Dealing with Dangerous Digitals *

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October 25, 2001

Abstract. Options with discontinuous payoffs are generally traded above their theoretical Black–Scholes prices because of the hedging difficulties created by their large delta and gamma values. A theoretical method for pricing these options is to constrain the hedging portfolio and incorporate this constraint into the pricing by computing the smallest initial capital which permits super-replication of the option. We develop this idea for exotic options, in which case the pricing problem becomes one of stochastic control. The high cost of exact super-replication coincides with market price quotations for dangerous derivatives such as reverse knock-out barrier options, which are often higher than their risk-neutral expected payoff (*theoretical value*). This paper illustrates how the theory of leverage constrained pricing can be successfully applied to compute close-to-market option values and serves as a practitioner's guide to derive explicit formulae and compute prices by finite difference methods.

*This article first appeared in *Foreign Exchange Risk*, Risk Publications, London 2001
[†]Research supported by Credit Suisse, Swiss Re and UBS through RiskLab, Switzerland.
[‡]Work supported by grant DMS-01-03814 of the U.S. National Science Foundation.

1 Introduction

The results reported in this paper were motivated by the problem of pricing and hedging a particular exotic option, a call which knocks out in the money. More specifically, we assume a geometric Brownian motion model

$$dS(t) = (r_d - r_f)S(t)\,dt + \sigma S(t)\,dW(t), \quad S(0) > 0, \tag{1}$$

for the exchange rate. The *domestic interest rate* $r_d \in \mathbb{R}$, the *foreign interest rate* $r_f \in \mathbb{R}$, the *volatility* $\sigma > 0$ and the planning horizon $T > 0$ are assumed to be constant. The process $(W(t); 0 \leq t \leq T)$ is a Brownian motion under a probability measure $\mathbb{P}$ which is *risk-neutral,* i.e., is chosen so that the foreign currency has mean rate of return $r \triangleq r_d - r_f$. In an equity model, where S denotes the stock price, we can think of r_f as a continuously paid dividend rate.

Consider an *up-and-out call* whose payoff is $(S(T) - K)^+ I_{\{\max_{0 \leq t \leq T} S(t) < B\}}$ at expiration date T, where the *strike price* K and the *knock-out barrier* B satisfy $0 < K < B$ and I_A denotes the indicator of the generic event A. This call "knocks out" in the money, which makes the implementation of the Black–Scholes hedging strategy difficult because it has large negative delta and gamma values near the barrier near expiration. A trader who is delta-hedging a short position in this option would take large short positions in the foreign currency and make large adjustments to this position.

A common pricing practice for such options is to *move the barrier*. If one prices and hedges the option as if the barrier were some number $B' > B$, then the dangerous region of large negative delta and gamma can be moved above B, and the option will knock out before the exchange rate reaches this region. Of course, the computed price of the option increases with increasing B', and there is no clear procedure for choosing an appropriate value for B'.

The risk of loss and the hedging problem of barrier options have been recognized by trading practitioners as well as academics. There are various ways to limit this risk and this hedging difficulty as for example

(i) Include rebates, see Section 4.2.2.

(ii) Modify the knock-out regulation as follows. The final payoff loses its value at a rate proportional to the total time the exchange rate spends above the barrier. Such barrier options are called *soft barrier options* or *step options* and are discussed, e.g., in [16] or [21].

(iii) Modify the knock-out regulation as follows. The final payoff loses its value only if the exchange rate spends a pre-specified time interval above the barrier without interruption. Such options are called *Parisian barrier options* and are discussed in [7] and [8] and [15].

All of these approaches have one common goal. The value of the option at the barrier is lifted to some positive number to ensure the holder does not face sudden loss of the entire option contract. The seller then has a smaller negative delta and gamma for the hedge. Rather than changing the contract, we address this problem by constraining the size of the short position allowed for hedging a short option position and by incorporating the cost of this constraint into the price of the option. We develop this method for general path-dependent options and show how to extend the methodology of Broadie, Cvitanić and Soner [5] to compute upper hedging prices. The method applies theoretically even to a book of options, although the computational issues for a book can be substantial as indicated in the case of a book of two barrier options in [24].

Our method is based on the idea of *super-replication*, developed by Cvitanić and Karatzas [9] and El Karoui and Quenez [13]. The price of a contingent claim obtained by this method is called the *upper hedging price*, and it is defined in terms of a minimization problem. It is informative to consider the dual problem, which is one of maximization over changes of measure, and the equivalence of the two problems was shown in [9] and [13].

The dual problem of [9] and [13] is not easily solved in the generality of those papers. However, Broadie, Cvitanić and Soner [5] showed that for a contingent claim whose payoff at expiration is a function of the final value of a single, geometric Brownian motion, the dual problem can be solved in two steps. One first computes a certain transform, which we call the *face-lift,* of the payoff function (see (15) and (21) below). One next prices the contingent claim whose payoff at the final time is the face-lifted version of the original payoff. One does this using the usual risk-neutral pricing formula, i.e., without regard to the portfolio constraint.

Schmock, Shreve and Wystup [24] extended this idea to the case of path-dependent options with a lower bound on the hedging portfolio. They provide a reformulation of the dual problem of [9] and [13] so that the solution can often be obtained by inspection.

The role of upper hedging prices in the presence of stochastic volatility and/or transaction costs is studied in [3], [10], [11], [26]. Gamma constraints are treated in [25]. Lower hedging prices are introduced in [17], and [18] treats perpetual American options using similar methodology. Classical Black–Scholes prices for a large number of exotic options are provided by Zhang [28]. Several authors, including [1], [4], [6] and [12], have suggested static hedges for dangerous exotic options. Since exact super-replication is in general too expensive, many authors including Föllmer and Leukert [14] examine hedging strategies which succeed with high probability (*quantile hedging*).

The paper is organized as follows. Section 2 discusses the motivating up-and-out call in more detail. Section 3 sets out the general model and presents a survey of super-replication with leverage constraints. Section 4 is a practitioner's guide to compute closed-form solutions and Section 5 illustrates how to combine finite-difference methods and leverage constraints.

2 Reverse Up-and-Out Call

We return to the up-and-out call of the previous section. If $S(t) = x > 0$ at time $t \in [0, T]$ and the call has not knocked out prior to t, then its value is given by the risk-neutral pricing formula

$$v(t,x) \triangleq \mathbb{E}^{t,x}\big[e^{-r_d(T-t)}(S(T)-K)^+ I_{\{M(t,T)<B\}}\big], \tag{2}$$

where $\mathbb{E}^{t,x}$ denotes expectation with respect to the solution S of (1) with initial value $S(t)=x$, i.e.,

$$S(u) = x\exp\big\{\sigma\big(W(u)-W(t)\big)+\mu(u-t)\big\}, \quad u\in[t,T], \tag{3}$$

with $\mu \triangleq r - \frac{1}{2}\sigma^2$, and

$$M(t,T) \triangleq \max_{u\in[t,T]} S(u), \quad t\in[0,T], \tag{4}$$

denotes the maximum of S during the time interval $[t,T]$ of length $\tau \triangleq T-t$. The joint distribution of the Brownian increment $Y = W(T)-W(t)+\theta(T-t)$ with drift $\theta\in\mathbb{R}$ and its maximum $Z=\max_{s\in[t,T]}(W(s)-W(t)+\theta(s-t))$ over the interval $[t,T]$ can be derived using Girsanov's theorem (see [2, Formula 1.1.8] or [19, Section 3.5]). A density of the joint distribution $\mathbb{P}(Y,Z)^{-1}$ with respect to the two-dimensional Lebesgue measure is given by

$$f_{\theta,\tau}(y,z) = \exp\Big\{-\frac{(2z-y)^2}{2\tau}+\theta y-\frac{1}{2}\theta^2\tau\Big\} \quad \text{if } z>0 \text{ and } y<z, \tag{5}$$

and $f_{\theta,\tau}(y,z)=0$ otherwise.

Let $\mathcal{N}$ denote the standard normal distribution function. Using formula (5), $v(t,x)$ can be computed explicitly (see Figure 1). For $t\in[0,T)$ and $x\in(0,B]$,

$$\begin{aligned} v(t,x) = {} & xe^{-r_f\tau}\big[\mathcal{N}(b-\theta_+)-\mathcal{N}(k-\theta_+)\big] \\ & + xe^{-r_f\tau+2b\theta_+}\big[\mathcal{N}(b+\theta_+)-\mathcal{N}(2b-k+\theta_+)\big] \\ & - Ke^{-r_d\tau}\big[\mathcal{N}(b-\theta_-)-\mathcal{N}(k-\theta_-)\big] \\ & - Ke^{-r_d\tau+2b\theta_-}\big[\mathcal{N}(b+\theta_-)-\mathcal{N}(2b-k+\theta_-)\big], \end{aligned} \tag{6}$$

where

$$b \triangleq \frac{1}{\sigma\sqrt{\tau}}\log\frac{B}{x}, \quad k \triangleq \frac{1}{\sigma\sqrt{\tau}}\log\frac{K}{x}, \quad \text{and} \quad \theta_\pm \triangleq \Big(\frac{r}{\sigma}\pm\frac{\sigma}{2}\Big)\sqrt{\tau}. \tag{7}$$

Definition (2) implies that $v(t,B)=0$ for $t\in[0,T]$. For $x\in(0,B]$, as $t\uparrow T$, we obtain from (6) that $v(t,x)$ approaches the discontinuous limit $v(T,x) = (x-K)^+ I_{\{x<B\}}$. Consequently, for x near B and t near T, the "delta" $v_x(t,x)$ and "gamma" $v_{xx}(t,x)$ of this option become large in absolute value as illustrated in Figure 1.

3 Model Formulation and Survey of Super-replication under Leverage Constraints

Throughout this paper, we work within the context of the canonical probability space for Brownian motion. In particular, we take Ω to be the set of continuous functions from $[0,T]$ to $\mathbb{R}$ taking the value zero at zero, we take $\mathbb{P}$ to be Wiener measure, and we take $W(t,\omega) = \omega(t)$ for all $t \in [0,T]$ and all $\omega \in \Omega$. For $0 \le t \le T$, we denote by $\mathcal{F}^W(t)$ the σ-algebra generated by $(W(s); 0 \le s \le t)$. The σ-algebra $\mathcal{F}(T)$ is the $\mathbb{P}$-completion of $\mathcal{F}^W(T)$, and for $0 \le t \le T$, $\mathcal{F}(t)$ is the augmentation of $\mathcal{F}^W(t)$ by the $\mathbb{P}$-null sets of $\mathcal{F}(T)$.

We introduce a contingent claim whose payoff at expiration date T is $g(S(\cdot))$. Let $C_+[0,T]$ denote the space of nonnegative continuous functions on $[0,T]$. We assume that the nonnegative function $g\colon C_+[0,T] \to [0,\infty)$ is lower semicontinuous in the supremum norm topology. The argument of g is the path of the exchange rate process S from date 0 to date T, and because this path is random, $g(S(\cdot))$ is a random variable on $(\Omega, \mathcal{F}(T), \mathbb{P})$.

The problem of super-replication of a short position in this option can be posed as follows. Let $X(0) > 0$ be a given nonrandom *initial wealth,* and choose an $(\mathcal{F}(t); 0 \le t \le T)$-adapted *portfolio process* $(\pi(t); 0 \le t \le T)$ and *cumulative consumption process* $(C(t); 0 \le t \le T)$. We interpret $\pi(t)$ as the proportion of wealth invested in the foreign currency at time t earning the continuous interest rate r_f. It is sometimes called the *gearing* or *leverage*. The remaining wealth is invested at domestic interest rate r_d, and $C(t)$ is the amount of wealth consumed up to time t. Hence, $C(t)$ is nondecreasing, right-continuous with left limits, and $C(0) = 0$. This leads us to model the differential of wealth as

$$\begin{aligned} dX(t) &= \pi(t)X(t)\frac{dS(t)}{S(t)} + r_f\pi(t)X(t)\,dt + r_d(1-\pi(t))X(t)\,dt - dC(t) \\ &= r_dX(t)\,dt + \sigma\pi(t)X(t)\,dW(t) - dC(t). \end{aligned} \tag{8}$$

If $X(T) \ge g(S(\cdot))$ almost surely (a.s.), we say that (π, C) *super-replicates* $g(S(\cdot))$ beginning with initial wealth $X(0)$. As in [20, Definition 2.2, p. 263], we do not impose any integrability condition on π, instead setting $X \equiv 0$ after any explosion of $\int_0^t \pi^2(s)\,ds$.

Next, given some fixed number $\alpha \in [0,\infty)$, we impose the *portfolio constraint*

$$\pi(t) \ge -\alpha, \quad 0 \le t \le T, \text{ a.s.} \tag{9}$$

The point of this constraint, in the context of the up-and-out call of the previous section, is to avoid short positions which are too large relative to the value of the contingent claim being hedged. The parameter α must be chosen by the person pricing the contingent claim; if $\alpha = 0$, then short positions in the underlying are prohibited.

The *upper hedging price* of the contingent claim $g(S(\cdot))$ is defined to be

$$v(0,S(0);\alpha) \triangleq \inf\left\{ X(0) \,\middle|\, \begin{array}{l}\text{There exist } \pi \text{ satisfying (9)} \\ \text{and } C \text{ such that } X(T) \ge g(S(\cdot)) \text{ a.s.}\end{array} \right\}. \tag{10}$$

Cvitanić and Karatzas [9] have shown that when $v(0,S(0);\alpha)$ is finite, there exists an $X(0)$, denoted $\widehat{X}(0)$, and corresponding portfolio and consumption processes $\hat{\pi}$ and $\widehat{C}$ attaining the infimum in (10). In the case of the up-and-out call option of the previous section, for each time $t \in [0,T]$ and exchange rate $S(t) \in (0, B)$, there is an upper hedging price $v(t, S(t); \alpha)$ computed under the assumption that the option has not been knocked out. For this option, the corresponding hedge portfolio process π given by Cvitanić and Karatzas [9] is

$$\pi(t) = \frac{S(t)\, v_x(t, S(t); \alpha)}{v(t, S(t); \alpha)}, \quad t \in [0, T], \tag{11}$$

(with $\pi(T)$ unspecified if $S(T) = K$) and the portfolio constraint (9) implies

$$\alpha v(t, x; \alpha) + x v_x(t, x; \alpha) \geq 0 \tag{12}$$

for all $(t, x) \in [0, T] \times (0, B) \setminus \{(T, K)\}$.

We denote the corresponding wealth process by $\widehat{X}$ and define the *upper hedging price at time* $t \in [0, T)$ of the contingent claim $g(S(\cdot))$ to be $\widehat{X}(t)$. Since the upper hedging price $\widehat{X}(t)$ includes a "reserve" to offset the portfolio constraint (9), it generally exceeds the risk-neutral price $\mathbb{E}[e^{-r_d(T-t)} g(S(\cdot)) \,|\, \mathcal{F}(t)]$. During the evolution of the process, some part of this reserve might be revealed to be unnecessary. The process $\widehat{C}$ is included in the formulation of the upper hedging price so that an unnecessary reserve can be removed and is thus no longer included in the upper hedging price.

Cvitanić and Karatzas [9] and El Karoui and Quenez [13] have shown that the problem of computing the upper hedging price, which is a minimization problem, can be transformed to a dual maximization problem. Their results apply to path-dependent contingent claims written on multiple currencies whose models may have random, time-varying volatilities, and they require only that π be constrained to lie in a closed, convex set. The dual problem is one of maximization over changes of probability measure, and in its full generality is not easy to solve. In our model, the dual problem takes the form of (13) below.

Theorem 1 (Cvitanić and Karatzas, El Karoui and Quenez) *The upper hedging price of* (10) *satisfies*

$$v(0, S(0); \alpha) = \sup_{\lambda} \mathbb{E}_\lambda\big[e^{-r_d T - \alpha\lambda(T)} g(S(\cdot))\big], \tag{13}$$

where the supremum is over all adapted, nondecreasing, processes which are Lipschitz continuous in t, uniformly in ω, and satisfy $\lambda(0) = 0$. Here $\mathbb{E}_\lambda$ denotes expectation under the probability measures $\mathbb{P}_\lambda$ whose Radon–Nikodým derivative with respect to $\mathbb{P}$ is

$$\frac{d\mathbb{P}_\lambda}{d\mathbb{P}} = \exp\left\{-\frac{1}{\sigma}\int_0^T \lambda'(t)\, dW(t) - \frac{1}{2\sigma^2}\int_0^T (\lambda'(t))^2 dt\right\}. \tag{14}$$

The supremum in (13) over Lipschitz continuous processes is often not attained, and Lipschitz continuity is not easily relaxed in Theorem 1 because of the need to define $\mathbb{P}_\lambda$ by (14).

Broadie, Cvitanić and Soner [5] specialized Theorem 1 to the case of a contingent claim whose payoff at expiration is a function of the final value of a single, geometric Brownian motion. A presentation of the results of both [9] and [5] in full generality may be found in [20].

Theorem 2 (Broadie, Cvitanić and Soner) *Let $\varphi\colon [0,\infty) \to [0,\infty)$ be lower semicontinuous, and suppose the contingent claim $g(S(\cdot))$ is given by $g(S(\cdot)) = \varphi(S(T))$. Define the "face-lifted" payoff function*

$$\widehat{\varphi}_\alpha(x) \triangleq \sup_{\lambda\geq 0} e^{-\alpha\lambda}\varphi(xe^{-\lambda}), \quad x \geq 0. \tag{15}$$

Then the upper hedging price under hedge-portfolio constraint (9) is given by

$$v(0, S(0); \alpha) = \mathbb{E}\big[e^{-r_d T}\widehat{\varphi}_\alpha(S(T))\big]. \tag{16}$$

The idea behind Theorem 2 is that the upper hedging price corresponds to the smallest function $[0,T] \times (0,\infty) \ni (t,x) \mapsto v(t,x;\alpha)$ which

(i) satisfies the Black–Scholes partial differential equation

$$v_t(t,x;\alpha) + rxv_x(t,x;\alpha) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x;\alpha) - r_d v(t,x;\alpha) = 0 \tag{17}$$

on $[0,T) \times (0,\infty)$,

(ii) dominates the final payoff, i.e., satisfies $v(T,x;\alpha) \geq \varphi(x)$ for $x > 0$, and

(iii) satisfies the portfolio constraint (12) on $[0,T) \times (0,\infty)$.

Since there is no guarantee that φ satisfies the constraint (12), we do not expect $v(T,\cdot;\alpha)$ to agree with φ. The function $\widehat{\varphi}_\alpha$ is the smallest function dominating φ and satisfying (12), namely

$$\alpha\widehat{\varphi}_\alpha(x) + x\frac{d}{dx}\widehat{\varphi}_\alpha(x) \geq 0 \quad \text{for all } x > 0, \tag{18}$$

cf. Subsection 4.2.1, hence $v(T,\cdot;\alpha)$ must be at least as large as $\widehat{\varphi}_\alpha$. If we set $v(T,\cdot;\alpha) \triangleq \widehat{\varphi}_\alpha$ and solve the Black–Scholes partial differential equation (17) on $[0,T) \times (0,\infty)$, then v satisfies (i) and (ii). The pleasant surprise is that v also satisfies (iii). This is because (i) and a bit of calculus shows that $(t,x) \mapsto \alpha v(t,x;\alpha) + xv_x(t,x;\alpha)$ satisfies the Black–Scholes equation (17) on $[0,T) \times (0,\infty)$, and since $\alpha v(T,x;\alpha) + xv_x(T,x;\alpha) \geq 0$ for all $x > 0$ by (18), we have also (12) on $[0,T) \times (0,\infty)$ by the maximum principle, see Appendix A. Put another way, we may regard $\alpha v(t,S(t);\alpha) + S(t)\,v_x(t,S(t);\alpha)$ as the price of a derivative security at time $t \in [0,T)$ with underlying S and nonnegative final payoff given by the left-hand side of (18). This price must be nonnegative at all times prior to expiration.

Schmock, Shreve and Wystup [24] formulated the dual problem (13) in such a way that no change of measure is required, and they could then extend the class of processes over which the supremum in the dual problem is computed. Their goal was to extend Theorem 2 to path-dependent options. Their main result is that in place of the "face-lifting" procedure (15), one must solve a singular stochastic control problem. This problem can sometimes be solved by inspection, and in particular, such a solution is possible for the up-and-out call option of the previous section. The solution of the stochastic control problem leads directly to a formula for the upper hedging price, in the spirit of (16).

The work by Schmock, Shreve and Wystup [24] is more general than [5] in that it allows path-dependent options, but more special in that the only portfolio constraint considered there is (9), whereas [5] permits a general convex constraint on π. Schmock, Shreve and Wystup [24] converted the computation of the supremum on the right-hand side of (13) to a singular stochastic control problem.

These authors also offered two interpretations for the parameter α. One of these was that $1/\alpha$ could be understood as the proportional transaction cost which the trader of the short option position incurs to close out his position in the foreign currency upon knock-out of the option. Taking this transaction cost into account when pricing the option, one attains the upper hedging price [24, Remark 5.4]. The other is related to moving the barrier. If one prices an option by the usual Black–Scholes method leading to (6), but with a knock-out barrier at $(1 + 1/\alpha)B$ rather than at B, then one obtains a price close to the upper hedging price with portfolio constraint (9) for the option with barrier at B, see [24, Remark 5.3].

Theorem 2 is a special case of a more general result of Broadie, Cvitanić and Soner [5]. Another special case is obtained by an upper portfolio constraint

$$\pi(t) \le \alpha, \quad 0 \le t \le T, \text{ a.s.} \tag{19}$$

with a nonnegative number α. In this case, the analogue of (12) is

$$\alpha v(t, x; \alpha) - x v_x(t, x; \alpha) \ge 0 \tag{20}$$

on $[0, T) \times (0, \infty)$ and (15) is replaced by

$$\widehat{\varphi}_\alpha(x) \triangleq \sup_{\lambda \ge 0} e^{-\alpha\lambda} \varphi\big(xe^{\lambda}\big), \quad x \ge 0. \tag{21}$$

4 Analytical Solutions

In this section we provide analytical formulae for the upper hedging prices of digital options, reverse barrier options and one-touch digital options. To model leverage constraints we will always take (9) for shortselling constraints and (19) for borrowing constraints with $\alpha \ge 0$. The first example of digital options is a straightforward application of face-lifting Theorem 2. As noted by Broadie, Cvitanić and Soner [5], who work out the example of a lookback option, this type of face-lifting can be extended to path-dependent options. These extensions are the subject of this paper. In particular, reverse barrier options and one-touch digital options are presented in Subsections 4.2 and 4.3, respectively.

4.1 Digital Options

Digital options, also called binary options, have the path-independent payoff

$$g(S) = \varphi(S_T) = I_{\{\phi S_T > \phi K\}}, \tag{22}$$

where K denotes the strike and ϕ is a binary variable taking the value $\phi = +1$ for a digital call and $\phi = -1$ for a digital put. We choose $\alpha \geq 0$ and impose the natural constraint $\alpha - \phi\pi(t) \geq 0$ for all $t \in [0, T]$, which leads to (cf. (12) and (20))

$$\alpha v(t, x; \alpha) - \phi x v_x(t, x; \alpha) \geq 0 \quad \text{on } [0, T) \times (0, \infty). \tag{23}$$

We then compute the constrained value function v by the face-lifting method of Theorem 2. Using (15) or (21), respectively, we obtain

$$\widehat{\varphi}_\alpha(x) = I_{\{\phi x > \phi K\}} + \left(\frac{x}{K}\right)^{\phi\alpha} I_{\{\phi x \leq \phi K\}}, \quad x \geq 0. \tag{24}$$

Evaluating the right-hand side of (16) using (3) and (24), we get

$$\begin{aligned} v(t, x; \alpha) &= \mathbb{E}^{t,x}\big[e^{-r_d\tau}\widehat{\varphi}_\alpha(S(T))\big] \\ &= e^{-r_d\tau}\Bigg[\int_{\{\phi x e^{\sigma\sqrt{\tau}y + \sigma\sqrt{\tau}\theta_-} > \phi K\}} \mathcal{N}'(y)\,dy \\ &\qquad + \int_{\{\phi x e^{\sigma\sqrt{\tau}y + \sigma\sqrt{\tau}\theta_-} \leq \phi K\}} \left(\frac{x e^{\sigma\sqrt{\tau}y + \sigma\sqrt{\tau}\theta_-}}{K}\right)^{\phi\alpha} \mathcal{N}'(y)\,dy\Bigg] \\ &= e^{-r_d\tau}\mathcal{N}(\phi d_-) + h(t, x; \alpha) \end{aligned} \tag{25}$$

for all $t \in [0, T)$ and $x > 0$, using the notation $\tau \triangleq T - t$,

$$h(t, x; \alpha) \triangleq e^{-r_d\tau}\left(\frac{x}{K}\right)^{\phi\alpha} e^{\phi\alpha\theta_-\sigma\sqrt{\tau} + \frac{1}{2}\alpha^2\sigma^2\tau}\mathcal{N}(-\phi d_- - \alpha\sigma\sqrt{\tau}),$$

$d_- \triangleq \theta_- - k$ and k, θ_- as in (7). The *danger-supplement* at time $t \in [0, T)$ is given by $h(t, S(t); \alpha)$.

4.2 Reverse Barriers

4.2.1 The Up-and-Out Call

We will now price an up-and-out call option subject to the shortselling constraint (9), which implied (12) on $[0, T] \times (0, B) \setminus \{(T, K)\}$. Before we proceed, let us understand the relation between the shortselling constraint and the face-lifting equation (15) on an intuitive level. Given a path-independent payoff $g(S) = \varphi(S_T) \geq 0$ with a differentiable φ, we want to compute the face-lifted $\widehat{\varphi}_\alpha$ as defined in (15). To do that, we need to maximize the real-valued function $[0, \infty) \ni \lambda \mapsto f(\lambda, x) \triangleq e^{-\alpha\lambda}\varphi(xe^{-\lambda})$ for every $x > 0$. Assume that there exists a differentiable function $x \mapsto \lambda(x) > 0$ such that $\lambda(x)$ maximizes $f(\cdot, x)$ for every $x > 0$, meaning that $\widehat{\varphi}_\alpha(x) = f(\lambda(x), x)$. By the first order condition,

$$0 = f_\lambda(\lambda(x), x) = -\alpha f(\lambda(x), x) - x f_x(\lambda(x), x), \quad x > 0.$$

Since $\widehat{\varphi}'_\alpha(x) = f_\lambda(\lambda(x), x)\lambda'(x) + f_x(\lambda(x), x)$, we obtain

$$\alpha\widehat{\varphi}_\alpha(x) + x\widehat{\varphi}'_\alpha(x) = 0, \quad x > 0, \tag{26}$$

which is (18) with equality. Hence, we see that the shortselling constraint (12) is imposed *with equality at the* final *boundary* of the region where v must satisfy the Black–Scholes partial differential equation (17). One can check that if $(t,x) \mapsto v(t,x;\alpha)$ satisfies the Black–Scholes equation, then the function $(t,x) \mapsto \alpha v(t,x;\alpha)+xv_x(t,x;\alpha)$ also satisfies the Black–Scholes equation (assuming enough differentiability). It is now a consequence of the *maximum principle* (see Appendix A) that the shortselling constraint (12) holds inside this region as well, but not necessarily with equality. The reason why the shortselling constraint is imposed with equality at the final time is to get the minimality of the value function.

This intuition leads to the following idea to determine the constrained value function v of the up-and-out call option. We impose the shortselling constraint *with equality on the boundary* of the region where the option is defined and where the unconstrained value function violates the shortselling constraint. Tedious calculation using (6) shows that this is the case for $(t,x) \in [0,T) \times \{B\}$. Hence we want to solve the Black–Scholes partial differential equation (17) on $[0,T) \times (0,B)$ subject to the boundary conditions

$$\alpha v(t,B;\alpha) + Bv_x(t,B;\alpha) = 0 \quad \forall\, t \in [0,T), \tag{27}$$

$$v(t,0;\alpha) = 0 \quad \forall\, t \in [0,T], \tag{28}$$

$$v(T,x;\alpha) = (x-K)^+ I_{\{x<B\}} \quad \forall\, x \in [0,B], \tag{29}$$

and then extend v to a function on $[0,T] \times [0,\infty)$ by setting $v(t,x;\alpha) \triangleq 0$ for all $t \in [0,T]$ and $x > B$. In (27) we mean the one-sided derivative

$$v_x(t,B;\alpha) \triangleq \lim_{h \downarrow 0} \frac{v(t,B;\alpha) - v(t,B-h;\alpha)}{h}.$$

Note that (27) makes the difference to the unconstrained case. We claim that the solution v is the upper hedging price of the constrained up-and-out call at time t if $S(t) = x$. To see this we define $M(t,T)$ by (4) and the value of an *auxiliary contingent claim* by

$$w(t,x;\alpha) \triangleq \mathbb{E}^{t,x}\Big[e^{-r_d(T-t)}\big[(1+\alpha)S(T) - \alpha K\big] I_{\{S(T)\geq K\}} I_{\{M(t,T)<B\}}\Big] \tag{30}$$

for $(t,x) \in [0,T] \times (0,\infty)$. The method for finding the auxiliary value function w is to consider the function $(t,x) \mapsto \alpha v(t,x;\alpha) + xv_x(t,x;\alpha)$. We would like to *define* this to be the auxiliary value function w. The problem is that v is yet to be computed, whence we cannot use it to define w. Instead, we use the desired identity

$$w(t,x;\alpha) = \alpha v(t,x;\alpha) + xv_x(t,x;\alpha) \tag{31}$$

only to compute terminal and boundary conditions for w and try to identify the auxiliary contingent claim w. *Then* we solve for each t the *ordinary* differential equation (31) to obtain a *candidate* for v in terms of w. Finally we have to verify that the value function v solves (17) on $[0,T) \times (0,B)$ and has the properties (27)–(29).

We will now see in an example how this procedure works in detail. Define the first hitting time $\tilde{\tau} \triangleq T \wedge \inf\{t \geq 0 : S(t) = B\}$. We list some properties of w which follow immediately from its definition (30).

(i) $\left\{e^{-r_d(t\wedge\tilde{\tau})}w(t \wedge \tilde{\tau}, S(t \wedge \tilde{\tau}); \alpha)\right\}_{t\geq 0}$ is a martingale, and therefore

(ii) w satisfies the Black–Scholes differential equation (17) on $[0, T) \times (0, B)$.

(iii) $0 \leq w(t, x; \alpha) \leq e^{-r_f(T-t)}(1 + \alpha)x$ for all $t \in [0, T]$ and $x > 0$ and thus we obtain a continuous extension of w by $w(t, 0; \alpha) \triangleq 0$ for $t \in [0, T]$.

(iv) $w(T, x; \alpha) = [(1 + \alpha)x - \alpha K]I_{\{x\geq K\}}I_{\{x<B\}}$ for all $x \geq 0$.

(v) w is nonnegative on $[0, T] \times [0, B]$ and also continuous there with the exception of the two points (T, B) and (T, K).

(vi) $w(t, x; \alpha) = 0$ for all $t \in [0, T]$ and $x \geq B$.

Now we can define, for all $(t, x) \in [0, T] \times [0, B]$,

$$\begin{aligned} v(t, x; \alpha) &\triangleq \int_0^1 y^{\alpha-1} w(t, xy; \alpha)\, dy \\ &= x^{-\alpha} \int_0^x z^{\alpha-1} w(t, z; \alpha)\, dz \quad \text{(if } x > 0\text{)} \end{aligned} \tag{32}$$

and list properties of v which follow from the definition (32):

(i) v satisfies the Black–Scholes differential equation (17) on $[0, T) \times (0, B)$.

(ii) $0 \leq v(t, x; \alpha) \leq e^{-r_f(T-t)}x$ for all $t \in [0, T]$ and $x \geq 0$, in particular $v(t, 0; \alpha) = 0$ for $t \in [0, T]$.

(iii) $\alpha v(t, x; \alpha) + xv_x(t, x; \alpha) = w(t, x; \alpha)$ for all $t \in [0, T)$ and $x \in (0, B)$ and therefore by left-continuity

(iv) $\alpha v(t, B; \alpha) + Bv_x(t, B; \alpha) = 0$ for all $t \in [0, T)$.

(v) $v(T, x; \alpha) = (x - K)^+$ for all $x \in [0, B]$.

(vi) v is continuous on $[0, T] \times [0, B]$.

(vii) $\lim_{x\downarrow 0} xv_x(t, x) = 0$ for all $t \in [0, T]$.

(viii) $v(t, x; \alpha) > v(t, x; \infty)$ (follows from the maximum principle).

(ix) $\lim_{\alpha\to\infty} v(t, x; \alpha) = v(t, x)$, defined by (2), as we would expect.

In particular we learn that this v solves the Black–Scholes partial differential equation (17) subject to the boundary conditions (27) and (28) and the terminal condition (29), and in addition $\pi(t, x) = xv_x(t, x; \alpha)/v(t, x; \alpha)$ super-replicates the payoff of an up-and-out call option and satisfies the shortselling constraint (9) during its lifetime.

We will now demonstrate that the function v derived above is the smallest function which super-replicates the payoff of an up-and-out call and satisfies the Black–Scholes partial differential equation and the shortselling constraint, which will complete the argument that $v(t, x; \alpha)$ is the upper hedging price. To do this, we show that any other function $\tilde{v}$, which satisfies

- the Black–Scholes partial differential equation (17) on $[0, T) \times (0, B)$,
- $\tilde{v}(T, x; \alpha) = v(T, x; \alpha)$ for $x \in [0, B]$,
- and the constraint $\alpha\tilde{v}(t, x; \alpha) + x\tilde{v}_x(t, x; \alpha) \geq 0$ for $t \in [0, T)$ and $x \in [0, B]$, where we take one-sided derivatives at the endpoints 0 and B,

cannot be less than $v(t, x; \alpha)$. Since $\tilde{v}$ also satisfies the shortselling constraint at the barrier, but perhaps not with equality, let

$$g(t) \triangleq \alpha\tilde{v}(t, B; \alpha) + B\tilde{v}_x(t, B; \alpha), \quad t \in [0, T), \tag{33}$$

for some nonnegative function g. Then $\tilde{v}$ can be characterized in the same way as v, namely by defining

$$\tilde{v}(t, x; \alpha) \triangleq \int_0^1 y^{\alpha-1}\tilde{w}(t, xy; \alpha)\, dy \tag{34}$$

where

(i) $\tilde{w}$ satisfies the Black–Scholes equation (17) on $[0, T) \times (0, B)$,

(ii) $\tilde{w}(T, x; \alpha) = w(T, x; \alpha)$ for $x \in [0, B]$,

(iii) $\tilde{w}(t, 0; \alpha) = w(t, 0; \alpha) = 0$ for $t \in [0, T]$,

(iv) $\tilde{w}(t, B; \alpha) = g(t) \geq 0 = w(t, B; \alpha)$ for $t \in [0, T)$.

As before we conclude that

(i) $\tilde{v}$ satisfies the Black–Scholes equation (17) on $[0, T) \times (0, B)$,

(ii) $\tilde{v}(T, x; \alpha) = v(T, x; \alpha)$ for $x \in [0, B]$,

(iii) $\tilde{v}(t, 0; \alpha) = v(t, 0; \alpha) = 0$ for $t \in [0, T]$,

(iv) $\alpha\tilde{v}(t, x; \alpha) + x\tilde{v}_x(t, x; \alpha) = \tilde{w}(t, x; \alpha)$ for $t \in [0, T)$ and $x \in [0, B]$ and hence

(v) $\alpha\tilde{v}(t, B; \alpha) + B\tilde{v}_x(t, B; \alpha) = \tilde{w}(t, B; \alpha) = g(t)$ for $t \in [0, T)$.

Since $\tilde{w} \geq w$ by the maximum principle (see Appendix A), we can deduce

$$\tilde{v}(t, x, \alpha) = \int_0^1 y^{\alpha-1}\tilde{w}(t, xy; \alpha) dy \geq \int_0^1 y^{\alpha-1} w(t, xy; \alpha) dy = v(t, x; \alpha). \tag{35}$$

Notice that $\tilde{w}$ can be viewed as an auxiliary up-and-out option with rebate $g(t)$, whereas w does not have a rebate. The option with the rebate must be worth at least as much as the option without the rebate. This is the maximum principle in terms of finance.

We conclude that $v(t, x; \alpha)$ is the upper hedging price at time $t \in [0, T]$ if $x = S(t) < B$. For $x = S(t) \geq B$ the upper hedging price is clearly zero, because the option is knocked out.

In the following, we will use definition (32) to compute v explicitly. To do this, we need w first. By definition, we know that $w(\cdot, x; \alpha) = 0$ for $x \geq B$. To find $w(t, x; \alpha)$ for $(t, x) \in [0, T) \times (0, B)$, we use the joint density $f_{\theta_-,\tau}$ from (5) for the random pair of a final time value and the running maximum of a Brownian motion with drift θ_- and compute the expected value (30) as an integral

$$\begin{aligned} w(t, x; \alpha) = e^{-r_d \tau} & \int_k^b \int_{0 \vee y}^b [(1+\alpha) x e^{\sigma y} - \alpha K] f_{\theta_-,\tau}(y, z) \, dz \, dy \\ &= (1+\alpha) x e^{-r_f \tau} \left[\mathcal{N}(b - \theta_+) - \mathcal{N}(k - \theta_+)\right] \\ &\quad + (1+\alpha) x e^{-r_f \tau + 2b\theta_+} \left[\mathcal{N}(b + \theta_+) - \mathcal{N}(2b - k + \theta_+)\right] \\ &\quad - \alpha K e^{-r_d \tau} \left[\mathcal{N}(b - \theta_-) - \mathcal{N}(k - \theta_-)\right] \\ &\quad - \alpha K e^{-r_d \tau + 2b\theta_-} \left[\mathcal{N}(b + \theta_-) - \mathcal{N}(2b - k + \theta_-)\right], \end{aligned} \tag{36}$$

where $\tau \triangleq T - t$ and b, k, $\theta_\pm$ are given by (7). Finally, it turns out that the integration of definition (32) needed to find the constrained value function v can be performed as well, and the result is given in equation (52). See Figure 1 for a graph of v.

4.2.2 Rebates

Rebates are discussed in most of the finance literature on barrier options. For a knock-in option a rebate agreement means that a sum R is paid at expiration by the seller of the option to the holder of the option if the option failed to knock in during its lifetime. For a knock-out option a rebate agreement means that a sum R is paid by the seller of the option to the holder of the option, if the option knocks out. There are two kinds of agreements in the knock-out case: (a) The rebate can be paid at expiration T, in which case the boundary condition of the Black–Scholes differential equation is $v(t, B) = Re^{-r_d(T-t)}$, or (b) the rebate can be paid at the first time τ the barrier is hit, in which case the corresponding boundary condition becomes $v(t, B) = R$. Both types can be viewed as an approximation to the function $v(t, B; \alpha)$. However, traded barrier options are normally sold without any rebate agreements, mainly because options without rebate are cheaper than options with rebate, and secondly because a rebate is actually just a path-dependent digital option which can be separated easily from the barrier option and will be sold separately, if the need really occurs. In any case, including such rebate features often makes delta hedging easier, which could be one of the reasons they were invented. The particular choice of the rebate $v(t, B; \alpha)$ is actually in favor of both the seller as well as the holder of the option. It is favorable for the seller, because it is exactly the kind of rebate one should specify in order to obey the portfolio constraint. It is favorable for the holder, because first of all she has an insurance against loss of her entire position, and secondly this insurance is cheap for times long before expiration. Of course, if an option knocks out in the money at a time near expiration the loss for the holder is substantially larger than at earlier times, and that is why a rebate $v(t, B; \alpha)$ is a suitable way to cover the holder's risk exposure.

4.2.3 Joint Formulae for all Reverse Barriers

There are four types of reverse barrier options:

(i) $(\phi = 1, \eta = 1)$: the down-and-out call,

(ii) $(\phi = 1, \eta = -1)$: the up-and-out call,

(iii) $(\phi = -1, \eta = 1)$: the down-and-out put,

(iv) $(\phi = -1, \eta = -1)$: the up-and-out put.

Since their analysis is similar to the up-and-out call, we just list the results covering all four types. The suitable constraints are $\pi \geq -\alpha$ for $\eta = -1$ and $\pi \leq \alpha$ for $\eta = 1$. The auxiliary value function $w(t, x; \alpha)$ satisfies the Black–Scholes partial differential equation, the boundary condition $w(t, B; \alpha) = 0$ and the terminal condition

(i) down-and-out call:

$$w(T, x; \alpha) = [(\alpha - 1)x - \alpha K] I_{\{x \geq (\frac{\alpha}{\alpha-1} K) \vee B\}} \tag{37}$$

(ii) up-and-out call:

$$w(T, x; \alpha) = [(\alpha + 1)x - \alpha K] I_{\{K \leq x < B\}} \tag{38}$$

(iii) down-and-out put:

$$w(T, x; \alpha) = [\alpha K - (\alpha - 1)x] I_{\{B < x \leq K\}} \tag{39}$$

(iv) up-and-out put:

$$w(T, x; \alpha) = [\alpha K - (\alpha + 1)x] I_{\{x \leq (\frac{\alpha}{\alpha+1} K) \wedge B\}}. \tag{40}$$

The solution is

$$\begin{aligned} w(t, x; \alpha) = (\alpha - \eta) x e^{-r_f \tau} &\Big[\frac{\phi - \eta}{2} \mathcal{N}(\phi(b - \theta_+)) + \eta \mathcal{N}(-\eta(k - \theta_+))\Big] \\ &+ (\alpha - \eta) x e^{-r_f \tau} e^{2b\theta_+} \Big[\frac{\phi - \eta}{2} \mathcal{N}(\phi(b + \theta_+)) - \phi \mathcal{N}(\phi(l + \theta_+))\Big] \\ &- \alpha K e^{-r_d \tau} \Big[\frac{\phi - \eta}{2} \mathcal{N}(\phi(b - \theta_-)) + \eta \mathcal{N}(-\eta(k - \theta_-))\Big] \\ &- \alpha K e^{-r_d \tau} e^{2b\theta_-} \Big[\frac{\phi - \eta}{2} \mathcal{N}(\phi(b + \theta_-)) - \phi \mathcal{N}(\phi(l + \theta_-))\Big], \end{aligned} \tag{41}$$

where $\tau \triangleq T - t$, b and $\theta_\pm$ are given by (7), and

$$k \triangleq \begin{cases} \frac{1}{\sigma\sqrt{\tau}} \ln \frac{K}{x} & \text{if } \phi\eta = -1, \\ \frac{1}{\sigma\sqrt{\tau}} \ln\left(\frac{\eta}{x} \max[\eta B, \frac{\eta\alpha}{\alpha-\eta} K]\right) & \text{if } \phi\eta = +1, \end{cases} \tag{42}$$

$$l \triangleq 2b - k. \tag{43}$$

The constrained value function $v(t, x; \alpha)$ is defined by

$$v(t, x; \alpha) \triangleq \int_0^1 y^{\alpha-1} w(t, xy^{-\eta}; \alpha)\, dy, \tag{44}$$

satisfies the relation

$$w(t, x; \alpha) = \alpha v(t, x; \alpha) - \eta x v_x(t, x; \alpha) \tag{45}$$

and the terminal condition

(i) down-and-out call:

$$K' \triangleq \frac{\alpha}{\alpha - 1} K \tag{46}$$

$$v(T, x; \alpha) = \begin{cases} x - K & \text{if } x \geq K' \vee B, \\ (K' - K)\left(\frac{x}{K'}\right)^\alpha & \text{if } B \leq x \leq K', \\ 0 & \text{if } x < B \end{cases} \tag{47}$$

(ii) up-and-out call:

$$v(T, x; \alpha) = [x - K]^+ I_{\{x \leq B\}} \tag{48}$$

(iii) down-and-out put:

$$v(T, x; \alpha) = [K - x]^+ I_{\{x \geq B\}} \tag{49}$$

(iv) up-and-out put:

$$K' \triangleq \frac{\alpha}{\alpha + 1} K \tag{50}$$

$$v(T, x; \alpha) = \begin{cases} K - x & \text{if } x \leq K' \wedge B, \\ (K - K')\left(\frac{K'}{x}\right)^\alpha & \text{if } K' \leq x \leq B, \\ 0 & \text{if } x > B, \end{cases} \tag{51}$$

and can be summarized with the notation $s \triangleq (1-\eta\alpha)\sigma\sqrt{\tau}$ and $\tilde{s} \triangleq -\eta\alpha\sigma\sqrt{\tau}$ as

$$\begin{aligned}
v(t,x;\alpha) &= xe^{-r_f\tau}\left[\frac{\phi-\eta}{2}\mathcal{N}\left(-\eta(b-\theta_+)\right)+\eta\mathcal{N}\left(-\eta(k-\theta_+)\right)+e^{\frac{1}{2}s\tau(s-2\theta_+)}\right.\\
&\quad\left.\times\left\{\frac{\phi-\eta}{2}e^{sb}\mathcal{N}\left(-\eta(-b+\theta_+-s)\right)+\eta e^{sk}\mathcal{N}\left(-\eta(-k+\theta_+-s)\right)\right\}\right]\\
&+xe^{-r_f\tau+2b\theta_+}\frac{s}{s-2\theta_+}\left[\frac{\phi-\eta}{2}\mathcal{N}\left(-\eta(b+\theta_+)\right)-\phi\mathcal{N}\left(\phi(l+\theta_+)\right)+e^{\frac{1}{2}s\tau(s-2\theta_+)}\right.\\
&\quad\left.\times\left\{\frac{\phi-\eta}{2}e^{(s-2\theta_+)b}\mathcal{N}\left(-\eta(-b+\theta_+-s)\right)+\eta e^{(s-2\theta_+)l}\mathcal{N}\left(-\eta(-l+\theta_+-s)\right)\right\}\right]\\
&-Ke^{-r_d\tau}\left[\frac{\phi-\eta}{2}\mathcal{N}\left(-\eta(b-\theta_-)\right)+\eta\mathcal{N}\left(-\eta(k-\theta_-)\right)+e^{\frac{1}{2}\tilde{s}\tau(\tilde{s}-2\theta_-)}\right.\\
&\quad\left.\times\left\{\frac{\phi-\eta}{2}e^{\tilde{s}b}\mathcal{N}\left(-\eta(-b+\theta_--\tilde{s})\right)+\eta e^{\tilde{s}k}\mathcal{N}\left(-\eta(-k+\theta_--\tilde{s})\right)\right\}\right]\\
&-Ke^{-r_d\tau}e^{2b\theta_-}\frac{\tilde{s}}{\tilde{s}-2\theta_-}\left[\frac{\phi-\eta}{2}\mathcal{N}\left(-\eta(b+\theta_-)\right)-\phi\mathcal{N}\left(\phi(l+\theta_-)\right)+e^{\frac{1}{2}\tilde{s}\tau(\tilde{s}-2\theta_-)}\right.\\
&\quad\left.\times\left\{\frac{\phi-\eta}{2}e^{(\tilde{s}-2\theta_-)b}\mathcal{N}\left(-\eta(-b+\theta_--\tilde{s})\right)+\eta e^{(\tilde{s}-2\theta_-)l}\mathcal{N}\left(-\eta(-l+\theta_--\tilde{s})\right)\right\}\right].
\end{aligned} \tag{52}$$

Notice that in the second and in the fourth summand the denominator $s-2\theta_+$ or $\tilde{s}-2\theta_-$ could be zero for $\alpha = 2r/\sigma^2$ or $\alpha = 2r/\sigma^2 - 1$, respectively. However, these are both removable discontinuities, and in fact one can apply l'Hôpital's rule to find the correct equation for these two points. We do not state the explicit result here, because it is not more illuminating than the above formula. Since a minor change in α can avoid hitting the two removable discontinuities, this does not cause any problems in practice.

4.2.4 Comparative Statics

For practical use it seems handy to list some derivatives of the constrained value function $v(t,x;\alpha)$, commonly known as the "Greeks". We use the already known auxiliary claim w and obtain

$$v_x = -\frac{\eta}{x}(w-\alpha v), \tag{53}$$

$$v_{xx} = -\frac{\eta}{x^2}[xw_x + (\alpha\eta-1)w + \alpha(1-\alpha\eta)v], \tag{54}$$

$$v_t = -\eta\frac{\sigma^2}{2}xw_x + \beta w + (r_d - \alpha\beta)v, \tag{55}$$

where we denote

$$\beta \triangleq -\eta\Big[\frac{\sigma^2}{2}(1-\eta\alpha) - r\Big]. \tag{56}$$

The sensitivity *theta* is most easily obtained via the Black–Scholes partial differential equation. The leverage is given by $\eta(\alpha - w/v)$. See Figure 1 for delta and gamma of the constrained value function of an up-and-out call option. The sensitivity *delta* of the auxiliary value function w can be derived as

$$\begin{aligned}
w_x(t,x;\alpha) = {} & (\alpha-\eta)e^{-r_f\tau}\Big[\frac{\phi-\eta}{2}\mathcal{N}(\phi(b-\theta_+)) + \eta\mathcal{N}(-\eta(k-\theta_+))\Big] \\
& + \frac{\alpha-\eta}{\sigma\sqrt{\tau}}e^{-r_f\tau}\Big[-\phi\frac{\phi-\eta}{2}\mathcal{N}'(b-\theta_+) + \mathcal{N}'(k-\theta_+)\Big] \\
& + (\alpha-\eta)e^{2b\theta_+}e^{-r_f\tau}\left(1-\frac{2\theta_+}{\sigma\sqrt{\tau}}\right) \\
& \quad \times \Big[\frac{\phi-\eta}{2}\mathcal{N}(\phi(b+\theta_+)) - \phi\mathcal{N}(\phi(l+\theta_+))\Big] \\
& + \frac{(\alpha-\eta)e^{2b\theta_+}e^{-r_f\tau}}{\sigma\sqrt{\tau}}\Big[-\phi\frac{\phi-\eta}{2}\mathcal{N}'(b+\theta_+) + \mathcal{N}'(l+\theta_+)\Big] \\
& - \frac{\alpha K e^{-r_d\tau}}{x\sigma\sqrt{\tau}}\Big[-\phi\frac{\phi-\eta}{2}\mathcal{N}'(b-\theta_-) + \mathcal{N}'(k-\theta_-)\Big] \\
& - \frac{-2\alpha\theta_- K e^{-r_d\tau+2b\theta_-}}{x\sigma}\Big[\frac{\phi-\eta}{2}\mathcal{N}(\phi(b+\theta_-)) - \phi\mathcal{N}(\phi(l+\theta_-))\Big] \\
& - \frac{\alpha K e^{-r_d\tau+2b\theta_-}}{x\sigma\sqrt{\tau}}\Big[-\phi\frac{\phi-\eta}{2}\mathcal{N}'(b+\theta_-) + \mathcal{N}'(l+\theta_-)\Big],
\end{aligned} \tag{57}$$

with b and $\theta_\pm$ given by (7), k given by (42), and l given by (43).

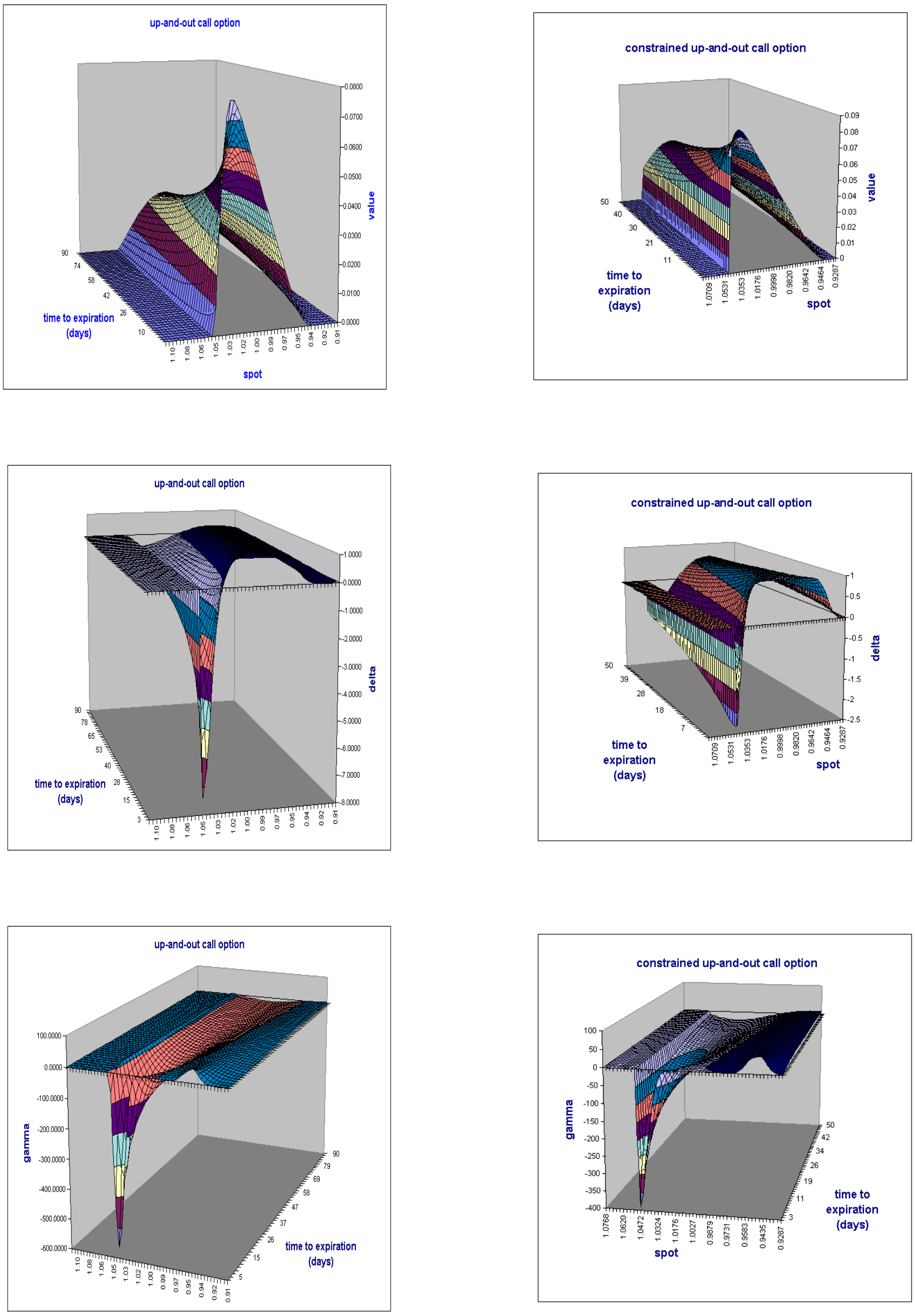

Figure 1: Value (top), delta (middle) and gamma (bottom) of an unconstrained (right) and constrained (right) up-and-out call option given by Equations (6), (52), (53), (54) with strike $K = 0.95$, knock-out barrier $B = 1.05$ and maturity $T = 90/365$. We used the interest rates $r_d = 5\%$, $r_f = 0\%$, volatility $\sigma = 10\%$ and $\alpha = 50$.

4.3 One-Touch Digitals

Given a hit-level or barrier B, there are two kinds of one-touch digital options, also called American digitals or hit options. In the first (second) kind the holder of the option receives an amount R, if the underlying hits the barrier B during the life time of the option from below (above). We define the binary variables η and ω to be

(i) $\eta = -1$, if B is to be hit from below,

(ii) $\eta = +1$, if B is to be hit from above,

(iii) $\omega = 1$, if R is to be paid at expiration time T,

(iv) $\omega = 0$, if R is to be paid the first time the underlying hits B.

In the case $\eta = -1$ we would want to impose the upper portfolio constraint (19), and in the case $\eta = +1$ we impose the shortselling constraint (9) for some real number $\alpha \geq 0$ and then find the upper hedging price. As before let us denote by $v(t,x)$ the unconstrained value of the option at time t when the exchange rate is x and by $v(t,x;\alpha)$ the corresponding constrained value function. Since raising the option value at the boundary, where $v(t,B) = R\exp(-\omega(T-t))$, would make the hedging problem worse, our only chance is to keep the boundary condition $v(t,B;\alpha) = R\exp(-\omega(T-t))$ as it is and increase the terminal condition $v(T,x) = 0$ in a minimal way such that $-\eta\pi \leq \alpha$ holds. The portfolio-constrained problem has already been solved for the path-independent digital option (Theorem 2). In that case, we must choose

$$v(T,x;\alpha) = R\left(\frac{B}{x}\right)^{\eta\alpha}, \quad \eta x \geq \eta B. \tag{58}$$

To solve for $v(t,x;\alpha)$ in the path-dependent case, we observe that it can be decomposed into the sum of the original hit option $v(t,x)$ plus a supplemental *power barrier option* $h(t,x;\alpha)$ defined by

(i) the boundary condition $h(t,B;\alpha) = 0$,

(ii) the terminal condition $h(T,x;\alpha) = R(B/x)^{\eta\alpha}, \quad \eta x \geq \eta B$,

(iii) $-r_d h + h_t + rxh_x + \frac{1}{2}\sigma^2 x^2 h_{xx} = 0$.

We present the detailed solution for the case $\eta = -1$ following the standard procedure to compute barrier option values. To compute the value at time $t \in [0,T)$ of the payoff random variable

$$R\left(\frac{S_T}{B}\right)^{\alpha} I_{\{\sup_{t\leq u\leq T} S_u < B\}}, \tag{59}$$

we use the joint probability density function $f_{\theta_-,\tau}$ with $\tau \triangleq T-t$ as in equation (5) and obtain

$$\begin{aligned} h(t,x;\alpha) &= \mathbb{E}^{t,x}\left[e^{-r_d\tau}R\left(\frac{S_T}{B}\right)^{\alpha} I_{\{\sup_{t\le u\le T} S_u<B\}}\right] \\ &= \frac{Re^{-r_d\tau}}{B^{\alpha}}\int_{-\infty}^{b}\int_{0\vee y}^{b}(xe^{\sigma y})^{\alpha} f_{\theta_-,\tau}(y,z)\,dz\,dy \\ &= Re^{(-r_d+\frac{1}{2}\alpha^2\sigma^2)\tau+\alpha\sigma\sqrt{\tau}\theta_-}\Big\{e^{-\alpha b\sigma\sqrt{\tau}}\mathcal{N}\big(b-\alpha\sigma\sqrt{\tau}-\theta_-\big) \\ &\qquad - e^{\alpha b\sigma\sqrt{\tau}+2b\theta_-}\mathcal{N}\big(-b-\alpha\sigma\sqrt{\tau}-\theta_-\big)\Big\}, \end{aligned} \tag{60}$$

where b and θ_- are given by (7). A similar computation can be done for the case $\eta=+1$. We summarize.

Theorem 3 *The supplement for one-touch digitals is given by*

$$\begin{aligned} h(t,x;\alpha) = &Re^{(-r_d+\frac{1}{2}\alpha^2\sigma^2)\tau-\eta\alpha\sigma\sqrt{\tau}\theta_-} \\ &\times\left\{e^{\eta\alpha\sigma\sqrt{\tau}b}\mathcal{N}(\eta d_-) - e^{-\eta\alpha\sigma\sqrt{\tau}b+2b\theta_-}\mathcal{N}(\eta d_+)\right\}, \end{aligned} \tag{61}$$

where $d_\pm \triangleq \pm b-(\theta_- - \eta\alpha\sigma\sqrt{\tau})$ *and* b, θ_- *are given by* (7).

Let us note that for $\alpha=0$ this formula simplifies to the rebate portion of a knock-in barrier option as presented in the Formula Catalogue of [27].

5 Numerical Solutions

5.1 Range Binaries

For many option payoffs it is difficult or impossible to compute the constrained value function analytically. If the boundary conditions are known, however, we compute the value function using a finite difference grid. As an example, we present the commonly traded range binary option whose payoff is

$$I_{\{\min_{0\le t\le T} S_t>L;\,\max_{0\le t\le T} S_t<U\}} \tag{62}$$

for some lower barrier $L>0$ and upper barrier $U>L$. If we impose the leverage constraint $\pi(t)\in[-\alpha_U,\alpha_L]$ for a given pair of nonnegative numbers $\vec{\alpha}=(\alpha_U,\alpha_L)$, then we are seeking a solution to the Black–Scholes equation which satisfies

$$-\alpha_U v(t,x;\vec{\alpha}) \le x v_x(t,x;\vec{\alpha}) \le \alpha_L v(t,x;\vec{\alpha}), \quad 0\le t\le T,\ L<x<U. \tag{63}$$

To find such a function, we solve the equations

$$-r_d v + v_t + r x v_x + \frac{1}{2}\sigma^2 x^2 v_{xx} = 0 \quad \forall\, t \in [0,T),\ x \in (L,U), \tag{64}$$
$$v(t,x;\vec{\alpha}) = 0 \quad \forall\, t \in [0,T],\ x \notin [L,U], \tag{65}$$
$$\alpha_L v(t,L;\vec{\alpha}) - L v_x(t,L;\vec{\alpha}) = 0 \quad \forall\, t \in [0,T], \tag{66}$$
$$\alpha_U v(t,U;\vec{\alpha}) + U v_x(t,U;\vec{\alpha}) = 0 \quad \forall\, t \in [0,T], \tag{67}$$
$$v(T,x;\vec{\alpha}) = 1 \quad \forall\, x \in (L,U). \tag{68}$$

For $x = U$, this solution satisfies

$$\alpha_U v + x v_x \geq 0. \tag{69}$$

by construction. At $x = L$, we use (66) to write

$$\alpha_U v + x v_x = \alpha_U v + \alpha_L v \geq 0, \tag{70}$$

and again (69) holds. Finally, for $t = T$ and $L < x < U$, $\alpha_U v + x v_x = \alpha_U \geq 0$. Since (69) holds on the entire boundary of the region in which we have solved the Black–Scholes equation, and $\alpha_U v + x v_x$ is itself a solution to the Black–Scholes equation, the maximum principle implies that relation (69) holds for all $t \in [0,T]$ and $x \in (L,U)$. This is the first inequality in (63). The proof of the second inequality in (63) is similar.

To numerically solve (64)–(68), the first step is to make this problem homogeneous by the change of variables $y = \ln x$. The function $u(t,y) \triangleq v(t,x;\vec{\alpha})$ is then uniquely determined by

$$-r_d u + u_t + \mu u_y + \frac{1}{2}\sigma^2 u_{yy} = 0 \quad \forall t \in [0,T),\ y \in (\ln L, \ln U), \tag{71}$$
$$u(t,y) = 0 \quad \forall t \in [0,T],\ y \notin [\ln L, \ln U], \tag{72}$$
$$\alpha_L\, u(t,\ln L) - u_y(t,\ln L) = 0 \quad \forall t \in [0,T], \tag{73}$$
$$\alpha_U\, u(t,\ln U) + u_y(t,\ln U) = 0 \quad \forall t \in [0,T], \tag{74}$$
$$u(T,y) = 1 \quad \forall y \in (\ln L, \ln U), \tag{75}$$

where we abbreviate $\mu \triangleq r - \frac{1}{2}\sigma^2$. The next step is to discretize the rectangle $[\ln L, \ln U] \times [0,T]$ into a uniformly spaced mesh with $M+2$ nodes along the t axis and $N+2$ nodes along the y axis:

$$y_i = y_0 + i\Delta_y = \ln L + i\frac{\ln U - \ln L}{N+1}, \quad i = 0,\ldots,N+1, \tag{76}$$
$$t_j = j\Delta_t = j\frac{T}{M+1}, \quad j = 0,\ldots,M+1. \tag{77}$$

This way the boundary conditions can be captured exactly, but the initial exchange rate value is most likely not a point in the mesh. To find the time zero value of the range binary option we must interpolate between the two neighboring mesh points of the initial exchange rate. The next step is to introduce the following difference approximations of the partial derivatives of u. We abbreviate $u_{i,j} \triangleq u(y_i, t_j)$ and approximate

$$u_t(y_i, t_j) \approx \frac{u_{i,j+1} - u_{i,j}}{\Delta_t}, \tag{78}$$

$$u_y(y_i, t_j) \approx (1-\Theta)\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta_y} + \Theta\frac{u_{i+1,j+1} - u_{i-1,j+1}}{2\Delta_y}, \tag{79}$$

$$\begin{aligned} u_{yy}(y_i, t_j) \approx (1-\Theta)&\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta_y^2} \\ &+ \Theta\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{\Delta_y^2}, \end{aligned} \tag{80}$$

where the parameter $\Theta \in [0, 1]$ denotes the degree of explicitness. Common values are

- $\Theta = 1$ for the fully explicit finite-difference method,
- $\Theta = 0$ for the fully implicit finite-difference method,
- $\Theta = \frac{1}{2}$ for the Crank–Nicholson scheme.

Plugging the finite difference approximations into Equation (71) yields for each $j = 0, \ldots, M$ the N linear equations

$$\begin{aligned} &u_{i-1,j}\Big(-\frac{1}{2}a(1-\Theta)(\sigma^2 - \Delta_y\mu)\Big) + u_{i,j}(1 + r_d\Delta_t + a(1-\Theta)\sigma^2) \\ &\quad + u_{i+1,j}\Big(-\frac{1}{2}a(1-\Theta)(\sigma^2 + \Delta_y\mu)\Big) \\ &\qquad\qquad = u_{i-1,j+1}\Big(\frac{1}{2}a\Theta(\sigma^2 - \Delta_y\mu)\Big) \\ &\qquad\qquad\quad + u_{i,j+1}(1 - a\Theta\sigma^2) + u_{i+1,j+1}\Big(\frac{1}{2}a\Theta(\sigma^2 + \Delta_y\mu)\Big), \end{aligned}$$

$i = 1, \ldots, N$. The boundary conditions translate into two more equations

$$\begin{aligned} (\Delta_y\alpha_L + (1-\Theta))u_{0,j} - (1-\Theta)u_{1,j} &= -\Theta(u_{1,j+1} - u_{0,j+1}), \\ (\Delta_y\alpha_U + (1-\Theta))u_{N+1,j} - (1-\Theta)u_{N,j} &= -\Theta(u_{N+1,j+1} - u_{N,j+1}). \end{aligned}$$

We obtain for each j a tridiagonal system of $N + 2$ linear equations in the unknowns $u_{i,j}, i = 0, \ldots, N+1$, which can be solved efficiently using an algorithm, e.g., from [22].

6 Summary

We have demonstrated by examples of digital, barrier and one-touch options how the option valuation problem with leverage constrained hedging portfolios can be solved. In some cases we provided closed form solutions, and in others have described how to set up the appropriate finite-difference method for numerical solution. The formulae provided here, as well as other formulae, are listed in the section "Dangerous Digitals" at `http://www.mathfinance.de`, and an online calculator there computes leverage constrained prices of reverse barrier options.

A Appendix: The Maximum Principle

Theorem 4 (The Maximum Principle) *(see, e.g., [19]). Suppose X is a diffusion of the form $dX(s) = a\,ds + \sigma\,dW(s)$ with second order differential operator $\mathcal{A}u(t,x) \triangleq au_x(t,x) + \frac{1}{2}\sigma^2 u_{xx}(t,x)$, $g(t,x) \geq 0$ a* potential*, $u(t,x)$ a function satisfying $u(T,x) \geq 0$ and $-u_t + r_d u = \mathcal{A}u + g$. Then $u(t,x) \geq 0$ for all $t \leq T$.*

For a quick proof, use Itô's rule to compute the differential

$$\begin{aligned} de^{-r_d s}u(s,X(s)) &= e^{-r_d s}[-r_d u\,ds + u_s\,ds + \mathcal{A}uds + \sigma u_x\,dW(s)] \\ &= e^{-r_d s}[-g\,ds + \sigma u_x\,dW(s)]. \end{aligned}$$

Now integrate between t and T and take expectations conditioned on $X(t) = x$ to get

$$\mathbb{E}^{t,x}\big[e^{-r_d T}u(T,X(T))\big] = e^{-r_d t}u(t,x) - \mathbb{E}^{t,x}\left[\int_t^T e^{-r_d s}g(s,X(s))\,ds\right], \quad (81)$$

which in turn implies

$$u(t,x) = \mathbb{E}^{t,x}\big[e^{-r_d(T-t)}u(T,X(T))\big] + \mathbb{E}^{t,x}\left[\int_t^T e^{-r_d(s-t)}g(s,X(s))\,ds\right]. \quad (82)$$

The assumed nonnegativity of both $u(T,x)$ and g shows thus, that $u(t,x)$ is nonnegative as well. We used this for the case $g = 0$.

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