{G}$-measurability of C,

$$\mathbb{E}[X1_B|\mathcal{G}] \le C\,\mathbb{E}[1_B|\mathcal{G}] = \mathbb{E}[C1_B|\mathcal{G}] \qquad \text{a. s.}$$

Taking expectations gives $\mathbb{E}[X1_B] \le \mathbb{E}[C1_B]$, which is a contradiction to (5.9).

(b) Since $Y_m \le Y_n \le \sup_{k\in\mathbb{N}} Y_k = X$ for all $m, n \in \mathbb{N}$ with $m \le n$, we obtain

$$\mathbb{E}[Y_m^n|\mathcal{G}]^{1/n} \le \mathbb{E}[Y_n^n|\mathcal{G}]^{1/n} \le \mathbb{E}[X^n|\mathcal{G}]^{1/n} \le X^{\mathcal{G}} \qquad \text{a. s.,}$$

using part (a) for the last inequality. Hence, by part (a), for every $m \in \mathbb{N}$,

$$\begin{aligned} Y_m \le Y_m^{\mathcal{G}} = \lim_{n\to\infty} \mathbb{E}[Y_m^n|\mathcal{G}]^{1/n} &\le \sup_{n\in\mathbb{N}} \mathbb{E}[Y_n^n|\mathcal{G}]^{1/n} \\ &\le \lim_{n\to\infty} \mathbb{E}[X^n|\mathcal{G}]^{1/n} = X^{\mathcal{G}} \qquad \text{a. s.} \end{aligned}$$

Therefore,

$$X = \sup_{m\in\mathbb{N}} Y_m \le \sup_{m\in\mathbb{N}} Y_m^{\mathcal{G}} \le \lim_{n\to\infty} \mathbb{E}[Y_n^n|\mathcal{G}]^{1/n} \le X^{\mathcal{G}} \qquad \text{a. s.}$$

[7] For a proof, cf. *Hölder's inequality* at en.wikipedia.org/wiki/, version of April 6, 2008.

Since $\sup_{m\in\mathbb{N}} Y_m^{\mathcal{G}}$ is $\mathcal{G}$-measurable and dominates X a. s., it also dominates $X^{\mathcal{G}}$ a. s., hence the last two inequalities are a. s. equalities.

For all $t > n$ we have $Y_n \le X_t$ a. s. Using Jensen's inequality for conditional expectations

$$\mathbb{E}[Y_n^n\,|\,\mathcal{G}]^{1/n} \le \mathbb{E}\big[Y_n^t\,|\,\mathcal{G}\big]^{1/t} \le \mathbb{E}\big[X_t^t\,|\,\mathcal{G}\big]^{1/t} \qquad \text{a. s.,}$$

hence

$$\mathbb{E}[Y_n^n\,|\,\mathcal{G}]^{1/n} \le \liminf_{t\to\infty} \mathbb{E}\big[X_t^t\,|\,\mathcal{G}\big]^{1/t} \qquad \text{a. s.}$$

Passing to the limit $n \to \infty$ gives the last inequality in (5.3).

(c) Since $X_t \le X^{\mathcal{G}} + \max\{X_t - X^{\mathcal{G}}, 0\}$ for all $t > 0$, the conditional Minkowski inequality and the $\mathcal{G}$-measurability of $X^{\mathcal{G}}$ imply for all $t \ge 1$ that

$$\mathbb{E}\big[X_t^t\,|\,\mathcal{G}\big]^{1/t} \le X^{\mathcal{G}} + \mathbb{E}\big[(\max\{X_t - X^{\mathcal{G}}, 0\})^t\,|\,\mathcal{G}\big]^{1/t} \qquad \text{a. s.}$$

By the assumption and Definition 2.20, the limit inferior of the last term is zero. □

With this lemma we show that the long-term spot rates never fall without the assumption that the spot rates converge.

Proof of Theorem 2.17. (a) In the discrete- and continuous-time case, it is sufficient to show that

$$\liminf_{T\to\infty} P(t,T)^{\frac{1}{T-t}} \le \liminf_{T\to\infty} P(s,T)^{\frac{1}{T-s}} \qquad \text{a. s.} \tag{5.10}$$

by definition of the zero-coupon rate in (2.1) and (2.2), respectively. Note that

$$\lim_{T\to\infty} P(s,t)^{1/(T-t)} = 1. \tag{5.11}$$

Using (5.3) from Lemma 5.1 (with $X_u = P(t, t+u)^{1/u}$ for $u > 0$ and $\mathcal{G} = \mathcal{F}_s$) and afterwards (5.11), it follows that

$$\begin{aligned}\liminf_{T\to\infty} P(t,T)^{\frac{1}{T-t}} &\le \liminf_{T\to\infty} \big(\mathbb{E}_{\mathbb{Q}_{s,t}}\big[P(t,T)\,\big|\,\mathcal{F}_s\big]\big)^{\frac{1}{T-t}} \\ &= \liminf_{T\to\infty} \big(P(s,t)\,\mathbb{E}_{\mathbb{Q}_{s,t}}\big[P(t,T)\,\big|\,\mathcal{F}_s\big]\big)^{\frac{1}{T-t}} \qquad \text{a. s.}\end{aligned} \tag{5.12}$$

For $\varepsilon > 0$ define f_ε: $[0,\infty] \to [0,\infty]$ by

$$f_\varepsilon(x) = \begin{cases} x & \text{for } x \in [0,1], \\ x^{1+\varepsilon} & \text{for } x > 1. \end{cases}$$

Then $x^{1+\delta} \le f_\varepsilon(x)$ for all $x \in [0,\infty)$, uniformly in $\delta \in [0,\varepsilon]$. Using the property in Condition 2.10 and this estimate, we obtain for all $T \ge \max\{T_0, t + (t-s)/\varepsilon\}$

$$\begin{aligned}\big(P(s,t)\,\mathbb{E}_{\mathbb{Q}_{s,t}}\big[P(t,T)\,\big|\,\mathcal{F}_s\big]\big)^{\frac{1}{T-t}} &\le P(s,T)^{\frac{1}{T-s}(1+\frac{t-s}{T-t})} \\ &\le f_\varepsilon\big(P(s,T)^{\frac{1}{T-s}}\big) \qquad \text{a. s.}\end{aligned} \tag{5.13}$$

Since f_ε is continuous and monotone increasing, we obtain with (5.12) that

$$\liminf_{T\to\infty} P(t,T)^{\frac{1}{T-t}} \le f_\varepsilon\Big(\liminf_{T\to\infty} P(s,T)^{\frac{1}{T-s}}\Big) \qquad \text{a. s.}$$

for all $\varepsilon > 0$, which implies (5.10).

(b) This follows from part (a) and the equivalence of the long-term forward and spot rates in Lemma 2.9. □

The following proof combines the ideas from the proofs of Lemma 5.1(c) and Theorem 2.17.

Proof of Theorem 2.21. Since $l(s)$ is $\mathcal{F}_s$-measurable by definition (2.5), Theorem 2.17 implies that $l(s) \le l(t)_{\mathcal{F}_s}$ a. s., hence it remains to show that $l(s) \ge l(t)_{\mathcal{F}_s}$ a. s. According to the definition (2.22) of the limiting annual discount factor V_t and Remark 2.25, it suffices to show that

$$\liminf_{T\to\infty} P(s,T)^{1/(T-s)} \le V_t^{\mathcal{F}_s} \qquad \text{a. s.} \tag{5.14}$$

For $\varepsilon \in (0,1)$ define $g_\varepsilon\colon [0,\infty] \to [0,\infty]$ by

$$g_\varepsilon(x) = \begin{cases} x^{1-\varepsilon} & \text{for } x \in [0,1], \\ x, & \text{for } x > 1. \end{cases}$$

The $x^{1-\delta} \le g_\varepsilon(x)$ for all $x \in [0,\infty)$, uniformly in $\delta \in [0,\varepsilon]$. Using the property in (2.11) and this estimate for $T \ge s + (t-s)/\varepsilon$, we obtain

$$\begin{aligned} P(s,T)^{\frac{1}{T-s}} &= P(s,t)^{\frac{1}{T-s}} \bigl(\mathbb{E}_{\mathbb{Q}_{s,t}}\bigl[P(t,T) \,\big|\, \mathcal{F}_s\bigr]\bigr)^{\frac{1}{T-s}} \\ &\le P(s,t)^{\frac{1}{T-s}} g_\varepsilon\Bigl(\bigl(\mathbb{E}_{\mathbb{Q}_{s,t}}\bigl[P(t,T) \,\big|\, \mathcal{F}_s\bigr]\bigr)^{\frac{1}{T-t}}\Bigr) \qquad \text{a. s.} \end{aligned} \tag{5.15}$$

Using (5.11), Lemma 5.1(c) (with $X_u = P(t,t+u)^{1/u}$ for $u > 0$ and $\mathcal{G} = \mathcal{F}_s$) and (2.26), it follows that

$$\liminf_{T\to\infty} P(s,T)^{1/(T-s)} \le g_\varepsilon\bigl(V_t^{\mathcal{F}_s}\bigr) \qquad \text{a. s.}$$

for every $\varepsilon \in (0,1)$, which implies (5.14). Using Lemma 2.9, the result for the long-term forward rates follows. □

6. Proofs for asymptotic monotonicity and minimality assuming absence of arbitrage in the limit

Proof of Lemma 2.32. Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a real-valued $\mathcal{F}_s$-measurable sequence, and $\{T_n\}_{n\in\mathbb{N}}$ be a sequence of $\mathcal{F}_s$-measurable random maturities $T_n : \Omega \to (n \vee t, \infty)$, each one taking only a finite number of values. Define the sequence $\{\psi_n\}_{n\in\mathbb{N}}$ as in Remark 2.30 to ensure part (a) of Definition 2.29. Then Condition 2.12 implies

$$\mathbb{E}_{\mathbb{Q}_{s,t}}[V_n(t) \,|\, \mathcal{F}_s] \overset{\text{a.s.}}{=} V_n(s)/P(s,t) \overset{\text{a.s.}}{=} 0, \qquad n \in \mathbb{N}. \tag{6.1}$$

Assume part (d) of Definition 2.29, in particular $\liminf_{n\to\infty} V_n(t) \ge 0$ a. s. In addition, there exists $n_1 \in \mathbb{N}$ such that $V_n(t) \ge -1$ a. s. for all $n \ge n_1$. Using Fatou's lemma for conditional expectations and (6.1), we obtain

$$\mathbb{E}_{\mathbb{Q}_{s,t}}\Bigl[\liminf_{n\to\infty} V_n(t) \,\Big|\, \mathcal{F}_s\Bigr] \le \liminf_{n\to\infty} \mathbb{E}_{\mathbb{Q}_{s,t}}[V_n(t) \,|\, \mathcal{F}_s] = 0 \qquad \text{a. s.}$$

So we must have $\liminf_{n\to\infty} V_n(t) \overset{\text{a.s.}}{=} 0$, hence part (b) of Definition 2.29 fails. Hence, there is no arbitrage opportunity in the limit with vanishing risk. □

Proof of Theorem 2.33. We prove asymptotic monotonicity, i. e. $l(s) \le l(t)$ a. s. It suffices to prove $l(s) \le l(t)_{\mathcal{F}_s}$ a. s., which we do by contradiction. Assume for the event $A := \{l(s) > l(t)_{\mathcal{F}_s}\}$ that $\mathbb{P}(A) > 0$. Since $A \subset \{l(s) > -\infty,\, l(t)_{\mathcal{F}_s} < \infty\}$, there exists $k \in \mathbb{N}$ such that $B := \{X > l(t)_{\mathcal{F}_s}\}$ with $X := \min\{k, l(s) - 2/k\}$ satisfies $\mathbb{P}(B) > 0$. Note that X is a real-valued, $\mathcal{F}_s$-measurable random variable and that $B \in \mathcal{F}_s$. Let $\{\tilde{T}_n\}_{n\in\mathbb{N}}$ denote an $\mathcal{F}_s$-measurable sequence satisfying Lemma 2.8, in particular

$$l(s) \overset{\text{a.s.}}{=} \lim_{n\to\infty} R(s,\tilde{T}_n).$$

Without loss of generality we assume that $\tilde{T}_n > t$ for all $n \in \mathbb{N}$. Since $l(s) > X+1/k$ by the definition of X, there exists $m \in \mathbb{N}$ such that

$$C := B \cap \bigcap_{n=m}^{\infty} \Bigl\{R(s,\tilde{T}_n) \geq X + \frac{1}{k}\Bigr\} \tag{6.2}$$

satisfies $\mathbb{P}(C) > 0$. Note that $C \in \mathcal{F}_s$. Define $D = \{X > l(t)\} \cap C$. If $\mathbb{P}(D) = 0$, then $X \leq l(t)$ a. s. on C, hence $X \leq l(t)_{\mathcal{F}_s}$ a. s. on C by the $\mathcal{F}_s$-measurability of C and X. This contradicts the strict inequality in the definition of B, which contains C, hence $\mathbb{P}(D) > 0$.

In the continuous-time case define

$$\varphi_n = 1_C \exp((\tilde{T}_n - s)X)P(s,t) \qquad \text{and} \qquad \psi_n = -1_C \exp((\tilde{T}_n - s)X)P(s,\tilde{T}_n),$$

in the discrete-time case, noting that $X > l(t)_{\mathcal{F}_s} \geq -1$ on C, define

$$\varphi_n = 1_C (X+1)^{\tilde{T}_n - s}P(s,t) \qquad \text{and} \qquad \psi_n = -1_C (X+1)^{\tilde{T}_n - s}P(s,\tilde{T}_n)$$

for all $n \in \mathbb{N}$. Then every (φ_n, ψ_n) is $\mathcal{F}_s$-measurable, the corresponding portfolio value $V_n(s) := \varphi_n P(s,\tilde{T}_n) + \psi_n P(s,t)$ is zero, and $V_n(t) = \varphi_n P(t,\tilde{T}_n) + \psi_n$. In the continuous-time case, using the definition of the zero-coupon rates in (2.2) these summands equal

$$\begin{aligned} \varphi_n P(t,\tilde{T}_n) &= 1_C \exp\bigl((\tilde{T}_n - t)(X - R(t,\tilde{T}_n)) + (t-s)(X - R(s,t))\bigr), \\ \psi_n &= -1_C \exp\bigl((\tilde{T}_n - s)(X - R(s,\tilde{T}_n))\bigr) \end{aligned}$$

for all $n \in \mathbb{N}$. Using the definition of C in (6.2) and $\tilde{T}_n > n$ from Lemma 2.8,

$$1_C \exp\bigl((\tilde{T}_n - s)(X - R(s,\tilde{T}_n))\bigr) \leq \exp\Bigl(-\frac{n-s}{k}\Bigr) \qquad \text{for all } n \geq m.$$

Since $\varphi_n \geq 0$ for all $n \in \mathbb{N}$, part (d) of Definition 2.29 holds. Since $X > l(t)$ on D and $l(t) \geq \limsup_{n\to\infty} R(t,\tilde{T}_n)$ a. s. by the definition of the long-term zero-coupon rate in (2.5), we obtain that

$$\liminf_{n\to\infty} 1_C \exp\bigl((\tilde{T}_n - t)(X - R(t,\tilde{T}_n))\bigr) \overset{\text{a.s.}}{=} \infty \qquad \text{on } D.$$

Therefore, we have an arbitrage opportunity in the limit with vanishing risk for times s and t, which is the desired contradiction. In the discrete-time case, we proceed in a similar way. The result for the long-term forward rates follows by using Lemma 2.9. □

The following proof has some similarities with the preceding one, however, the stronger no-arbitrage assumption from Definition 2.29 is needed, because the downside risk of the constructed portfolios might be unbounded.

Proof of Theorem 2.34. We want to show that the set, where (2.35) is violated, is a $\mathbb{P}$-null set. For this purpose, define the event

$$C = \{Y > l(s)\}, \qquad \text{where} \qquad Y := \Bigl(\liminf_{n\to\infty} R(t,T_n)\Bigr)_{\mathcal{F}_s}, \tag{6.3}$$

and the portfolio compositions

$$\varphi_n = -1_C \frac{P(s,t)}{P(s,T_n)} \qquad \text{and} \qquad \psi_n = 1_C,$$

for all $n \in \mathbb{N}$. Then every (φ_n, ψ_n) is $\mathcal{F}_s$-measurable, the corresponding portfolio value $V_n(s) := \varphi_n P(s,T_n) + \psi_n P(s,t)$ is zero, and $V_n(t) = \varphi_n P(t,T_n) + \psi_n$. In the continuous-time case, the first summand can be rewritten using (2.2) as

$$\varphi_n P(t,T_n) = -1_C \exp\bigl((T_n - t)(R(s,T_n) - R(t,T_n)) + (t-s)(R(s,T_n) - R(s,t))\bigr),$$

for all $n \in \mathbb{N}$. Since $\limsup_{n\to\infty} R(s,T_n) \le l(s) < \infty$ a. s. on C and

$$\limsup_{n\to\infty}\bigl(R(s,T_n) - R(t,T_n)\bigr) \le l(s) - \liminf_{n\to\infty} R(t,T_n) \le l(s) - Y < 0 \quad \text{a. s. on } C,$$

we get $\liminf_{n\to\infty} \varphi_n P(t,T_n) \overset{\text{a.s.}}{=} 0$. Since $\psi_n = 1_C \ge 0$ for all $n \in \mathbb{N}$, this implies part (c) of Definition 2.29. Since an arbitrage opportunity in the limit for times s and t is excluded by assumption, Definition 2.29(b) implies $\mathbb{P}(C) = 0$. In the discrete-time case, using the representation

$$\varphi_n P(t,T_n) = -1_C \left(\frac{1+R(s,T_n)}{1+R(t,T_n)}\right)^{T_n - t} \left(\frac{1+R(s,T_n)}{1+R(s,t)}\right)^{t-s},$$

we conclude in a similar way that $\mathbb{P}(C) = 0$. □

References

1. L. Arnold, *Stochastic Differential Equations: Theory and Applications*, Wiley, New York, 1974.
2. T. Björk, *Arbitrage Theory in Continuous Time*, 2 ed., Oxford University Press, New York, 2004.
3. A. J. G. Cairns, *Interest Rate Models: An Introduction*, Princeton University Press, New Jersey, 2004.
4. J. F. Carriere, *Long-term yield rates for actuarial valuations*, North American Actuarial Journal **3** (1999), no. 3, 13–24.
5. J. C. Cox, J. E. Ingersoll, and S. A. Ross, *A theory of the term-structure of interest rates*, Econometrica **53** (1985), 385–407.
6. P. Dybvig, J. Ingersoll, and S. Ross, *Long forward and zero coupon rates can never fall*, Journal of Business **69** (1996), 1–25.
7. H. Föllmer and A. Schied, *Stochastic Finance. An Introduction in Discrete Time*, 2 ed., Walter de Gruyter, Berlin, New York, 2004.
8. S. He, J. Wang, and J. Yan, *Semimartingale Theory and Stochastic Calculus*, CRC Press, Beijing, 1992.
9. D. Heath, R. Jarrow, and A. Morton, *Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation*, Econometrica **60** (1992), 77–105.
10. F. Hubalek, I. Klein, and J. Teichmann, *A general proof of the Dybvig–Ingersoll–Ross theorem: long forward rates can never fall*, Mathematical Finance **12** (2002), 447–451.
11. J. Hull and A. White, *Pricing interest–rate–derivative securities*, The Review of Financial Studies **3** (1990), 573–592.
12. N. El Karoui, A. Frachot, and H. Geman, *On the behavior of long zero coupon rates in a no arbitrage framework*, Review of Derivatives Research **1** (1998), no. 4, 351–369.
13. J. H. McCulloch, *Long forward and zero-coupon rates indeed can never fall, but are indeterminate: a comment on Dybvig, Ingersoll and Ross*, Working paper, Ohio State University, 2000.
14. M. Musiela and M. Rutkowski, *Martingale Methods in Financial Modelling*, 2 ed., Springer-Verlag, Berlin, Heidelberg, 2005.
15. S. R. Pliska, *Introduction to Mathematical Finance: Discrete Time Models*, Blackwell, Malden, Oxford, 1997.

16. P. E. Protter, *Stochastic Integration and Differential Equations*, 2 ed., Springer-Verlag, New York, 2004.
17. L. C. G. Rogers and M. R. Tehranchi, *The implied volatility surface does not move by parallel shifts*, Working Paper, University of Cambridge, 2006.
18. K. Schulze, *Asymptotic maturity behavior of bond markets*, Working paper, University of Bonn, 2007.
19. O. Vasiček, *An equilibrium characterization of the term structure*, Journal of Financial Economics **5** (1977), 177–188.
20. Y. Yao, *Term structure models: A perspective from the long rate*, North American Journal **3** (1999), 122–138.

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