Convergence of path measures arising from a mean field or polaron type interaction*

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Summary. We discuss the limiting path measures of Markov processes with either a mean field or a polaron type interaction of the paths. In the polaron type situation the strength is decaying at large distances on the time axis, and so the interaction is of short range in time. In contrast, in the mean field model, the interaction is weak, but of long range in time. Donsker and Varadhan proved that for the partition functions, there is a transition from the polaron type to the mean field interaction when passing to a limit by letting the strength tend to zero while increasing the range. The discussion of the path measures is more subtle. We treat the mean field case as an example of a differentiable interaction and discuss the transition from the polaron type to the mean field interaction for two instructive examples.

Keywords. maximum entropy principle – large deviations – weak convergence – polaron problem – mean field interaction – interacting Markov processes

1. Introduction

Let $\{X_t\}_{t\geq 0}$ be a Markov process with Polish state space E. We assume that the sample paths are in $(C([0,\infty),E),\mathcal{F})$ or $(D([0,\infty),E),\mathcal{F})$, where $\mathcal{F}$ denotes the corresponding Borel σ-algebra. Let $V : E^2 \to \mathbb{R}$ be a suitable function. A mean field interaction between positions at different points of time, which has often been considered, is given by the "Hamiltonian"

$$H_T = \frac{1}{T}\int_0^T \int_0^T V(X_s, X_t)\, ds\, dt\ , \qquad T > 0, \tag{1.1}$$

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which introduces an interaction which is weak but of long range in time. With this Hamiltonian, we can transform the original path measure $\mathbb{P}$ of the process by defining

$$\widehat{\mathbb{P}}_T(A) = \mathbb{E}[1_A \exp(H_T)]/Z_T , \qquad A \in \mathcal{F}, \tag{1.2}$$

where $Z_T = \mathbb{E}[\exp(H_T)]$. In contrast to this long-range but weak interaction, there is a model with a strong but short-range (with respect to time) interaction, which we call polaron type interaction, given by the Hamiltonian

$$H_{\alpha,T} = \frac{1}{2}\int_0^T \int_0^T \alpha e^{-\alpha|s-t|} V(X_s, X_t)\, ds\, dt , \qquad \alpha, T > 0, \tag{1.3}$$

and the corresponding transformed path measure

$$\widehat{\mathbb{P}}_{\alpha,T}(A) = \mathbb{E}[1_A \exp(H_{\alpha,T})]/Z_{\alpha,T} , \qquad A \in \mathcal{F}, \tag{1.4}$$

where $Z_{\alpha,T} = \mathbb{E}[\exp(H_{\alpha,T})]$. The aim of this paper is to describe and compare

$$\lim_{T\to\infty} \widehat{\mathbb{P}}_T \qquad \text{and} \qquad \lim_{\alpha\downarrow 0}\lim_{T\to\infty} \widehat{\mathbb{P}}_{\alpha,T} , \tag{1.5}$$

provided that these limits exist.

Under some technical conditions, one knows that

$$\lim_{\alpha\downarrow 0}\lim_{T\to\infty} \frac{1}{T}\log Z_{\alpha,T} = \lim_{T\to\infty}\frac{1}{T}\log Z_T = \sup_{\mu\in\mathcal{M}_1(E)} \bigl(\widetilde{V}(\mu,\mu) - J(\mu)\bigr) , \tag{1.6}$$

where $\mathcal{M}_1(E)$ denotes the probability measures on E, the abbreviation $\widetilde{V}(\mu,\nu)$ stands for the integral of V with respect to $\mu\otimes\nu$ for $\mu,\nu\in\mathcal{M}_1(E)$, and J, defined in (2.3), is the Donsker-Varadhan rate function which governs the large deviations of the empirical process. Equation (1.6) has been proved by Donsker and Varadhan [6] for the Fröhlich polaron, where $\{X_t\}_{t\geq 0}$ is the Brownian motion in $\mathbb{R}^3$ and $V(x,y) = |x-y|^{-1}$ for $x,y\in\mathbb{R}^3$ with $x\neq y$ is the Coulomb potential. Recently, (1.6) has been generalized in [14] and [15].

An investigation of the limiting mean field path measure with a Hamiltonian, which is more general than (1.1), has been done for discrete-time Markov processes in [2] and [20], and for (possibly non-symmetric) continuous-time Markov processes it is given in Section 2 below. The case of a symmetric continuous-time Markov process with a Hamiltonian given by (1.1) has already been treated in [12], and, with more general Hamiltonians having at least C^2-regularity, corresponding results have recently been obtained in [13]. A discussion of the limiting polaron path measure is given in [22] and [23], partially on a heuristic level.

Let us summarize our results about the mean field model contained in Section 2. We consider a uniformly mixing Markov process on a compact state space E, a real-valued continuous function Ψ on $\mathcal{M}_1(E)$, which is differentiable in a suitable sense (Condition 2.12), and define a Hamiltonian by $H_T^\Psi = T\Psi(L_T)$, where L_T is the empirical distribution of the process $\{X_t\}_{t\geq 0}$ up to time T. The Hamiltonian in (1.1) corresponds to $\Psi(\mu) := \widetilde{V}(\mu,\mu)$ for $\mu\in\mathcal{M}_1(E)$, see Example 2.15. In Theorem 2.20 we show that $\{\widehat{\mathbb{P}}_T\}_{T>0}$ is relatively compact in the weak topology as $T\to\infty$ and that each accumulation point $\widehat{\mathbb{P}}$ is a mixture of homogeneous Markovian path measures $\{\mathbb{Q}^\mu\}_{\mu\in K_\Psi}$, where K_Ψ is the set of all $\mu\in\mathcal{M}_1(E)$ which maximize $\Psi - J$. In particular, for the

Hamiltonian given by (1.1), the set K_Ψ consists of all solutions of the variational problem in (1.6) and we obtain a characterization of the first limit in (1.5). For $\mu \in K_\Psi$, the measure $\mathbb{Q}^\mu$ is given in terms of $\Psi(\mu)$ and the derivative of Ψ at μ. If additional symmetry assumptions are satisfied, then Theorem 2.32 shows that $\{\widehat{\mathbb{P}}_T\}_{T>0}$ converges weakly to a specified mixture of $\{\mathbb{Q}^\mu\}_{\mu\in K_\Psi}$ as $T \to \infty$.

Given the first equality in (1.6), one could expect that for large T and small α the measure $\widehat{\mathbb{P}}_{\alpha,T}$ should be close to $\widehat{\mathbb{P}}_T$. There is, however, a somewhat subtle boundary effect for the measures $\widehat{\mathbb{P}}_{\alpha,T}$ at the starting point. This boundary effect shows up because the Hamiltonian in (1.3) is not defined in terms of translations of paths which are periodic continuations of the paths in $[0,T]$. Such periodic paths are used in [5] to treat process-level large deviations.

We conjecture that, quite generally, the limits in (1.5) are different and that

$$\lim_{\alpha\downarrow 0}\lim_{T\to\infty} \widehat{\mathbb{P}}_{\alpha,T} = \lim_{R\to\infty}\lim_{\alpha\downarrow 0}\widehat{\mathbb{P}}_{\alpha,R/\alpha} \,. \tag{1.7}$$

Let us give, on a heuristic level, a description of the right-hand side of (1.7) and explain, why we expect this equality to hold. We are not yet able to prove (1.7) in any generality, especially not for the Fröhlich polaron, but we discuss two instructive examples in Sections 3 and 4.

The crucial point is that the right-hand side in (1.7) is a mean field type limit which, however, is more delicate to handle than the limit of $\{\widehat{\mathbb{P}}_T\}_{T>0}$ as $T \to \infty$. To show this, let us compare the limit behaviour of the partition functions $\{Z_T\}_{T>0}$ and $\{Z_{\alpha,R/\alpha}\}_{\alpha,R>0}$. After the time transformations $s \mapsto s/\alpha$ and $t \mapsto t/\alpha$, the latter ones are given by

$$Z_{\alpha,R/\alpha} = \mathbb{E}\left[\exp\left(\frac{1}{2R\alpha}\int_0^R\int_0^R \mathrm{e}^{-|s-t|}V(X_{s/\alpha},X_{t/\alpha})\,ds\,dt\right)\right] . \tag{1.8}$$

One should remark that, for fixed $R>0$, the limit of $\{Z_{\alpha,R/\alpha}\}_{\alpha>0}$ as $\alpha\downarrow 0$ is a mean field type limit. To see this, divide the time interval $[0,R]$ into small intervals such that $\mathrm{e}^{-|s-t|}$ is approximately constant for (s,t) in rectangles formed by these intervals. Then the restriction of the integration in (1.8) to such rectangles is essentially just a mean field type expression. Patching things together, one gets, under appropriate conditions,

$$\begin{aligned}&\lim_{\alpha\downarrow 0}\alpha\log Z_{\alpha,R/\alpha} \\ &\quad= \sup_{\mu_R}\left(\frac{1}{2R}\int_0^R\int_0^R \mathrm{e}^{-|s-t|}\widetilde{V}(\mu_{R,s},\mu_{R,t})\,ds\,dt - \frac{1}{R}\int_0^R J(\mu_{R,t})\,dt\right),\end{aligned} \tag{1.9}$$

where μ_R runs over an appropriate space of functions which map $[0,R]$ into $\mathcal{M}_1(E)$, compare [3, Exercise 4.2.70].

We expect that the limit of $\{\widehat{\mathbb{P}}_{\alpha,R/\alpha}\}_{\alpha>0}$ for $\alpha\downarrow 0$ is related to the solutions of the variational problem in (1.9) in a similar way as the limit of $\{\widehat{\mathbb{P}}_T\}_{T>0}$ for $T\to\infty$ is related to the solutions of the variational problem in (1.6). Although the solutions of the variational problem in (1.9) are in general not constant in time, the limit of $\{\widehat{\mathbb{P}}_{\alpha,R/\alpha}\}_{\alpha>0}$ as $\alpha\downarrow 0$ should be a mixture of homogeneous Markov processes, because, due to the transformations leading to (1.8), the inhomogeneity of these solutions is on the time scale $1/\alpha$. However, it turns out that, for the description of the Markov processes appearing in this mixture, the inhomogeneity of the solutions of the variational problem is important.

It is reasonable to expect that, as $R \to \infty$, the right-hand side of (1.9) approaches the right-hand side of (1.6) and the solutions $(\mu_{R,t})_{t\in[0,R]}$ of the variational problem in (1.9) converge to the corresponding solutions of the mean field variational problem in (1.6) for t around the center of the interval $[0, R]$. The above mentioned boundary effect shows up because this convergence does not take place for t near the boundary of the time interval $[0, R]$ as $R \to \infty$. This behaviour originates from the factor $(1/2)\mathrm{e}^{-|s-t|}$ in (1.9), which, for $t \in \{0, R\}$, gives less than $1/2$ when integrated over $[0, R]$ instead of converging to 1 as $R \to \infty$.

If (1.7) is true, then one has a characterization of the iterated limit in (1.5). Equality in (1.7) would follow if one can prove that $\widehat{\mathbb{P}}_{\alpha,T}$ is close to $\widehat{\mathbb{P}}_{\alpha,R/\alpha}$ for large R uniformly in $T \geq R/\alpha$. To achieve this, one actually needs quite precise information about the solutions of the variational problem in (1.9); to prove the uniformity in $T \geq R/\alpha$, one has to determine and control the limiting behaviour of these solutions in terms of an added condition for $\mu_{R,R}$ as $R \to \infty$, see Section 3 for details.

As already mentioned, we are far away from proving (1.7) and characterizing the iterated limit in (1.5) in a general setting, but the two examples in Sections 3 and 4 fully confirm the picture presented above. We actually do not prove (1.7) for these examples, but (1.7) is our guide for identifying the iterated limit in (1.5).

The first model, treated in Section 3, is a symmetric Markovian jump process on $E = \{0, 1\}$ with exponential waiting times of expectation one and an interaction function V which is given by $V(1, 0) = V(0, 1) = 0$ and $V(0, 0) = V(1, 1) = \tau$, where $\tau \in \mathbb{R}$ denotes a strength parameter. We determine the limiting path measures in (1.5) explicitly. If $\tau \leq 1$, then they are equal to $\mathbb{P}$. If $\tau > 1$, then the limiting measures in (1.5) are different but they are both mixtures of asymmetric Markovian jump processes. It turns out that the second limiting measure in (1.5) corresponds to the first one with an adjusted strength parameter, namely $\tilde{\tau} = (\tau + 1/\tau)/2$.

The second model, treated in Section 4, is the one-dimensional Brownian motion with $V(x, y) = \tau^2(x - y)^2/4$ for $x, y \in \mathbb{R}$ and a strength parameter $\tau > 0$. Since Brownian motion is not sufficiently mixing, the results of Section 2 are not applicable. Because V is quadratic, everything is in the realm of Gaussian processes and we can determine the limits in (1.5) explicitly by investigating the corresponding covariances. It turns out that the limits in (1.5) exist, that they are different, and that they are both mixtures of Ornstein-Uhlenbeck processes with a normally distributed random center. Similar to the result in Section 3, the second limiting measure in (1.5) corresponds to the first one with an adjusted strength parameter, namely $\tilde{\tau} = \tau/\sqrt{2}$.

2. Convergence of Path Measures in a Mean Field Model

Let E be a compact metric space with Borel σ-algebra $\mathcal{E}$. If I is a non-empty subset of $\mathbb{R}$, then $C(E, I)$ denotes the set of all I-valued continuous functions on E and $C(E)$ is an abbreviation for $C(E, \mathbb{R})$. We write $\|\cdot\|$ for the supremum norm. Let $\mathcal{M}_1(E)$ be the set of all probability measures on $(E, \mathcal{E})$. Note that $\mathcal{M}_1(E)$ with the Prohorov metric, which induces the weak topology [10, Chap. 3, Theorem 3.1], is a compact metric space [10, Chap. 3, Theorem 1.7 and 2.2]. Let $\mathcal{B}(\mathcal{M}_1(E))$ denote the corresponding Borel σ-algebra.

Consider the path space $\Omega = D([0, \infty), E)$ and, for each $t \in [0, \infty)$, define the evaluation map $X_t : \Omega \to E$ by $X_t(\omega) = \omega(t)$. Let $\mathcal{F}$ be the σ-algebra on Ω generated by $\{X_t\}_{t\geq 0}$. Note that Ω with the metric given by [10, Chap. 3, (5.2)] is a Polish space

[10, Chap. 3, Theorem 5.6] and that $\mathcal{F}$ coincides with the Borel σ-algebra generated by this metric [10, Chap. 3, Proposition 7.1]. For $t \geq 0$ let $\mathcal{F}_t$ denote the sub-σ-algebra generated by $\{X_s\}_{s\in[0,t]}$.

We consider an $\mathcal{E}$-measurable family $\{\mathbb{P}_x\}_{x\in E}$ of time-homogeneous Markovian probability measures on $(\Omega, \mathcal{F})$ with $\mathbb{P}_x(X_0 = x) = 1$ for all $x \in E$. Let $\{P_t\}_{t\geq 0}$ denote the corresponding semigroup of stochastic transition kernels as well as the semigroup of bounded linear operators on $C(E)$. We assume:

Condition 2.1 There exists a $\{P_t\}_{t\geq 0}$-invariant probability measure $\pi \in \mathcal{M}_1(E)$ with $\operatorname{supp}(\pi) = E$ and, for each $t > 0$, there exists a jointly continuous transition density $p_t \in C(E^2, (0,\infty))$ of P_t with respect to π.

Since $\operatorname{supp}(\pi) = E$, the continuous transition densities are unique. As an abbreviation define $c_t = \|\max\{p_t, 1/p_t\}\| = \exp(\|\log p_t\|)$ for all $t > 0$. Note that $\{P_t\}_{t\geq 0}$ is Feller continuous by Condition 2.1. For $t \geq 0$ the empirical distribution process of the position process after t is defined by

$$\Omega \times (t,\infty) \ni (\omega, T) \mapsto L_{t,T}(\omega) = \frac{1}{T-t}\int_t^T \delta_{X_u(\omega)}\, du \in \mathcal{M}_1(E) \,. \tag{2.2}$$

If $t = 0$, then we write L_T instead of $L_{0,T}$. The right-continuity of the paths in Ω implies that $P_t(x,\cdot)$ converges weakly to δ_x as $t \downarrow 0$, for every $x \in E$. Therefore, the operator semigroup $\{P_t\}_{t\geq 0}$ is strongly continuous [25, p. 115]. Let L denote the corresponding (strong) generator with domain $\mathcal{D}(L)$ and define the position level rate function $J\colon \mathcal{M}_1(E) \to [0,\infty]$ by

$$J(\mu) = \sup\Big\{ -\int_E \frac{L\phi}{\phi}\, d\mu \,\Big|\, \phi \in \mathcal{D}(L) \cap C(E,[1,\infty)) \Big\} \,. \tag{2.3}$$

Examination of [3, pp. 123–126] shows that this definition coincides with J_P in [3]. According to [3, Theorem 4.2.43], the measures $\{\, \mathbb{P}_x L_T^{-1} \mid T > 0,\ x \in E \,\}$ satisfy a uniform full large deviation principle with the rate function J.

For $\mu \in \mathcal{M}_1(E)$, let $\mathbb{P}_\mu \in \mathcal{M}_1(\Omega)$ denote the path measure with starting distribution μ, hence $\mathbb{P}_\mu(A) = \int_E \mathbb{P}_x(A)\,\mu(dx)$ for all $A \in \mathcal{F}$. In particular, $\mathbb{P}_{\delta_x} = \mathbb{P}_x$. We denote the expectation with respect to $\mathbb{P}_\mu$ or $\mathbb{P}_x$ by $\mathbb{E}_\mu$ and $\mathbb{E}_x$, respectively.

Given $\varphi \in C(E)$, define the semigroup of transition kernels $\{P_t^\varphi\}_{t\geq 0}$ by

$$P_t^\varphi(x,A) = \mathbb{E}_x\left[\exp\left(\int_0^t \varphi(X_u)\, du\right) 1_A(X_t)\right] , \qquad x \in E,\, A \in \mathcal{E},\, t \geq 0. \tag{2.4}$$

The corresponding semigroup of bounded linear operators on $C(E)$ is denoted by $\{P_t^\varphi\}_{t\geq 0}$, too. The logarithmic spectral radius of P_1^φ, given by

$$\Lambda^\varphi = \lim_{t\to\infty} \frac{1}{t} \log \|P_t^\varphi\|_{\mathrm{op}} \,, \tag{2.5}$$

satisfies $|\Lambda^\varphi| \leq \|\varphi\|$. If $t \geq 0$, then $\exp(\Lambda^\varphi t)$ is the spectral radius of P_t^φ. Using [3, Exercise 2.1.15, (4.2.21), and Corollary 4.2.27], it follows that

$$\Lambda^\varphi = \sup\{\, \langle \varphi, \mu\rangle - J(\mu) \mid \mu \in \mathcal{M}_1(E) \,\} \,, \tag{2.6}$$

where $\langle \varphi, \mu\rangle$ denotes the integral of φ with respect to μ. Let $\|\cdot\|_p$ denote the usual norm on $L^p(\pi)$.

Lemma 2.7 *Let $\varphi \in C(E)$ be given.*

(a) *For each $t > 0$, there exists a transition density $p_t^\varphi \in C(E^2, (0,\infty))$ of P_t^φ with respect to π satisfying $\| \log p_t^\varphi \| \le \|\varphi\| t + \log c_t$.*

(b) *There exists a unique function $h^\varphi \in C(E, (0,\infty))$ which satisfies $\|h^\varphi\|_2 = 1$ and $P_t^\varphi h^\varphi = \mathrm{e}^{\Lambda^\varphi t} h^\varphi$ for all $t \ge 0$.*

(c) *There exists an $\mathcal{E}$-measurable set $\{\mathbb{Q}_x^\varphi\}_{x\in E}$ of time-homogeneous Markovian probability measures on $(\Omega, \mathcal{F})$ such that, for all $x \in E$, $t \ge 0$, and $A \in \mathcal{F}_t$,*

$$\mathbb{Q}_x^\varphi(A) = \frac{\mathrm{e}^{-\Lambda^\varphi t}}{h^\varphi(x)} \mathbb{E}_x \left[1_A \exp\left(\int_0^t \varphi(X_u)\, du \right) h^\varphi(X_t) \right] . \tag{2.8}$$

(d) *Transition densities of $\{\mathbb{Q}_{(\cdot)}^\varphi X_t^{-1}\}_{t>0}$ with respect to π are given by*

$$q_t^\varphi(x,y) = \frac{\mathrm{e}^{-\Lambda^\varphi t}}{h^\varphi(x)} p_t^\varphi(x,y) h^\varphi(y) , \qquad x,y \in E,\ t > 0. \tag{2.9}$$

(e) *There exists a unique $\{\mathbb{Q}_{(\cdot)}^\varphi X_t^{-1}\}_{t\ge 0}$-invariant distribution $\mu^\varphi \in \mathcal{M}_1(E)$.*

(f) *There exists a unique $l^\varphi \in C(E, (0,\infty))$ such that $d\mu^\varphi / d\pi = h^\varphi l^\varphi$.*

(g) $\| \log h^\varphi \| \le 2\|\varphi\| + \log c_1$ *and* $\| \log l^\varphi \| \le 4\|\varphi\| + \log c_1^2$.

(h) *For every $C > 0$ there exist $\varepsilon, C' > 0$ such that, for all $t \ge 1$,*

$$\sup_{\varphi : \|\varphi\| \le C} \sup_{x\in E} \| q_t^\varphi(x,\cdot) - l^\varphi h^\varphi \| \le C' \mathrm{e}^{-\varepsilon t} .$$

Remark 2.10 From Lemma 2.7(b) we see that h^φ is the $\|\cdot\|_2$-normalized, positive eigenfunction associated with the principle eigenvalue Λ^φ of the (strong) generator $L + \varphi$ of $\{P_t^\varphi\}_{t\ge 0}$. If π is $\{P_t\}_{t\ge 0}$-reversing [3, p. 128], then $l^\varphi = h^\varphi$.

Remark 2.11 The transition densities $\{q_t^\varphi\}_{t>0}$ in part (d) of Lemma 2.7 are the Doob h^φ-transforms of $\{\mathrm{e}^{\Lambda^\varphi t} p_t^\varphi\}_{t>0}$, and the corresponding measures $\{\mathbb{Q}_x^\varphi\}_{x\in E}$ on $(\Omega, \mathcal{F})$ describe the h^φ-path process [8, Section 2.VI.13].

The proof of Lemma 2.7 is given after Theorem 2.20. We consider a real-valued function Ψ on $\mathcal{M}_1(E)$ which satisfies the following condition. Note that this condition determines $D\Psi$ only up to a constant but that this constant does not enter into φ^μ given by (2.18).

Condition 2.12 Let the function $\Psi : \mathcal{M}_1(E) \to \mathbb{R}$ be continuous and differentiable in the sense that there exists a continuous map $D\Psi : \mathcal{M}_1(E) \to C(E)$ such that the map $R : (0,1] \times (\mathcal{M}_1(E))^2 \to \mathbb{R}$, defined by

$$R_\lambda(\mu,\nu) = \frac{1}{\lambda} \big(\Psi((1-\lambda)\mu + \lambda\nu) - \Psi(\mu) - \lambda \langle D\Psi(\mu), \nu - \mu \rangle \big) \tag{2.13}$$

for all $\lambda \in (0,1]$ and $\mu, \nu \in \mathcal{M}_1(E)$, is bounded and satisfies

$$\lim_{\lambda \downarrow 0} \sup_{\mu \in \mathcal{M}_1(E)} |R_\lambda(\mu,\nu)| = 0 \tag{2.14}$$

for each $\nu \in \mathcal{M}_1(E)$.

Example 2.15 Given any $V \in C(E^2)$, define the quadratic functional Ψ by $\Psi(\mu) = \langle V, \mu^{\otimes 2} \rangle$ for all $\mu \in \mathcal{M}_1(E)$. Without loss of generality we may assume that Ψ is given by a symmetric V. To prove that Ψ is continuous, note that V is bounded and uniformly continuous on the compact set E^2. Hence, given any $\varepsilon > 0$, there exist $n \in \mathbb{N}$ and $x_1, \dots, x_n \in E$ such that the n balls in $C(E)$ with centers $V(x_1, \cdot), \dots, V(x_n, \cdot)$ and radius ε cover $\{ V(x, \cdot) \mid x \in E \}$. Hence, if $|\langle V(x_j, \cdot), \mu - \nu\rangle| \le \varepsilon$ for all $j \in \{1, \dots, n\}$, then $|\langle V(x, \cdot), \mu - \nu\rangle| \le 3\varepsilon$ for all $x \in E$ and $|\Psi(\mu) - \Psi(\nu)| \le 6\varepsilon$. Using the same argument, the continuity of $D\Psi$, given by $D\Psi(\mu)(x) = 2\langle V(x, \cdot), \mu\rangle$, follows. Since $R_\lambda(\mu, \nu) = \lambda\langle V, (\mu - \nu)^{\otimes 2}\rangle$ for all $\lambda \in (0, 1]$ and $\mu, \nu \in \mathcal{M}_1(E)$, Condition 2.12 holds for Ψ.

We fix a starting distribution $m \in \mathcal{M}_1(E)$. For every function $\varphi \in C(E)$ define $m^\varphi \in \mathcal{M}_1(E)$ and $\mathbb{Q}^\varphi \in \mathcal{M}_1(\Omega)$, for all $A \in \mathcal{F}$ and $B \in \mathcal{E}$, by

$$m^\varphi(B) = \frac{\langle 1_B h^\varphi, m\rangle}{\langle h^\varphi, m\rangle} \qquad \text{and} \qquad \mathbb{Q}^\varphi(A) = \int_E \mathbb{Q}^\varphi_x(A)\, m^\varphi(dx) \ . \tag{2.16}$$

Let

$$K_\Psi = \left\{ \mu \in \mathcal{M}_1(E) \;\middle|\; \Psi(\mu) - J(\mu) = \sup_{\nu \in \mathcal{M}_1(E)} (\Psi(\nu) - J(\nu)) \right\} . \tag{2.17}$$

For each $\mu \in \mathcal{M}_1(E)$ define $\varphi^\mu \in C(E)$ by

$$\varphi^\mu = D\Psi(\mu) + \Psi(\mu) - \langle D\Psi(\mu), \mu\rangle \ . \tag{2.18}$$

To simplify the notation, we replace the superscript φ^μ by μ.

Let $\mathbb{P} \in \mathcal{M}_1(\Omega)$ be the path measure with starting distribution m. Similar to (1.2) we define the transformed path measures $\{\widehat{\mathbb{P}}_T\}_{T>0} \subset \mathcal{M}_1(\Omega)$ by

$$\widehat{\mathbb{P}}_T(A) = \mathbb{E}[1_A \exp(T\Psi(L_T))]/Z_T \ , \qquad A \in \mathcal{F}, T > 0, \tag{2.19}$$

where $Z_T = \mathbb{E}[\exp(T\Psi(L_T))]$. The main result of this section is the following theorem; for further results about the mixture Σ see Theorem 2.32.

Theorem 2.20 *The set K_Ψ is non-empty and compact, $\{\widehat{\mathbb{P}}_T\}_{T>0}$ is relatively compact in the weak topology on $\mathcal{M}_1(\Omega)$ as $T \to \infty$, and for each accumulation point $\widehat{\mathbb{P}}$ there exists $\Sigma \in \mathcal{M}_1(K_\Psi)$ such that*

$$\widehat{\mathbb{P}}(A) = \int_{K_\Psi} \mathbb{Q}^\mu(A)\, \Sigma(d\mu) \ , \qquad A \in \mathcal{F}.$$

Proof of Lemma 2.7. (a) Since $P^\varphi_t(x, \cdot) \ll \pi$ by Condition 2.1, we can define $p^\varphi_t(x, \cdot) = dP^\varphi_t(x, \cdot)/d\pi$ for each $x \in E$ and $t > 0$. For $s, t > 0$ with $2s < t$ define

$$p^\varphi_{s,t}(x, y) = \int_E p_s(x, \tilde{x}) \int_E p^\varphi_{t-2s}(\tilde{x}, \tilde{y}) p_s(\tilde{y}, y)\, \pi(d\tilde{y})\, \pi(d\tilde{x}) \ , \qquad x, y \in E.$$

Then $p^\varphi_{s,t} \in C(E^2, (0, \infty))$ by Condition 2.1. Furthermore,

$$|P^\varphi_t(x, A) - (P_s P^\varphi_{t-2s} P_s)(x, A)| \le c_t \mathrm{e}^{(t-2s)\|\varphi\|}(\mathrm{e}^{2s\|\varphi\|} - 1)\pi(A) \ ,$$

for each $A \in \mathcal{E}$ and $x \in E$, hence

$$\sup_{x \in E} \|p^\varphi_t(x, \cdot) - p^\varphi_{s,t}(x, \cdot)\|_\infty \le c_t \mathrm{e}^{t\|\varphi\|}(\mathrm{e}^{2s\|\varphi\|} - 1)$$

whenever $0 < 2s < t$. Since supp(π) = E by Condition 2.1, we may and will assume that $p_t^\varphi \in C(E^2,(0,\infty))$ for all $t > 0$. The estimate for p_t^φ follows from Condition 2.1 and the definition of P_t^φ.

(b) Since $p_t^\varphi \in C(E^2,(0,\infty))$, it follows that P_t^φ is a compact (compare [17, VI.5]), strictly positive [19, Def. II.2.4], and irreducible [19, p. 186] operator on $C(E)$. The logarithmic spectral radius of $\mathrm{e}^{-\Lambda^\varphi t}P_t^\varphi$ is 0, hence 1 is in the spectrum of $\mathrm{e}^{-\Lambda^\varphi t}P_t^\varphi$ [19, Proposition V.4.1] and therefore a pole of the resolvent [9, Theorem V.5.2]. By [19, Theorem V.5.2 and its corollary], there exists a unique $h_t^\varphi \in C(E,(0,\infty))$ which satisfies $\|h_t^\varphi\|_2 = 1$ and $h_t^\varphi = \mathrm{e}^{-\Lambda^\varphi t}P_t^\varphi h_t^\varphi$. Define $h^\varphi = h_1^\varphi$. Since $\{P_t^\varphi\}_{t\ge 0}$ is a semigroup, $P_t^\varphi h^\varphi = \mathrm{e}^{\Lambda^\varphi t}h^\varphi$ for all rational t in $[0,\infty)$. Using the right-continuity of the paths in Ω and the dominated convergence theorem, it follows that $(P_t^\varphi f)(x) = \mathbb{E}_x[\exp(t\langle\varphi, L_t\rangle)f(X_t)]$ and converges to $f(x)$ as $t \downarrow 0$ for every $x \in E$ and $f \in C(E)$. Applying [25, proof on p. 115] to $\{\mathrm{e}^{-\|\varphi\|t}P_t^\varphi\}_{t\ge 0}$, it follows that $\{P_t^\varphi\}_{t\ge 0}$ is strongly continuous. Hence, $P_t^\varphi h^\varphi = \mathrm{e}^{\Lambda^\varphi t}h^\varphi$ for all $t \ge 0$.

(c) Given $x \in E$, it follows from (b) that (2.8) defines a probability measure on $(\Omega,\mathcal{F}_t)$ for each $t \ge 0$ and that these measures are consistent. Hence, they can be uniquely extended to a measure $\mathbb{Q}_x^\varphi$ on $(\Omega,\mathcal{F})$ [16, Chap. V, Theorem 4.2]. The other properties of $\{\mathbb{Q}_x^\varphi\}_{x\in E}$ follow from those of $\{\mathbb{P}_x\}_{x\in E}$.

(d) follows from (a), (c), and the definition of P_t^φ.

(e) Let $\{Q_t^\varphi\}_{t\ge 0}$ denote the semigroup of stochastic transition kernels corresponding to $\{\mathbb{Q}_x^\varphi\}_{x\in E}$ in (c). According to [3, Exercise 4.1.48], there exists a unique Q_t^φ-invariant $\mu_t^\varphi \in \mathcal{M}_1(E)$ for each $t > 0$. Define $\mu^\varphi = \mu_1^\varphi$. The semigroup property of $\{Q_t^\varphi\}_{t\ge 0}$ implies that $\mu^\varphi Q_t^\varphi = \mu^\varphi$ for all rational t in $[0,\infty)$. Since $\{P_t^\varphi\}_{t\ge 0}$ is a strongly continuous semigroup by the proof of (b), the semigroup $\{Q_t^\varphi\}_{t\ge 0}$ of linear operators on $C(E)$ is strongly continuous, too. Choose $\varepsilon, t > 0$ and $f \in C(E)$. Using [3, (4.1.50)], there exist $n \in \mathbb{N}$ and a rational s in $(0, nt)$ such that $\|\mu^\varphi Q_{nt}^\varphi - \mu_t^\varphi\|_{\mathrm{var}} \le \varepsilon$ and $\|Q_{nt-s}^\varphi f - f\| \le \varepsilon$. Since $\mu^\varphi Q_{nt}^\varphi = \mu^\varphi Q_{nt-s}^\varphi$, it follows that $|\langle\mu_t^\varphi, f\rangle - \langle\mu^\varphi, f\rangle| \le 2\varepsilon$. Using [1, Theorem 1.3], it follows that $\mu_t^\varphi = \mu^\varphi$ for all $t > 0$, hence $\mu^\varphi Q_t^\varphi = \mu^\varphi$ for all $t \ge 0$.

(f) Using $\mu^\varphi Q_1^\varphi = \mu^\varphi$, (2.9), the continuity of p_1^φ and h^φ, and supp(π) = E, it follows that there exists a unique $f^\varphi \in C(E,(0,\infty))$ with $f^\varphi = d\mu^\varphi/d\pi$. Define $l^\varphi = f^\varphi/h^\varphi$.

(g) Note that $h^\varphi = \mathrm{e}^{-\Lambda^\varphi}P_1^\varphi h^\varphi$ by (b). Using the estimate for p_1^φ in (a) as well as $|\Lambda^\varphi| \le \|\varphi\|$ and $\|h^\varphi\|_1 \le \|h^\varphi\|_2 = 1$, the estimate for h^φ follows. Note that $l^\varphi h^\varphi = d(\mu^\varphi Q_1^\varphi)/d\pi$ by (e) and (f). Rewriting with (2.9), dividing by h^φ, using $|\Lambda^\varphi| \le \|\varphi\|$ and the estimates for p_1^φ and h^φ, the estimate for l^φ follows.

(h) Choose $\varphi \in C(E)$ with $\|\varphi\| \le C$. For $s, t > 0$ define

$$f(t) = \sup_{\mu\in\mathcal{M}_1(E)}\left\|\int_E q_t^\varphi(x,\cdot)\,\mu(dx) - l^\varphi h^\varphi\right\| \qquad \text{and} \qquad \alpha_s = \inf_{x,y\in E}\frac{q_s^\varphi(x,y)}{l^\varphi(y)h^\varphi(y)}\,.$$

Note that $\alpha_s \in (0,1]$. If $\alpha_s < 1$, then, for each $\mu \in \mathcal{M}_1(E)$,

$$\left\|\int_E q_{s+t}^\varphi(x,\cdot)\,\mu(dx) - l^\varphi h^\varphi\right\| = (1-\alpha_s)\left\|\int_E q_t^\varphi(y,\cdot)\,\tilde{\mu}_s(dy) - l^\varphi h^\varphi\right\|,$$

where $\tilde{\mu}_s \in \mathcal{M}_1(E)$ is given by

$$\tilde{\mu}_s(A) = \frac{1}{1-\alpha_s}\int_A\left(\int_E q_s^\varphi(x,y)\,\mu(dx) - \alpha_s l^\varphi(y)h^\varphi(y)\right)\pi(dy)\,, \qquad A \in \mathcal{E}.$$

Therefore, $f(s+t) \le (1-\alpha_s)f(t)$ in both cases. Using (2.9), (a), (g), and the estimate $|\Lambda^\varphi| \le \|\varphi\|$, it follows that $\alpha_1 \ge \exp(-8C)/c_1^4$ and $f(1) \le 2c_1^3\exp(6C)$. Hence, part (h) follows. □

Lemma 2.21 *Let M be the set of all functions $f\colon \Omega \to \mathbb{R}$, which are bounded, continuous, and $\mathcal{F}_s$-measurable for some $s \ge 0$. Then M is convergence determining [10, p. 112] for the weak topology on $\mathcal{M}_1(\Omega)$.*

Proof. By [10, Chap. 3, Theorem 3.1], the set of all bounded $f\colon \Omega \to \mathbb{R}$, which are uniformly continuous with respect to the metric d given by [10, Chap. 3, (5.2)], is convergence determining. We show that M is dense in this set.

Given f as above, choose $\varepsilon > 0$. There exists $s \ge 1$ such that $|f(\omega) - f(\tilde\omega)| \le \varepsilon$ for all $\omega, \tilde\omega \in \Omega$ with $d(\omega,\tilde\omega) \le \mathrm{e}^{1-s}$. Define $\pi_t\colon \Omega \to \Omega$ by $\pi_t(\omega)(u) = \omega(s \wedge u)$ for all $t, u \ge 0$ and $\omega \in \Omega$. Note that $d(\omega, \pi_t(\omega)) \le \mathrm{e}^{-t}$ by the definition of d. If $g\colon \Omega \to \mathbb{R}$ is defined by $g(\omega) = \int_{[s-1,s]} f(\pi_t(\omega))\,dt$, then g is bounded and measurable with respect to $\mathcal{F}_s$. Furthermore, $\|g - f\| \le \varepsilon$. Given $t \ge 0$ and $\omega \in \Omega$, it follows from [10, Chap. 3, (5.2), (5.3), and Proposition 5.2] that π_t is continuous at ω if t is a continuity point of ω. Since ω has at most countably many points of discontinuity [10, Chap. 3, Lemma 5.1], it follows with the dominated convergence theorem that g is continuous at ω. □

Lemma 2.22
(a) *The map $C(E) \ni \varphi \mapsto \Lambda^\varphi$ is continuous.*
(b) *The map $C(E) \ni \varphi \mapsto h^\varphi \in C(E,(0,\infty))$ is continuous.*
(c) *The map $C(E) \ni \varphi \mapsto \mathbb{Q}^\varphi \in \mathcal{M}_1(\Omega)$ is continuous.*
(d) *The map $C(E) \ni \varphi \mapsto l^\varphi \in C(E,(0,\infty))$ is continuous.*

Proof. (a) If $\varphi, \psi \in C(E)$, then $|\Lambda^\varphi - \Lambda^\psi| \le \|\varphi - \psi\|$ by (2.6).

(b) If $\varphi, \psi \in C(E)$, then $|P_t^\varphi(x,A) - P_t^\psi(x,A)| \le c_t \mathrm{e}^{t\|\varphi\|}(\mathrm{e}^{t\|\varphi-\psi\|} - 1)\pi(A)$ for all $x \in E$ and $A \in \mathcal{E}$, hence

$$\|p_t^\varphi - p_t^\psi\| \le c_t \mathrm{e}^{t\|\varphi\|}(\mathrm{e}^{t\|\varphi-\psi\|} - 1) \tag{2.23}$$

for all $t > 0$. If $t > 0$ and $\varphi \in C(E)$, then (2.9) shows that

$$g_t^\varphi(x,y,z) := \frac{q_1^\varphi(x,y)q_t^\varphi(y,z)}{q_{1+t}^\varphi(x,z)} = \frac{p_1^\varphi(x,y)p_t^\varphi(y,z)}{p_{1+t}^\varphi(x,z)}\,, \qquad x,y,z \in E. \tag{2.24}$$

Lemma 2.7(a) and (2.23) imply the continuity of $\varphi \mapsto g_t^\varphi$. Define g^φ in $C(E^3)$ by $g^\varphi(x,y,z) = q_1^\varphi(x,y)$. If $C > 0$, then Lemma 2.7(h) implies that

$$\sup_{\varphi\,:\,\|\varphi\|\le C} \|g_t^\varphi - g^\varphi\| \le \sup_{\varphi\,:\,\|\varphi\|\le C} \|q_1^\varphi\| \left\| \frac{C'\mathrm{e}^{-\varepsilon t} + C'\mathrm{e}^{-\varepsilon(1+t)}}{l^\varphi h^\varphi - C'\mathrm{e}^{-\varepsilon(1+t)}} \right\| \tag{2.25}$$

for all sufficiently large $t \ge 1$. Since this estimate is uniform on bounded subsets of $C(E)$, it follows by using Lemma 2.7(a), (g), (2.9), and $|\Lambda^\varphi| \le \|\varphi\|$ and letting $t \to \infty$, that the map $\varphi \mapsto q_1^\varphi$ is continuous. Since

$$\frac{1}{h^\varphi(x)} = \frac{\|h^\varphi\|_2}{h^\varphi(x)} = \left\| \frac{q_1^\varphi(x,\cdot)}{p_1^\varphi(x,\cdot)} \mathrm{e}^{\Lambda^\varphi} \right\|_2\,, \qquad x \in E, \tag{2.26}$$

by (2.9), it follows with (a) that $\varphi \mapsto 1/h^\varphi$ and $\varphi \mapsto h^\varphi$ are continuous.

(c) By part (b), the map $\varphi \mapsto \langle h^\varphi, m\rangle$ is continuous. Using (a) and (b), it follows that the map $\varphi \mapsto \langle \exp(-\Lambda^\varphi t)\mathbb{E}_{(\cdot)}[f\exp(t\langle\varphi, L_t\rangle)h^\varphi(X_t)], m\rangle$ is continuous for each $t > 0$

and each bounded, continuous, and $\mathcal{F}_t$-measurable $f\colon \Omega \to \mathbb{R}$. Using (2.8), (2.16), and Lemma 2.21, part (c) follows.

(d) If $C > 0$, then (2.9) and Lemma 2.7(g), (h) show that

$$\lim_{t\to\infty} \sup_{\varphi:\|\varphi\|\le C} \sup_{x\in E} \left\| \frac{\mathrm{e}^{-\Lambda^\varphi t}}{h^\varphi(x)} p_t^\varphi(x,\cdot) - l^\varphi \right\| = 0 \,.$$

Since this limit is uniform on bounded subsets of $C(E)$, part (d) follows from (a), (b), and (2.23). □

Lemma 2.27 *There exists a constant $M \ge 1$ such that $\mathbb{E}_x[\exp(T\Psi(L_T))] \le M\mathbb{E}_\pi[\exp(T\Psi(L_T))] \le M^2\mathbb{E}_y[\exp(T\Psi(L_T))]$ for all $x, y \in E$ and $T > 1$.*

Proof. Let C denote the common bound of the maps $D\Psi$ and R given by Condition 2.12. Define $M = c_1 \exp(4C)$. Note that

$$\begin{aligned} T|\Psi(L_T) - \Psi(L_{1,T+1})| &= |\langle D\Psi(L_{1,T}), L_1 - L_{T,T+1}\rangle \\ &\quad + R_{1/T}(L_{1,T}, L_1) - R_{1/T}(L_{1,T}, L_{T,T+1})| \le 4C \,. \end{aligned}$$

Using the Markov property and Condition 2.1, it follows that

$$\begin{aligned} \mathbb{E}_x[\exp(T\Psi(L_T))] &\le \mathrm{e}^{4C}\mathbb{E}_x[\exp(T\Psi(L_{1,T+1}))] \\ &= \mathrm{e}^{4C}\mathbb{E}_x[\mathbb{E}_{X_1}[\exp(T\Psi(L_T))]] \le M\mathbb{E}_\pi[\exp(T\Psi(L_T))] \end{aligned}$$

and $\mathbb{E}_y[\exp(T\Psi(L_T))] \ge M^{-1}\mathbb{E}_\pi[\exp(T\Psi(L_T))]$ for all $x, y \in E$. □

By Condition 2.12 and Lemma 2.22(d), the map $\mathcal{M}_1(E) \times E \ni (\mu, y) \mapsto l^\mu(y)$ is continuous in each argument and therefore jointly measurable. For $t, T > 0$ define $\Gamma_{t,T} \in \mathcal{M}_1(\mathcal{M}_1(E))$ by

$$\Gamma_{t,T}(A) = \frac{1}{c_{t,T}} \int_E \int_A l^\mu(y) \exp(t\Lambda^\mu + T\Psi(\mu))\, (\mathbb{P}_y L_T^{-1})(d\mu)\, \pi(dy) \tag{2.28}$$

for all $A \in \mathcal{B}(\mathcal{M}_1(E))$, where $c_{t,T}$ denotes the appropriate normalizing constant.

Proof of Theorem 2.20. Due to Condition 2.12 and [3, Theorem 4.2.43], the map $\Psi - J$ is upper semi-continuous. Therefore, $\Psi - J$ attains its supremum on a closed subset of the compact set $\mathcal{M}_1(E)$. Hence, K_Ψ is non-empty and compact.

Condition 2.12 and (2.18) imply that the map $\mu \mapsto \varphi^\mu$ is bounded. Take any $t > 0$. Using Lemma 2.7(g) and $|\Lambda^\varphi| \le \|\varphi\|$, it follows that there exists a constant $C > 0$ such that $\|t\Lambda^\mu + \log l^\mu\| \le C$ for all $\mu \in \mathcal{M}_1(E)$. Since Ψ is bounded and continuous by Condition 2.12, it follows from the full large deviation principle for $\{\, \mathbb{P}L_T^{-1} \mid T > 0,\ x \in E\,\}$ and Varadhan's theorem (compare [3, Exercise 2.1.24] and [2, Lemma 4.4]) that $\Gamma_{t,T}(U^{\mathrm{c}}) \to 0$ as $T \to \infty$ for every $t > 0$ and every neighbourhood $U \subset \mathcal{M}_1(E)$ of K_Ψ.

Since $\mathcal{M}_1(E)$ is compact, the set $\{\Gamma_{t,T}\}_{T\ge1}$ is relatively compact for each $t > 0$. By the above paragraph, each accumulation point of $\{\Gamma_{t,T}\}_{T\ge1}$ as $T \to \infty$ is concentrated on K_Ψ.

Let $\{T_n\}_{n\in\mathbb{N}}$ be a strictly increasing sequence tending to infinity. By the preceeding paragraph and a diagonal argument we may assume that, for each $k \in \mathbb{N}$, the sequence $\{\Gamma_{T_k,T_n-T_k}\}_{n>k}$ converges weakly to a measure Γ_{T_k} in $\mathcal{M}_1(\mathcal{M}_1(E))$ as $n \to \infty$. Note that $\Gamma_{T_k}(K_\Psi) = 1$. By choosing a further subsequence if necessary, we may assume that $\{\Gamma_{T_k}\}_{k\in\mathbb{N}}$ converges weakly to a measure Γ with $\Gamma(K_\Psi) = 1$. Let $f\colon \Omega \to [1,2]$ be a continuous function, which is $\mathcal{F}_s$-measurable for some $s \geq 0$. By Lemma 2.21, it suffices to show that

$$\lim_{n\to\infty} \widehat{\mathbb{E}}_{T_n}[f] = \int_{K_\Psi} \langle h^\mu, m\rangle \int_\Omega f\, d\mathbb{Q}^\mu\, \Gamma(d\mu) \Big/ \int_{K_\Psi} \langle h^\mu, m\rangle\, \Gamma(d\mu)\ . \qquad (2.29)$$

If $s < t < T$, then $L_T = (1 - t/T)L_{t,T} + (t/T)L_t$ and, by (2.13) and (2.18),

$$T\Psi(L_T) = t\langle \varphi^{L_{t,T}}, L_t\rangle + (T-t)\Psi(L_{t,T}) + tR_{t/T}(L_{t,T}, L_t)\ . \qquad (2.30)$$

If $A \in \mathcal{F}_t$ and $B \in \mathcal{B}(\mathcal{M}_1(E))$, then the Markov property for $\mathbb{P}$ implies that

$$\begin{aligned}\mathbb{E}[1_{A\times B}(\cdot, L_{t,T})|X_t = y] &= \mathbb{P}(A|X_t = y)\mathbb{P}_y(L_{T-t} \in B)\\ &= \int_{\mathcal{M}_1(E)} \mathbb{E}[1_{A\times B}(\cdot,\mu)|X_t = y]\, (\mathbb{P}_y L_{T-t}^{-1})(d\mu)\end{aligned}$$

for $\mathbb{P}X_t^{-1}$-almost all $y \in E$; therefore, by (2.30),

$$\begin{aligned}\mathbb{E}[f\exp(T\Psi(L_T))] = \int_E \int_{\mathcal{M}_1(E)} &\mathbb{E}[f \exp(t\langle\varphi^\mu, L_t\rangle + tR_{t/T}(\mu, L_t))|X_t = y]\tilde{p}_t(y)\\ &\times \exp((T-t)\Psi(\mu))\, \mathbb{P}_y L_{T-t}^{-1}(d\mu)\, \pi(dy)\ ,\end{aligned}$$

where $\tilde{p}_t = d(\mathbb{P}X_t^{-1})/d\pi$. Using (2.14) and the dominated convergence theorem, it follows that

$$\lim_{T\to\infty} \mathbb{E}\Big[\sup_{\mu\in\mathcal{M}_1(E)} \big|\exp(tR_{t/T}(\mu, L_t)) - 1\big|\Big] = 0$$

for every $t > 0$. Since $\mu \mapsto \varphi^\mu$ is bounded, it follows with Lemma 2.27 that, for each $t > s$, there exist $\{\varepsilon_{t,T}\}_{T>t} \subset (0,\infty)$ with $\varepsilon_{t,T} \to 1$ as $T \to \infty$ such that

$$\begin{aligned}\mathbb{E}[f\exp(T\Psi(L_T))] = \varepsilon_{t,T} \int_E \int_{\mathcal{M}_1(E)} &\mathbb{E}[f \exp(t\langle\varphi^\mu, L_t\rangle)|X_t = y]\tilde{p}_t(y)\\ &\times \exp((T-t)\Psi(\mu))\, \mathbb{P}_y L_{T-t}^{-1}(d\mu)\, \pi(dy)\ .\end{aligned}$$

By Lemma 2.7(c), (d), and (2.16),

$$\mathbb{E}[f\exp(t\langle\varphi^\mu, L_t\rangle)|X_t = y]\tilde{p}_t(y) = \mathrm{e}^{\Lambda^\mu t}\langle h^\mu, m\rangle l^\mu(y) \int_\Omega f \frac{q^\mu_{t-s}(X_s, y)}{h^\mu(y) l^\mu(y)}\, d\mathbb{Q}^\mu$$

for π-almost all $y \in E$. Since $\mu \mapsto \varphi^\mu$ is bounded, it follows with Lemma 2.7(g) and (h) that there exists $\{\tilde{\varepsilon}_t\}_{t>s} \subset (0,\infty)$ with $\tilde{\varepsilon}_t \to 1$ as $t \to \infty$ such that

$$\begin{aligned}\mathbb{E}[f\exp(T\Psi(L_T))] = \varepsilon_{t,T}\tilde{\varepsilon}_t \int_E \int_{\mathcal{M}_1(E)} &\left(\int_\Omega f\, d\mathbb{Q}^\mu\right)\langle h^\mu, m\rangle l^\mu(y)\\ &\times \exp(t\Lambda^\mu + (T-t)\Psi(\mu))\, \mathbb{P}_y L_{T-t}^{-1}(d\mu)\, \pi(dy)\ .\end{aligned}$$

Using (2.19) and (2.28), it follows that, for $s < t < T$,

$$\widehat{\mathbb{E}}_T[f] = \varepsilon'_{t,T}\tilde{\varepsilon}'_t \int_{\mathcal{M}_1(E)} \langle h^\mu, m\rangle \int_\Omega f\, d\mathbb{Q}^\mu\, \Gamma_{t,T-t}(d\mu) \Big/ \int_{\mathcal{M}_1(E)} \langle h^\mu, m\rangle\, \Gamma_{t,T-t}(d\mu)\ ,$$

where $\{\tilde{\varepsilon}'_t\}_{t>s}$ and $\{\varepsilon'_{t,T}\}_{T>t}$ have the same properties as $\{\tilde{\varepsilon}_t\}_{t>s}$ and $\{\varepsilon_{t,T}\}_{T>t}$. Condition 2.12 and Lemma 2.22(c) imply that the map $\mu \mapsto \mathbb{Q}^\mu$ is continuous. Using the assumptions about $\{T_n\}_{n\in\mathbb{N}}$, equation (2.29) follows. $\square$

Usually, it is difficult to determine whether $\{\widehat{\mathbb{P}}_T\}_{T>0}$ converges as $T \to \infty$ and, if it converges, to which limit law. There is, however, a special case, where this limit can be determined.

Let G be a compact and metrizable topological group. Then there exists a unique normalized left-invariant Haar measure σ [4, 14.1.5, 14.2.3], which is also right-invariant [4, 14.3.3]. Let $\{T_a\}_{a\in G}$ be a collection of bijective transformations of E onto itself such that the map $G \times E \ni (a, x) \mapsto T_a x$ is continuous and $T_a T_b = T_{ab}$ for all $a, b \in G$. If e denotes the neutral element of G, then $T_e = \mathrm{id}_E$, hence $T_a^{-1} = T_{a^{-1}}$. For each $a \in G$ define $\mathcal{T}_a : \mathcal{M}_1(E) \to \mathcal{M}_1(E)$ by $(\mathcal{T}_a\mu)(A) = \mu(T_a^{-1}(A))$ for all $A \in \mathcal{E}$ and $\mu \in \mathcal{M}_1(E)$. Note that $\mathcal{T}_a$ is continuous, $\mathcal{T}_a\mathcal{T}_b = \mathcal{T}_{ab}$, and $\mathcal{T}_a^{-1} = \mathcal{T}_{a^{-1}}$ for all $a, b \in G$. We assume:

Condition 2.31

(a) The function Ψ and the transition kernels $\{P_t\}_{t\geq 0}$ are G-invariant, which means that $\Psi = \Psi \circ \mathcal{T}_a$ and $P_t(T_a x, A) = P_t(x, T_a^{-1}(A))$ for all $t \geq 0$, $x \in E$, $A \in \mathcal{E}$, and $a \in G$.

(b) There exists $\nu \in K_\Psi$ such that the map $G \ni a \mapsto \Phi(a) := \mathcal{T}_a\nu$ is injective and $K_\Psi = \{\, \mathcal{T}_a\nu \mid a \in G \,\}$.

Typical examples satisfying (a) are Brownian motion and suitable Markovian jump processes on groups. Condition 2.31(b) is more restrictive and usually delicate to investigate. Of course, if K_Ψ contains just one measure, then Condition 2.31 is satisfied with $G = \{e\}$ and $T_e = \mathrm{id}_E$. The next section contains an example with $G = \mathbb{Z}_2$.

Theorem 2.32 *If the Conditions 2.1, 2.12, and 2.31 hold, then*

$$\lim_{T\to\infty} \widehat{\mathbb{P}}_T = \int_G \mathbb{Q}^{\mathcal{T}_a\nu} \langle h^{\mathcal{T}_a\nu}, m\rangle \, \sigma(da) \Big/ \int_G \langle h^{\mathcal{T}_a\nu}, m\rangle \, \sigma(da) \,. \tag{2.33}$$

Proof. Choose $a \in G$ arbitrarily. Define $\mathbb{T}_a : \Omega \to \Omega$ by $\mathbb{T}_a(\omega)(t) = T_a X_t(\omega)$ for all $t \geq 0$ and $\omega \in \Omega$. Then $X_t \circ \mathbb{T}_a = T_a \circ X_t$ and, since $\{P_t\}_{t\geq 0}$ is T_a-invariant, $\mathbb{P}_{T_a x} = \mathbb{P}_x \mathbb{T}_a^{-1}$ for each $x \in E$. By Condition 2.1, the measure $\mathcal{T}_a\pi$ is $\{P_t\}_{t\geq 0}$-invariant and the invariant measure is unique [3, Exercise 4.1.48], hence $\mathcal{T}_a\pi = \pi$. Since $\mathcal{T}_a$ is continuous and bijective, $J \circ \mathcal{T}_a^{-1}$ is the rate function for $\{\, \mathbb{P}_x L_T^{-1}\mathcal{T}_a^{-1} \mid x \in E,\, T > 0 \,\}$ [3, Lemma 2.1.4]. Since $L_T \circ \mathbb{T}_a = \mathcal{T}_a \circ L_T$ and $\mathbb{P}_{T_a x} L_T^{-1} = \mathbb{P}_x L_T^{-1}\mathcal{T}_a^{-1}$ for each $T > 0$, the uniqueness of the rate function [3, Lemma 2.1.1] implies that $J = J \circ \mathcal{T}_a^{-1}$, hence $J = J \circ \mathcal{T}_a$. Choose $\varphi \in C(E)$. Then $\Lambda^\varphi = \Lambda^{\varphi\circ T_a}$ by (2.6). Since $h^\varphi \circ T_a = \exp(-\Lambda^{\varphi\circ T_a} t) P_t^{\varphi\circ T_a}(h^\varphi \circ T_a)$ for all $t \geq 0$ and $\|h^\varphi \circ T_a\|_2 = 1$, Lemma 2.7(b) shows that $h^{\varphi\circ T_a} = h^\varphi \circ T_a$. Since $\mathbb{Q}^\varphi_{T_a x} = \mathbb{Q}^{\varphi\circ T_a}_x \mathbb{T}_a^{-1}$ for all $x \in E$, the measure $\mathcal{T}_a\mu^{\varphi\circ T_a}$ is $\{\mathbb{Q}^\varphi_{(\cdot)} X_t^{-1}\}_{t>0}$-invariant and Lemma 2.7(e) implies that $\mathcal{T}_a\mu^{\varphi\circ T_a} = \mu^\varphi$. Since $T_a^{-1} = T_{a^{-1}}$ and $d\mu^{\varphi\circ T_a}/d\pi = (h^\varphi l^\varphi) \circ T_a$, Lemma 2.7(f) implies that $l^{\varphi\circ T_a} = l^\varphi \circ T_a$.

Since $\Psi = \Psi \circ \mathcal{T}_a$ by assumption, $\mathcal{T}_a\mu \in K_\Psi$ for each $\mu \in K_\Psi$. Since $D\Psi$ is (up to a constant) uniquely determined by (2.13) and (2.14), it follows that $D\Psi(\mathcal{T}_a\mu)(T_a x) - D\Psi(\mu)(x)$ is constant and, by (2.18), $\varphi^{\mathcal{T}_a\mu} \circ T_a = \varphi^\mu$ for all $\mu \in \mathcal{M}_1(E)$. Therefore, $\Lambda^{\mathcal{T}_a\mu} = \Lambda^\mu$ and $l^{\mathcal{T}_a\mu} \circ T_a = l^\mu$. Finally, $\Gamma_{t,T}\mathcal{T}_a^{-1} = \Gamma_{t,T}$ for all $T > t > 0$. Since $\mathcal{T}_a$ is continuous, $\Gamma\mathcal{T}_a^{-1} = \Gamma$ for the limit Γ in the proof of Theorem 2.20.

Since $G \times E \ni (a, x) \mapsto T_a x$ is continuous, Φ is continuous, too. By Condition 2.31(b), the map Φ is injective. Therefore, if $\mathcal{G}$ denotes the Borel σ-algebra of G, then $\Phi(A) \in \mathcal{B}(\mathcal{M}_1(E))$ for every $A \in \mathcal{G}$ by Kuratowski's theorem [16, Chap. I, Corollary 3.3]. Since $\Gamma(K_\Psi) = 1$ by the proof of Theorem 2.20 and $K_\Psi = \Phi(G)$ by Condition 2.31(b), the measure $\tilde{\sigma} \in \mathcal{M}_1(G)$ with $\tilde{\sigma}(A) = \Gamma(\Phi(A))$ for all $A \in \mathcal{G}$ is well-defined. Since $\Phi(a^{-1}b) = \mathcal{T}_{a^{-1}}\Phi(b)$, $\mathcal{T}_{a^{-1}} = \mathcal{T}_a^{-1}$, and $\Gamma\mathcal{T}_a^{-1} = \Gamma$ for all $a, b \in G$, it follows that

$$\tilde{\sigma}(a^{-1}A) = \Gamma\Phi(a^{-1}A) = \Gamma(\mathcal{T}_{a^{-1}}\Phi(A)) = \Gamma\mathcal{T}_a^{-1}(\Phi(A)) = \Gamma(\Phi(A)) = \tilde{\sigma}(A)$$

for all $A \in \mathcal{G}$. Therefore, $\sigma = \tilde{\sigma}$ and the theorem follows from (2.29). □

3. Convergence of Path Measures Arising from a Jump Process

Let $\{X_t\}_{t\geq 0}$ denote a symmetric jump process on $E = \{0, 1\}$ with exponential holding times of expectation one. We denote by $\mathbb{P}_x$ the law of the process on the path space $\Omega = D([0,\infty), \{0,1\})$ which starts in $x \in \{0,1\}$. For $t > 0$ define $l_t = (1/t)\int_0^t X_s\, ds$. Since the measures in $\mathcal{M}_1(\{0,1\})$ can be parametrized by $\mu_p = p\delta_1 + (1-p)\delta_0$ for $p \in [0,1]$, the empirical distribution L_t, defined after (2.2), is given by μ_{l_t} for $t > 0$. The random variables $\{l_t\}_{t>0}$ satisfy a uniform large deviation principle [3, Theorem 4.2.43]. The corresponding rate function $J : \mathbb{R} \to [0,\infty]$ can either be calculated via (2.3) or, since the generator of the jump process is symmetric, via [3, Theorem 4.2.58 and (4.2.49)]. Both ways show that

$$J(p) = \begin{cases} 1 - 2\sqrt{p(1-p)} & \text{if } p \in [0,1], \\ \infty & \text{if } p \in \mathbb{R} \setminus [0,1]. \end{cases}$$

Fix a constant $\tau \in \mathbb{R}$ and define an interaction function $V : \{0,1\}^2 \to \mathbb{R}$ by $V(x,y) = \tau(xy + (1-x)(1-y))$ for $x, y \in \{0,1\}$. As in Example 2.15 define $\Psi(\mu) = \langle V, \mu^{\otimes 2}\rangle$. Then $\Psi(\mu_p) = \tau(p^2 + (1-p)^2)$ for $p \in [0,1]$ and $H_T = T\Psi(\mu_{l_T})$. We choose $m = \delta_0$ as starting distribution, hence $\mathbb{P} = \mathbb{P}_0$.

The transformed probability measures $\widehat{\mathbb{P}}_T$ and $\widehat{\mathbb{P}}_{\alpha,T}$ corresponding to $\mathbb{P}$ are defined by (1.2) and (1.4), respectively. Then the right-hand side of (1.6) is given by $\sup\{\,\Psi(\mu_p) - J(p) \mid p \in [0,1]\,\}$. If $\tau \leq 1$, then the supremum is attained only for $p = 1/2$. If $\tau > 1$, then there are exactly two maxima at $p_\pm = (1 \pm \sqrt{1 - 1/\tau^2})/2$.

Compared to the situation considered in [6], it would be easy to prove (1.6) in the present setting, but we do not need (1.6) explicitly.

For $\gamma > 0$ let $\mathbb{Q}^\gamma$ be the path measure of a jump process on $\{0,1\}$ starting in 0 with generator

$$L^\gamma = \begin{pmatrix} L^\gamma_{0,0} & L^\gamma_{0,1} \\ L^\gamma_{1,0} & L^\gamma_{1,1} \end{pmatrix} = \begin{pmatrix} -\gamma & \gamma \\ 1/\gamma & -1/\gamma \end{pmatrix} ,$$

and define the mixture

$$\widehat{\mathbb{P}}^\gamma = \frac{1}{1+\gamma}\mathbb{Q}^\gamma + \frac{\gamma}{1+\gamma}\mathbb{Q}^{1/\gamma} .$$

Theorem 3.1 *Define*

$$\gamma(\tau) = \begin{cases} 1 & \text{if } \tau \leq 1, \\ \tau + \sqrt{\tau^2 - 1} & \text{if } \tau > 1, \end{cases} \quad \text{and} \quad \tilde{\tau} = \begin{cases} \tau & \text{if } \tau \leq 1, \\ (\tau + 1/\tau)/2 & \text{if } \tau > 1. \end{cases}$$

Then

$$\lim_{T\to\infty} \widehat{\mathbb{P}}_T = \widehat{\mathbb{P}}^{\gamma(\tau)} \tag{3.2}$$

and

$$\lim_{\alpha\downarrow 0} \lim_{T\to\infty} \widehat{\mathbb{P}}_{\alpha,T} = \widehat{\mathbb{P}}^{\gamma(\tilde\tau)} \ . \tag{3.3}$$

Remark 3.4 Given $\alpha > 0$, we do not prove explicitly that $\{\widehat{\mathbb{P}}_{\alpha,T}\}_{T>0}$ converges to a limit as $T \to \infty$. Instead of (3.3) we prove that

$$\lim_{R\to\infty} \limsup_{\alpha\downarrow 0} \sup_{T\geq R/\alpha} r_\Omega(\widehat{\mathbb{P}}_{\alpha,T}, \widehat{\mathbb{P}}^{\gamma(\tilde\tau)}) = 0 \ , \tag{3.5}$$

where r_Ω denotes the Prohorov metric on $\mathcal{M}_1(\Omega)$.

Proof of (3.2). If $\tau \leq 1$, then $K_\Psi = \{\mu_{1/2}\}$ by (2.17). Example 2.15 and (2.18) give $D\Psi(\mu_{1/2})(x) = \tau$ and $\varphi^{\mu_{1/2}}(x) = \tau/2$ for $x \in \{0,1\}$, hence $\varphi^{\mu_{1/2}} - \Lambda^{\mu_{1/2}} = 0$ by (2.6). Therefore, $h^{\mu_{1/2}}$ from Lemma 2.7(b) is given by $h^{\mu_{1/2}}(x) = 1$, hence $m^{\mu_{1/2}} = \delta_0$ and $\mathbb{Q}^{\mu_{1/2}} = \mathbb{P}$ by (2.16), and (3.2) follows from Theorem 2.20.

If $\tau > 1$, then we apply Theorem 2.32 as follows: Abbreviating $\mu_{p_\pm}$ by $\mu_\pm$, we obtain $K_\Psi = \{\mu_+, \mu_-\}$, and $D\Psi(\mu_\pm)(x) = 2\tau(p_\pm x + (1-p_\pm)(1-x))$ according to Example 2.15. Using (2.18) and (2.6) and writing $\varphi_\pm$ for $\varphi^{\mu_\pm}$ and $\Lambda_\pm$ for $\Lambda^{\mu_\pm}$, it follows that $\varphi_\pm(x) = \tau p_\pm(2x - p_\pm) + \tau(1-p_\pm)(2(1-x) - (1-p_\pm))$ and $\Lambda_\pm = \tau p_\pm^2 + \tau(1-p_\pm)^2 - 1 + 1/\tau$. By the Feynman-Kac formula [18, Example IV.22.11], the semigroups $\{\exp(-\Lambda_\pm t)P_t^{\varphi_\pm}\}_{t\geq 0}$ with $P_t^{\varphi_\pm}$ given by (2.4) have the generators

$$\begin{pmatrix} \varphi_\pm(0) - \Lambda_\pm - 1 & 1 \\ 1 & \varphi_\pm(1) - \Lambda_\pm - 1 \end{pmatrix} = \begin{pmatrix} -2\tau p_\pm & 1 \\ 1 & -2\tau p_\mp \end{pmatrix}$$

with eigenvalues 0 and -2τ. The non-negative $\|\cdot\|_2$-normalized eigenfunction corresponding to 0 is given by $h^{\varphi_\pm}(0) = \sqrt{2(1-p_\pm)}$ and $h^{\varphi_\pm}(1) = \sqrt{2p_\pm}$. Since $h^{\varphi_\pm}(1)/h^{\varphi_\pm}(0) = 2\tau p_\pm = \gamma(\tau)^{\pm 1}$, it follows from (2.8) that $L^{\gamma(\tau)}$ and $L^{1/\gamma(\tau)}$ are the generators of $\{\mathbb{Q}_x^{\varphi_\pm}\}_{x\in\{0,1\}}$. □

We prove (3.5) only in the case $\tau > 1$, the other one is simpler. Notice that $\gamma(\tilde\tau) = \tau$ for all $\tau \geq 1$. We prove (3.5) essentially by the technique used in Section 2. There are, however, additional difficulties. We proceed as outlined in the introduction starting with (1.9), see Lemma 3.6 below. The crucial estimate is given in Lemma 3.11. It would not be difficult to prove the existence of the limit of $\widehat{\mathbb{P}}_{\alpha,R/\alpha}$ as $\alpha \downarrow 0$, which appears in (1.7). We do not need the limit explicitely, therefore we only study the behaviour for small α and large R. The uniformity stated in Lemma 3.11 is crucial for the interchange of the limits in (1.7). Lemma 3.11 depends on analytic considerations in Lemma 3.13. Once we have Lemma 3.11, the rest of the proof follows along the lines of Section 2.

For $R > 0$ define $C_R = \{\, f \in C([0,R],\mathbb{R}) \mid f(0) = 0 \,\}$, which is a Banach space with respect to the supremum norm. Let B_R denote the set of all Borel measurable functions from $[0,R]$ to $[0,1]$. For $\alpha, R > 0$ define the map $\mathcal{L}_{\alpha,R}\colon \Omega \to C_R$ by

$\mathcal{L}_{\alpha,R}(t) = tl_{t/\alpha}$ for all $t \in [0,R]$. Note that the random variable $\mathcal{L}_{\alpha,R}$ takes values in the set $\mathcal{G}_R = \{\, f \in C_R \mid 0 \le f(t) - f(s) \le t - s$ for all $s,t \in [0,R]$ with $s \le t\,\}$, which is compact by Ascoli's theorem. If $f \in \mathcal{G}_R$, then we denote by $f' \in B_R$ the density with respect to Lebesgue measure.

For $\alpha, R > 0$ and $x,y \in \{0,1\}$ define the probability measure $Q^{x,y}_{\alpha,R}$ on $\mathcal{G}_R$ by

$$Q^{x,y}_{\alpha,R} = \mathbb{P}_x(\,\mathcal{L}^{-1}_{\alpha,R}(\cdot) \mid X_{R/\alpha} = y\,)\ .$$

Lemma 3.6 *If $x,y \in \{0,1\}$ and $R > 0$, then the measures $\{Q^{x,y}_{\alpha,R}\}_{\alpha>0}$ satisfy a large deviation principle on $\mathcal{G}_R$ with the good rate function*

$$J_R(f) = \int_0^R J(f'(t))\,dt\ , \qquad f \in \mathcal{G}_R,$$

in the sense of [3], *i. e. the rate function J_R is convex, $\{\, f \in \mathcal{G}_R \mid J_R(f) \le r\,\}$ is compact for each $r \in \mathbb{R}$, and*

$$-\inf_{f\in A^\circ} J_R(f) \le \liminf_{\alpha\downarrow 0} \alpha \log Q^{x,y}_{\alpha,R}(A) \le \limsup_{\alpha\downarrow 0} \alpha \log Q^{x,y}_{\alpha,R}(A) \le -\inf_{f\in \bar{A}} J_R(f) \tag{3.7}$$

for each Borel subset A of $\mathcal{G}_R$. Here the interior A° and the closure $\bar{A}$ of A are taken with respect to the relative topology on $\mathcal{G}_R$.

Proof. For the measures $\mathbb{P}_x\mathcal{L}^{-1}_{\alpha,R}$, $\alpha > 0$, the function space large deviation principle on C_R follows from [3, Exercise 4.2.70] with the good rate function $\widetilde{J}_R\colon C_R \to [0,\infty]$ given by

$$\widetilde{J}_R(f) = \lim_{n\to\infty} \frac{R}{2^n} \sum_{k=1}^{2^n} J\left(\frac{f(2^{-n}kR) - f(2^{-n}(k-1)R)}{2^{-n}R}\right)\ . \tag{3.8}$$

If $f \in C_R \setminus \mathcal{G}_R$, then there exists $k_n \in \{1,\ldots,2^n\}$ for all sufficiently large $n \in \mathbb{N}$ such that the corresponding term in (3.8) is infinite, hence $\widetilde{J}_R(f) = \infty$. If $f \in \mathcal{G}_R$, then [7, VII.8] and the dominated convergence theorem show that $\widetilde{J}_R(f) = J_R(f)$.

To prove (3.7), first note that $\lim_{\alpha\downarrow 0} \mathbb{P}_x(X_{R/\alpha} = y) = 1/2$. The upper estimate in (3.7) then follows from $\mathbb{P}_x(\mathcal{L}_{\alpha,R} \in A, X_{R/\alpha} = y) \le \mathbb{P}_x(\mathcal{L}_{\alpha,R} \in A)$ and the corresponding estimate for the measures $\{\mathbb{P}_x\mathcal{L}^{-1}_{\alpha,R}\}_{\alpha>0}$. To show the lower estimate in (3.7), let A be an open subset of $\mathcal{G}_R$ and $\varepsilon \in (0,R)$. Let A_ε denote the set of all $g \in \mathcal{G}_{R-\varepsilon}$ such that there exists $f \in A$ satisfying $f|_{[0,R-\varepsilon]} = g$ and $\mathrm{dist}(f, \mathcal{G}_R \setminus A) > \varepsilon$. Since $\{\, f \in \mathcal{G}_R \mid f|_{[0,R-\varepsilon]} \in A_\varepsilon\,\} \subset A$ and since $\mathcal{L}_{\alpha,R}$ takes values in $\mathcal{G}_R$, it follows by using the Markov property that

$$\begin{aligned}\mathbb{P}_x(\mathcal{L}_{\alpha,R} \in A, X_{R/\alpha} = y) &\ge \mathbb{P}_x(\mathcal{L}_{\alpha,R-\varepsilon} \in A_\varepsilon, X_{R/\alpha} = y)\\ &\ge \mathbb{P}_x\mathcal{L}^{-1}_{\alpha,R-\varepsilon}(A_\varepsilon)\min\{\,\mathbb{P}_0(X_{\varepsilon/\alpha} = y), \mathbb{P}_1(X_{\varepsilon/\alpha} = y)\,\}\ .\end{aligned}$$

Since A_ε is an open subset of $\mathcal{G}_{R-\varepsilon}$, there exists an open subset B_ε of $C_{R-\varepsilon}$ such that $A_\varepsilon = B_\varepsilon \cap \mathcal{G}_{R-\varepsilon}$. Using the lower estimate for $\mathbb{P}_x(\mathcal{L}_{\alpha,R-\varepsilon} \in B_\varepsilon)$ as $\alpha \downarrow 0$, it only remains to show that

$$\inf_{\varepsilon\in(0,R)} \inf_{g\in A_\varepsilon} J_{R-\varepsilon}(g) \le \inf_{f\in A} J_R(f)\ . \tag{3.9}$$

Take any $f \in A$ and define $\varepsilon = \min\{R, \operatorname{dist}(f, \mathcal{G}_R \setminus A)\}/2$. Since A is open, $\varepsilon > 0$. If $g := f|_{[0,R-\varepsilon]}$, then $g \in A_\varepsilon$ and $J_{R-\varepsilon}(g) \le J_R(f)$, hence (3.9) holds. □

To prepare the application of Varadhan's theorem, define, for each $R > 0$, the mappings $g_R : B_R \to \mathbb{R}$ and $\bar{g}_R : \mathcal{G}_R \to \mathbb{R}$ by

$$g_R(\phi) = \int_0^R \mathrm{e}^{-t}\phi(t)\,dt \qquad \text{and} \qquad \bar{g}_R(f) = \mathrm{e}^{-R} f(R) + \int_0^R \mathrm{e}^{-t} f(t)\,dt$$

for all $\phi \in B_R$ and $f \in \mathcal{G}_R$. Extend the interaction function V to $[0,\infty)^2$ by defining $V(x,y) = \tau(xy + (1-x)(1-y))$ for all $x, y \in [0,\infty)$. Furthermore, define $F_R : [-1,1] \times B_R \to \mathbb{R}$ and $\bar{F}_R : [-1,1] \times \mathcal{G}_R \to \mathbb{R}$ by

$$F_R(\varrho,\phi) = \frac{1}{2}\int_0^R \int_0^R \mathrm{e}^{-|s-t|} V(\phi(s),\phi(t))\,ds\,dt + \tau\varrho \int_0^R \mathrm{e}^{-(R-t)}\phi(t)\,dt$$

and

$$\begin{aligned}\bar{F}_R(\varrho,f) = {} & \frac{\tau}{2} f^2(R) + \frac{\tau}{2}(R - f(R))^2 + \tau\varrho f(R) - F_R(\varrho,f) \\ & + \frac{\tau}{2}\int_0^R \left\{ f^2(t) + (t - f(t))^2 \right\} dt \\ & - \tau \int_0^R \mathrm{e}^{-(R-t)} \left\{ f(R) f(t) + (R - f(R))(t - f(t)) \right\} dt\end{aligned}$$

for all $\varrho \in [-1,1]$, $\phi \in B_R$, and $f \in \mathcal{G}_R$. Integration by parts shows that

$$\bar{g}_R(f) = g_R(f') \qquad \text{and} \qquad \bar{F}_R(\varrho,f) = F_R(\varrho,f') \tag{3.10}$$

for all $\varrho \in [-1,1]$ and $f \in \mathcal{G}_R$. Note that $\bar{g}_R$ and $\bar{F}_R$ are continuous.

For $x \in \{0,1\}$ let $\widehat{\mathbb{P}}^x_{\alpha,T}$ be the transformed path measure arising from $\mathbb{P}_x$ via (1.4). For $\delta > 0$ define $U_\delta = (\xi_0 - \delta, \xi_0 + \delta) \cup (\xi_1 - \delta, \xi_1 + \delta)$, where $\xi_0 = 1/(2\tau^2)$ and $\xi_1 = 1 - \xi_0$.

Lemma 3.11 *If $\delta > 0$ and $x \in \{0,1\}$, then*

$$\lim_{\alpha \downarrow 0} \sup_{T \ge R/\alpha} \widehat{\mathbb{P}}^x_{\alpha,T}(Y_{\alpha,R} \notin U_\delta) = 0 \tag{3.12}$$

for all sufficiently large R, where $Y_{\alpha,R} = \int_0^R \mathrm{e}^{-t} X_{t/\alpha}\,dt$.

Proof. Given $\alpha, R > 0$, define $X^{\alpha,R} : \Omega \to B_R$ by $X^{\alpha,R}(t) = X_{t/\alpha}$ for all $t \in [0,R]$. If $T \ge R/\alpha$, then

$$H_{\alpha,T} = H_{\alpha,[R/\alpha,T]} + \frac{\tau}{\alpha} Z_0(1 - \mathrm{e}^{-R}) + \frac{1}{\alpha} F_R(Z_1 - Z_0, X^{\alpha,R}),$$

where $H_{\alpha,[R/\alpha,T]}$ is given by (1.3) but with integration over $[R/\alpha, T]$ and

$$Z_1 = \int_R^{\alpha T} \mathrm{e}^{-(t-R)} X_{t/\alpha}\,dt \qquad \text{and} \qquad Z_0 = \int_R^{\alpha T} \mathrm{e}^{-(t-R)}(1 - X_{t/\alpha})\,dt\,.$$

Using (3.10), it follows that

$$\begin{aligned}
&\mathbb{E}_x[\,\exp(H_{\alpha,T});\ Y_{\alpha,R}\notin U_\delta\,]\\
&\quad=\mathbb{E}_x\Big[\exp\big(H_{\alpha,[R/\alpha,T]}+\alpha^{-1}\tau Z_0(1-\mathrm{e}^{-R})\big)\\
&\qquad\times\mathbb{E}_x\big[\exp\big(\alpha^{-1}\bar F_R(Z_1-Z_0,\mathcal{L}_{\alpha,R})\big);\ \bar g_R(\mathcal{L}_{\alpha,R})\notin U_\delta\ \big|\ \mathcal{F}^{R/\alpha}\big]\Big]\ ,
\end{aligned}$$

where $\mathcal{F}^{R/\alpha}$ is the σ-algebra generated by $\{X_t\}_{t\geq R/\alpha}$. Since a similar formula holds for $\mathbb{E}_x[\exp(H_{\alpha,T})]$, we obtain the estimate

$$\begin{aligned}
&\widehat{\mathbb{P}}^x_{\alpha,T}\big(Y_{\alpha,R}\notin U_\delta\big)\\
&\qquad\leq\sup_{\substack{\varrho\in[-1,1]\\ y\in\{0,1\}}}\frac{\mathbb{E}_x\big[\exp\big(\alpha^{-1}\bar F_R(\varrho,\mathcal{L}_{\alpha,R})\big);\ \bar g_R(\mathcal{L}_{\alpha,R})\notin U_\delta\ \big|\ X_{R/\alpha}=y\big]}{\mathbb{E}_x\big[\exp\big(\alpha^{-1}\bar F_R(\varrho,\mathcal{L}_{\alpha,R})\big)\ \big|\ X_{R/\alpha}=y\big]}\ .
\end{aligned}$$

Since $\bar F_R(\cdot,f)\colon[-1,1]\to\mathbb{R}$ is increasing for every $f\in\mathcal{G}_R$, it follows that

$$\widehat{\mathbb{P}}^x_{\alpha,T}\big(Y_{\alpha,R}\notin U_\delta\big)\leq\max_{\substack{k\in\{-n,\dots,n-1\}\\ y\in\{0,1\}}}\frac{\int_{G_{\delta,R}}\exp\big(\alpha^{-1}\bar F_R((k+1)/n,\cdot)\big)\,dQ^{x,y}_{\alpha,R}}{\int_{\mathcal{G}_R}\exp\big(\alpha^{-1}\bar F_R(k/n,\cdot)\big)\,dQ^{x,y}_{\alpha,R}}\ ,$$

for every $n\in\mathbb{N}$, where $G_{\delta,R}=\{\,f\in\mathcal{G}_R\mid\bar g_R(f)\notin U_\delta\,\}$.

For each $\varrho\in[-1,1]$ the function $\bar F_R(\varrho,\cdot)\colon\mathcal{G}_R\to\mathbb{R}$ is continuous. Hence, it is bounded on the compact set $\mathcal{G}_R$ and an analogue of [3, (2.1.9)] holds. Since $\bar g_R$ is continuous, $G_{\delta,R}$ is closed. Therefore, Lemma 3.6 and a version of Varadhan's theorem [3, Exercise 2.1.24(i)] imply that, for every $\varrho\in[-1,1]$,

$$\limsup_{\alpha\downarrow 0}\alpha\log\int_{G_{\delta,R}}\exp\big(\alpha^{-1}\bar F_R(\varrho,\cdot)\big)\,dQ^{x,y}_{\alpha,R}\leq\sup_{f\in G_{\delta,R}}\big(\bar F_R(\varrho,f)-J_R(f)\big)\ .$$

Analogously, by [3, Lemma 2.1.7], it follows that

$$\liminf_{\alpha\downarrow 0}\alpha\log\int_{\mathcal{G}_R}\exp\big(\alpha^{-1}\bar F_R(\varrho,\cdot)\big)\,dQ^{x,y}_{\alpha,R}\geq\kappa(\varrho):=\sup_{f\in\mathcal{G}_R}\big(\bar F_R(\varrho,f)-J_R(f)\big)\ .$$

The last three estimates together imply that

$$\begin{aligned}
&\limsup_{\alpha\downarrow 0}\frac{1}{\alpha}\log\sup_{T\geq R/\alpha}\widehat{\mathbb{P}}^x_{\alpha,T}\big(Y_{\alpha,R}\notin U_\delta\big)\\
&\qquad\leq\lim_{n\to\infty}\sup_{(\varrho,f)\in[-1,1]\times G_{\delta,R}}\big(\bar F(\min\{1,\varrho+1/n\},f)-J_R(f)-\kappa(\varrho)\big)\\
&\qquad=\sup_{(\varrho,f)\in[-1,1]\times G_{\delta,R}}\big(\bar F(\varrho,f)-J_R(f)-\kappa(\varrho)\big)\ ,
\end{aligned}$$

where the last equality follows from the uniform continuity of $\bar F_R$ on the compact set $[-1,1]\times\mathcal{G}_R$. Furthermore, the uniform continuity of $\bar F_R$ implies that the map $[-1,1]\ni\varrho\mapsto\kappa(\varrho)$ is continuous, too. Hence, $\bar\Lambda_R\colon[-1,1]\times\mathcal{G}_R\to(-\infty,0]$, defined by $\bar\Lambda_R(\varrho,f)=\bar F_R(\varrho,f)-J_R(f)-\kappa(\varrho)$, is upper semi-continuous and, therefore, attains its supremum on the compact set $[-1,1]\times G_{\delta,R}$. To prove the lemma, it suffices to show that

$$\{\,(\varrho,f)\in[-1,1]\times\mathcal{G}_R \mid \bar{\Lambda}_R(\varrho,f)=0 \text{ and } \bar{g}_R(f)\notin U_\delta\,\}=\emptyset$$

for all sufficiently large R. Using (3.10), this follows from the next lemma. □

Define $\Lambda_R\colon[-1,1]\times B_R\to(-\infty,0]$ by

$$\Lambda_R(\varrho,\phi)=F_R(\varrho,\phi)-\int_0^R J(\phi(t))\,dt-\sup_{\tilde\phi\in B_R}\left(F_R(\varrho,\tilde\phi)-\int_0^R J(\tilde\phi(t))\,dt\right).$$

Lemma 3.13 *For each $\delta>0$ there exists $R_\delta>0$ such that, for all $R>R_\delta$,*

$$\{\,(\varrho,\phi)\in[-1,1]\times B_R \mid \Lambda_R(\varrho,\phi)=0 \textit{ and } g_R(\phi)\notin U_\delta\,\}=\emptyset\,.$$

Proof. Define $x_*=\tau-1/\tau$. Since $\tau>1$, it follows that $x_*>0$. Since $U_{\delta'}\subset U_\delta$ for $0<\delta'<\delta$, it suffices to prove the lemma for $\delta\in(0,x_*]$. Define $c\colon\mathbb{R}\to\mathbb{R}$ by $c(x)=-4(\tau^2-\tau\sqrt{4+x^2}+1)$. Note that c is even and strictly increasing on $[0,\infty)$ with $c(0)=-4(\tau-1)^2$ and $c(x_*)=0$. Define $R_\delta>0$ by

$$R_\delta=\max\left\{\log\frac{2}{\delta},\ \frac{\tau}{\tau-1},\ \frac{4\tau}{\sqrt{c(x_*+\delta)}},\ 2\left(\frac{2\tau}{2\tau-\sqrt{-c(x_*-\delta)}}-1\right)^{-1}+1\right\}.$$

Fix $R\in(R_\delta,\infty)$. Suppose that $\varrho\in[-1,1]$ and $\phi\in B_R$ satisfy $\Lambda_R(\varrho,\phi)=0$, which is equivalent to saying that Λ_R attains its maximal value at (ϱ,ϕ). To prove the lemma, it suffices to show that $g_R(\phi)\in U_\delta$.

Define $p^\pm=1/2\pm\sqrt{\tau^2/(1+4\tau^2)}$ and $\phi^*(t)=p^-\vee(p^+\wedge\phi(t))$ for all $t\in[0,R]$. Since

$$J'(p)=2\,\frac{2p-1}{\sqrt{1-(2p-1)^2}}\,,\qquad p\in(0,1),$$

it follows that $J(p)-J(p^-)\geq 4\tau|p-p^-|$ for all $p\in[0,p^-]$ and $J(p)-J(p^+)\geq 4\tau|p-p^+|$ for all $p\in[p^+,1]$. Since $|F_R(\varrho,\phi_1)-F_R(\varrho,\phi_2)|\leq 3\tau\|\phi_1-\phi_2\|_{L^1}$ for all $\phi_1,\phi_2\in B_R$, it follows that $\Lambda_R(\varrho,\phi^*)-\Lambda_R(\varrho,\phi)\geq\tau\|\phi^*-\phi\|_{L^1}$. Since (ϱ,ϕ) maximizes Λ_R, we therefore may assume that $\phi(t)\in[p^-,p^+]$ for all $t\in[0,R]$.

For each $\tilde\phi\in B_R$ the function $(-p^-,p^-)\ni\varepsilon\mapsto\Lambda_R(\varrho,\phi+\varepsilon\tilde\phi)$ attains its maximum at $\varepsilon=0$. Considering the derivatives, we see that ϕ satisfies

$$\tau\int_0^R(2\phi(s)-1)\mathrm{e}^{-|s-t|}\,ds+\tau\varrho\mathrm{e}^{-(R-t)}-J'(\phi(t))=0 \tag{3.14}$$

for almost all $t\in[0,R]$. Define $\psi=J'\circ\phi$. Then $\phi(t)=(g(\psi(t))+1)/2$ for all t in $[0,R]$, where the bounded function $g\in C^\infty(\mathbb{R})$ is given by $g(x)=x(4+x^2)^{-1/2}$. Equation (3.14) implies that

$$\psi(t)=\tau\varrho\mathrm{e}^{-(R-t)}+\tau\int_0^R g(\psi(s))\mathrm{e}^{-|s-t|}\,ds \tag{3.15}$$

for almost all $t\in[0,R]$. Since the right-hand side of (3.15) is continuous, we may assume that ψ is continuous and that (3.15) holds for all $t\in[0,R]$. It then follows that ψ and ϕ are in $C^\infty([0,R],\mathbb{R})$. Hence, (3.14) holds for $t = 0$ and shows that $g_R(\phi) = (\psi(0)+\tau)/(2\tau) - (1+\varrho)\mathrm{e}^{-R}/2$. If we can prove that

$$|\psi(0) - x_*| < \delta \qquad \text{or} \qquad |\psi(0) + x_*| < \delta\ , \tag{3.16}$$

then $\min\{\,|g_R(\phi)-\xi_1|,\ |g_R(\phi)-\xi_0|\,\} \le \delta/(2\tau)+\mathrm{e}^{-R} < \delta$, because $R > R_\delta \ge \log(2/\delta)$. This means that $g_R(\phi) \in U_\delta$. Therefore, it remains to prove (3.16).

If $a, b \in [0,1/2)$ or $a, b \in (1/2,1]$, then $a(1-b) + (1-a)b < ab + (1-a)(1-b)$. Furthermore, note that $J(a) = J(1-a)$ for all $a \in [0,1]$. If there were $s, t \in [0,R]$ with $\phi(s) < 1/2$ and $\phi(t) > 1/2$, then $F_R(0,\phi) < F_R(0,\phi_+) = F_R(0,\phi_-)$, where $\phi_+ := \max\{\phi, 1-\phi\}$ and $\phi_- := \min\{\phi, 1-\phi\}$; hence $\Lambda_R(\varrho,\phi) < \Lambda_R(\varrho,\phi_+)$ if $\varrho \in [0,1]$, and $\Lambda_R(\varrho,\phi) < \Lambda_R(\varrho,\phi_-)$ if $\varrho \in [-1,0]$, which contradicts the choice of (ϱ,ϕ). This proves that $\phi \ge 1/2$ if $\varrho \in (0,1]$, that $\phi \le 1/2$ if $\varrho \in [-1,0)$, and that $\phi - 1/2$ does not change its sign if $\varrho = 0$. Since $p^- \le \phi \le p^+$, as we proved above, it follows that $0 \le \psi \le 4\tau$ if $\varrho \in (0,1]$, that $-4\tau \le \psi \le 0$ if $\varrho \in [-1,0)$, and that either $0 \le \psi \le 4\tau$ or $-4\tau \le \psi \le 0$ in the case $\varrho = 0$.

Computing the first and the second derivative of (3.15), we see that ψ is a solution of the boundary value problem

$$y''(t) = y(t) - 2\tau g(y(t))\ , \quad t \in [0,R], \tag{3.17}$$

$$y'(0) = y(0)\ , \tag{3.18}$$

$$y'(R) = 2\tau\varrho - y(R)\ . \tag{3.19}$$

Since ψ solves (3.17) and (3.18), it follows that, for all $t \in [0,R]$,

$$\psi'^2(t) = \Big(2\tau - \sqrt{4+\psi^2(t)}\Big)^2 + c(\psi(0))\ . \tag{3.20}$$

If $\psi(0)$ is given, then ψ is uniquely determined by (3.17) and (3.18).

If $\psi(0) = 0$, then ψ is identically zero and (3.19) shows that $\varrho = 0$. Note that $\psi = 0$ corresponds to $\phi = 1/2$. According to the special choice of ϕ, the function $(-1,1) \ni \varepsilon \mapsto \Lambda_R(0, \phi+\varepsilon/2)$ attains its maximum at $\varepsilon = 0$, hence, since $\phi = 1/2$,

$$0 \ge \frac{d^2}{d\varepsilon^2}\Lambda_R(0,\phi+\varepsilon/2)\Big|_{\varepsilon=0} = (\tau-1)R - \tau(1-\mathrm{e}^{-R})\ ,$$

which is impossible because $R > R_\delta \ge \tau/(\tau-1)$.

If $|\psi(0)| \ge x_* + \delta$, then $|\psi'(t)| \ge c^{1/2}(x_*+\delta) > c^{1/2}(x_*) = 0$ for all $t \in [0,R]$ by (3.20). Using (3.18) and the continuity of ψ', it follows that $|\psi(R)| \ge c^{1/2}(x_*+\delta)R$. This is impossible because $R > R_\delta \ge 4\tau c^{-1/2}(x_*+\delta)$ and $|\psi(R)| \le 4\tau$.

If $\psi(0) \in (0, x_*-\delta]$, then $0 > c(x_*-\delta) \ge c(\psi(0)) \ge c(0) = -4(\tau-1)^2$. Define

$$y_0 = \sqrt{\big(2\tau - \sqrt{-c(\psi(0))}\,\big)^2 - 4}\ ,$$

$$y_1 = \sqrt{\big(2\tau - \sqrt{-c(x_*-\delta)}\,\big)^2 - 4}\ ,$$

$$t_1 = \left(\frac{2\tau}{\sqrt{4+y_1^2}} - 1\right)^{-1}\ ,$$

and note that $t_1 \le (R_\delta - 1)/2$ and

$$\psi(0) = \sqrt{\big(2\tau - \sqrt{-c(\psi(0)) + \psi^2(0)}\,\big)^2 - 4} < y_0 \le y_1 < 2\sqrt{\tau^2-1}\ .$$

If $\psi(t) \in [\psi(0), y_0]$, then (3.17) implies that

$$\psi''(t) = -\psi(t)\left(\frac{2\tau}{\sqrt{4+\psi^2(t)}} - 1\right) \leq -\frac{\psi(0)}{t_1} < 0 \; .$$

Hence, since $\psi(0) = \psi'(0) > 0$ by (3.18), there exists a smallest $t_0 \in [0, t_1]$ such that $\psi'(t_0) = 0$ or $\psi(t_0) = y_0$. By (3.20), each of these two equations implies the other one. According to (3.17), the functions $\psi|_{[t_0,2t_0]}$ and $\bar{\psi}\colon [t_0, 2t_0] \to \mathbb{R}$, given by $\bar{\psi}(t) = \psi(2t_0 - t)$, solve the initial value problem

$$y''(t) = y(t) - 2\tau g(y(t)) \; , \qquad y'(t_0) = 0 \; , \qquad y(t_0) = y_0 \; .$$

Due to the uniqueness of the solution, it follows that $\psi(2t_0) = \psi(0)$ and $\psi'(2t_0) = -\psi'(0)$. Since $\psi(0) = \psi'(0) > 0$ and $\psi''(t) \leq 0$ if $\psi(t) \in [0, 2\sqrt{\tau^2 - 1}]$, there exists $t_2 \in [2t_0, 2t_0 + 1]$ where ψ changes its sign. This is impossible because we proved above that ψ does not change its sign in $[0, R]$. A similar argument shows that $\psi(0) \in [-x_* + \delta, 0)$ is impossible.

The preceding three paragraphs prove (3.16). □

Proof of (3.5) for $\tau > 1$. Define $\pi = (1/2, 1/2)$. For $x \in \{0, 1\}$ and $\varrho \in [0, 1]$ define $\varphi_\varrho(x) = \tau\varrho x + \tau(1-\varrho)(1-x)$. Since $p \mapsto \langle \varphi_\varrho, \mu_p \rangle - J(p)$ attains its supremum at

$$p_\varrho = \frac{1}{2}\left(1 + \tau\frac{2\varrho - 1}{\sqrt{4 + \tau^2(2\varrho-1)^2}}\right) ,$$

(2.6) yields $\Lambda^\varrho := \Lambda^{\varphi_\varrho} = (\tau - 2 + \sqrt{4 + \tau^2(2\varrho - 1)^2})/2$.

Next we determine h^{φ_ϱ}, μ^{φ_ϱ}, and l^{φ_ϱ}, which appear in Lemma 2.7. We use the abbreviations h^ϱ, μ^ϱ, and l^ϱ, respectively. According to the Feynman-Kac formula [18, Example IV.22.11], the semigroup $\{\exp(-\Lambda^\varrho t)P_t^{\varphi_\varrho}\}_{t\geq 0}$, where $P_t^{\varphi_\varrho}$ given by (2.4), has the generator

$$\begin{pmatrix} \varphi_\varrho(0) - \Lambda^\varrho - 1 & 1 \\ 1 & \varphi_\varrho(1) - \Lambda^\varrho - 1 \end{pmatrix} = \begin{pmatrix} -\gamma_\varrho & 1 \\ 1 & -1/\gamma_\varrho \end{pmatrix}$$

with $\gamma_\varrho := (\sqrt{4 + \tau^2(2\varrho-1)^2} + \tau(2\varrho - 1))/2$. The non-negative $\|\cdot\|_2$-normalized eigenfunction corresponding to the eigenvalue 0 is given by $h^\varrho(0) = \sqrt{2/(1+\gamma_\varrho^2)}$ and $h^\varrho(1) = \gamma_\varrho h^\varrho(0)$. Since $h^\varrho(1)/h^\varrho(0) = \gamma_\varrho$, it follows from (2.8) that L^{γ_ϱ} is the generator of $\{\mathbb{Q}_x^{\varphi_\varrho}\}_{x\in\{0,1\}}$. Note that $p_\varrho = (h^\varrho(1))^2\pi(1)$ and $1 - p_\varrho = (h^\varrho(0))^2\pi(0)$. Since $(1, \gamma_\varrho^2)L^{\gamma_\varrho} = 0$, it follows that $\mu^\varrho := (1 - p_\varrho, p_\varrho)$ is the $\{\mathbb{Q}_{(\cdot)}^{\varphi_\varrho} X_t^{-1}\}_{t\geq 0}$-invariant distribution and that $l^\varrho = h^\varrho$. Note that $\gamma_{\xi_1} = \tau$ and $\gamma_{\xi_0} = 1/\tau$.

We now follow the proof of Theorem 2.20. For $\varrho \in [0, 1]$ and $\alpha, t, T > 0$ with $t < T$ define

$$\Theta_{\alpha,[t,T]} = \int_t^T \alpha \mathrm{e}^{-\alpha(s-t)} X_s \, ds + \frac{1}{2}\mathrm{e}^{-\alpha(T-t)}$$

and

$$R_{\alpha,t,T}(\varrho) = H_{\alpha,t} - \tau\int_0^t (1 - \mathrm{e}^{-\alpha(t-s)})(\varrho X_s + (1-\varrho)(1 - X_s))\, ds - \frac{\tau}{2\alpha}(\mathrm{e}^{\alpha t} - 1)\mathrm{e}^{-\alpha T}.$$

We use the abbreviation $\Theta_{\alpha,T} = \Theta_{\alpha,[0,T]}$. Then, corresponding to (2.30),

$$H_{\alpha,T} = t\langle \varphi^{\Theta_{\alpha,[t,T]}}, \mu_{l_t}\rangle + H_{\alpha,[t,T]} + R_{\alpha,t,T}(\Theta_{\alpha,[t,T]}) \; , \tag{3.21}$$

where $H_{\alpha,[t,T]}$ is given by (1.3) but with integration over $[t,T]$ instead of $[0,T]$. Define $\varGamma_{\alpha,t,T} \in \mathcal{M}_1([0,1])$, for all Borel sets A of $[0,1]$, by

$$\varGamma_{\alpha,t,T}(A) = \sum_{y\in\{0,1\}} \frac{\pi(y)}{c_{\alpha,t,T}} \mathbb{E}_y[1_A(\Theta_{\alpha,T}) l^{\Theta_{\alpha,T}}(y) \exp(t\Lambda^{\Theta_{\alpha,T}} + H_{\alpha,T})] , \qquad (3.22)$$

where $c_{\alpha,t,T}$ denotes the appropriate normalizing constant. Since $\Lambda^\varrho = \Lambda^{1-\varrho}$ and $V(x,y) = V(1-x,1-y)$ as well as $\mathbb{P}_x(X_s = y) = \mathbb{P}_{1-x}(X_s = 1-y)$ and $h^\varrho(x) = h^{1-\varrho}(1-x)$ for all $s \geq 0$, $\varrho \in [0,1]$, and $x,y \in \{0,1\}$, it follows that $\varGamma_{\alpha,t,T}$ is symmetric, which means that $\varGamma_{\alpha,t,T}(A) = \varGamma_{\alpha,t,T}(\{\, 1-a \mid a \in A \,\})$ for all Borel sets A of $[0,1]$. Since $\sup\{\, |\Theta_{\alpha,T} - Y_{\alpha,R}| : T \geq R/\alpha \,\} \to 0$ as $\alpha \downarrow 0$ and $R \to \infty$, it follows from Lemma 3.11 that, for each $t > 0$, the measures $\{\varGamma_{\alpha,t,T}\}_{\alpha,T>0}$ converge weakly to the symmetric measure $(\delta_{\xi_1} + \delta_{\xi_0})/2$, uniformly for $T \geq R/\alpha$ as $\alpha \downarrow 0$ and then $R \to \infty$.

Let $f\colon \Omega \to [1,2]$ be a continuous function which is $\mathcal{F}_s$-measurable for some $s \geq 0$. By Lemma 2.21 it suffices to show that

$$\lim_{R\to\infty} \limsup_{\alpha\downarrow 0} \sup_{T\geq R/\alpha} \left| \widehat{\mathbb{E}}_{\alpha,T}[f] - \frac{1}{1+\tau}\int_\Omega f\, d\mathbb{Q}^\tau - \frac{\tau}{1+\tau}\int_\Omega f\, d\mathbb{Q}^{1/\tau} \right| = 0 , \quad (3.23)$$

which would prove (1.7) for this model, too. For $\alpha, T > 0$ and $y \in \{0,1\}$ define the measure $\bar{\varGamma}^y_{\alpha,T}$ by $\bar{\varGamma}^y_{\alpha,T}(A) = \mathbb{E}_y[1_A(\Theta_{\alpha,T}) \exp(H_{\alpha,T})]$ for all Borel sets A of $[0,1]$. Using (3.21) and the Markov property, it follows that, for every $t \in (s,T)$,

$$\mathbb{E}[f \exp(H_{\alpha,T})] = \sum_{y\in\{0,1\}} \int_0^1 \mathbb{E}[\, f \exp(t\langle \varphi_\varrho, \mu_{l_t}\rangle + R_{\alpha,t,T}(\varrho)); X_t = y\,]\, \bar{\varGamma}^y_{\alpha,T-t}(d\varrho).$$

Since $|R_{\alpha,t,T}(\varrho)| \leq \alpha\tau t^2 + 2\tau(\mathrm{e}^{-\alpha t} - 1 + \alpha t)/\alpha + \tau \mathrm{e}^{-R}(\mathrm{e}^{\alpha t} - 1)/(2\alpha)$ for all $\varrho \in [0,1]$ and $\alpha, R, t, T > 0$ with $T \geq R/\alpha$, there exists, for each $t \geq s$, a suitable subset $\{\, \varepsilon_{\alpha,t,T} \mid \alpha > 0, T > t \,\}$ of $(0,\infty)$ with $\sup\{\, |\varepsilon_{\alpha,t,T} - 1| : T \geq R/\alpha \,\} \to 0$ as $\alpha \downarrow 0$ and $R \to \infty$ such that

$$\mathbb{E}[f \exp(H_{\alpha,T})] = \varepsilon_{\alpha,t,T} \sum_{y\in\{0,1\}} \int_0^1 \mathbb{E}[\, f \exp(\langle \varphi_\varrho, \mu_{l_t}\rangle);\ X_t = y\,]\, \bar{\varGamma}^y_{\alpha,T-t}(d\varrho) .$$

By Lemma 2.7(c) and (d),

$$\mathbb{E}[\, f \exp(t\langle \varphi_\varrho, \mu_{l_t}\rangle);\ X_t = y\,] = \mathrm{e}^{\Lambda^\varrho t} h^\varrho(0) l^\varrho(y) \pi(y) \int_\Omega f \frac{q^\varrho_{t-s}(X_s, y)}{h^\varrho(y) l^\varrho(y)}\, d\mathbb{Q}^\varrho$$

for $y \in \{0,1\}$. Since $[0,1] \ni \varrho \mapsto \varphi^\varrho$ is bounded, it follows from Lemma 2.7(g) and (h) that there exists $\{\tilde{\varepsilon}_t\}_{t>s} \subset (0,\infty)$ with $\tilde{\varepsilon}_t \to 1$ as $t \to \infty$ such that

$$\mathbb{E}[f \exp(H_{\alpha,T})] = \varepsilon_{\alpha,t,T}\tilde{\varepsilon}_t \sum_{y\in\{0,1\}} \int_0^1 \Bigl(\int_\Omega f\, d\mathbb{Q}^\varrho\Bigr) h^\varrho(0)\pi(y) l^\varrho(y) \mathrm{e}^{\Lambda^\varrho t}\, \bar{\varGamma}^y_{\alpha,T-t}(d\varrho).$$

Using (1.4) and (3.22), it follows that, for $s < t < T$ and $\alpha > 0$,

$$\widehat{\mathbb{E}}_{\alpha,T}[f] = \varepsilon'_{\alpha,t,T}\tilde{\varepsilon}'_t \int_0^1 h^\varrho(0) \int_\Omega f\, d\mathbb{Q}^\varrho\, \varGamma_{\alpha,t,T-t}(d\varrho) \Big/ \int_0^1 h^\varrho(0)\, \varGamma_{\alpha,t,T-t}(d\varrho) ,$$

where $\{\tilde{\varepsilon}'_t\}_{t>s}$ and $\{\, \varepsilon'_{\alpha,t,T} \mid \alpha > 0, T > t \,\}$ have the same properties as $\{\tilde{\varepsilon}_t\}_{t>s}$ and $\{\, \varepsilon_{\alpha,t,T} \mid \alpha > 0, T > t \,\}$. Since $\varrho \mapsto \varphi_\varrho$ is continuous, Lemma 2.22(b) and (c) show that $\varrho \mapsto h^\varrho(0)$ and $\varrho \mapsto \mathbb{Q}^\varrho$ are continuous. Hence, (3.23) follows. □

4. Convergence of Path Measures in a Gaussian Model

Let $\Omega := \{\, \omega \in C([0,\infty),\mathbb{R}) \mid \omega(0) = 0 \,\}$ be the vector space of all continuous paths starting at the origin. The space Ω equipped with the usual invariant metric which induces the uniform convergence on compact intervals is a Polish space. Let $\mathcal{F}$ be the Borel σ-algebra on Ω, let $\mathbb{P}$ be the Wiener measure on $(\Omega,\mathcal{F})$ and denote by $\mathbb{E}$ the expectation with respect to $\mathbb{P}$. For each $t \geq 0$, define the evaluation map $\beta_t\colon \Omega \to \mathbb{R}$ by $\beta_t(\omega) = \omega(t)$. Let λ denote the Lebesgue measure on $\mathbb{R}$, and let $\langle\cdot,\cdot\rangle$ and $\|\cdot\|_2$ denote the usual inner product and norm of $L^2([0,\infty),\lambda)$, respectively. Furthermore, define $s \wedge t = \min\{s,t\}$ and $s \vee t = \max\{s,t\}$ for all $s,t \in \mathbb{R}$. Fix a positive real constant τ and define the function $V\colon \mathbb{R}^2 \to \mathbb{R}$ by $V(x,y) = -\tau^2(x-y)^2/4$. Let the transformed probability measures $\{\widehat{\mathbb{P}}^\tau_T\}_{T>0}$ and $\{\widehat{\mathbb{P}}^\tau_{\alpha,T}\}_{\alpha,T>0}$ on $(\Omega,\mathcal{F})$ be defined by (1.2) and (1.4), respectively.

It follows from Lemma 4.5 below that $\{\widehat{\mathbb{P}}^\tau_T\}_{T>0}$ and $\{\widehat{\mathbb{P}}^\tau_{\alpha,T}\}_{\alpha,T>0}$ are tight. In analogy with the previous sections, we can try to identify the limiting probability measures in (1.5) via variational problems. Of course, this will be a heuristic argument, since the Wiener measure is not sufficiently mixing. Nevertheless, it allows us to guess the correct limiting measures.

Let $J\colon \mathcal{M}_1(\mathbb{R}) \to [0,\infty]$ be the rate function associated with Brownian motion:

$$J(\mu) = \begin{cases} \frac{1}{2}\int_{\mathbb{R}} (f'(x))^2\,dx & \text{if } d\mu/d\lambda = f^2 \text{ and } f \in H^1(\mathbb{R}),\\ +\infty & \text{otherwise.}\end{cases}$$

Since J and $\widetilde{V}$ are translation invariant, the sets of solutions of the variational problems (1.6) and (1.9), respectively, are translation invariant. Denote by $m(\mu)$ and $\mathrm{var}(\mu)$ the mean and the variance, respectively, of $\mu \in \mathcal{M}_1(\mathbb{R})$. Then

$$\begin{aligned}\widetilde{V}(\mu,\nu) &= -\frac{\tau^2}{4}\int_{\mathbb{R}}\int_{\mathbb{R}} (x-y)^2\,\mu(dx)\,\nu(dy)\\ &= -\frac{\tau^2}{4}\big(\mathrm{var}(\mu) + \mathrm{var}(\nu) + (m(\mu)-m(\nu))^2\big)\end{aligned}$$

for all $\mu,\nu \in \mathcal{M}_1(\mathbb{R})$, in particular, it follows that $\widetilde{V}(\mu,\mu) = -\tau^2\,\mathrm{var}(\mu)/2$. Furthermore, if $(\mu_{R,t})_{t\in[0,R]}$ is a solution of the variational problem (1.9), then $m(\mu_{R,s}) = m(\mu_{R,t})$ for almost all $s,t \in [0,R]$, hence

$$\frac{1}{2R}\int_0^R\int_0^R \mathrm{e}^{-|s-t|}\,\widetilde{V}(\mu_{R,s},\mu_{R,t})\,ds\,dt = -\frac{1}{R}\int_0^R \frac{\tau_R^2(t)}{2}\,\mathrm{var}(\mu_{R,t})\,dt\,, \tag{4.1}$$

where $\tau_R^2(t) = \tau^2(2 - \mathrm{e}^{-t} - \mathrm{e}^{-(R-t)})/2$ for $t \in [0,R]$. Therefore, a solution μ of the variational problem (1.6) has to maximize $-\tau^2\,\mathrm{var}(\mu)/2 - J(\mu)$ and a solution $(\mu_{R,t})_{t\in[0,R]}$ of (1.9) should maximize $-\tau_R^2(t)\,\mathrm{var}(\mu_{R,t})/2 - J(\mu_{R,t})$ for every $t \in [0,R]$. Due to (4.1), we only need the possible limits of $\{\mu_{R,0}\}_{R>0}$ as R tends to infinity. Hence, since $\tau_R^2(0) \to \tau^2/2$ as $R \to \infty$, we need those $\mu \in \mathcal{M}_1(\mathbb{R})$ which maximize $-\tau^2\,\mathrm{var}(\mu)/4 - J(\mu)$.

Given $c \in \mathbb{R}$ and $\tau > 0$, define $h_{c,\tau}(x) = (\tau/\pi)^{1/4} \exp(-\tau(x-c)^2/2)$ for all $x \in \mathbb{R}$. Note that $h_{c,\tau}^2$ is a density of the normal distribution $\mu_{c,\tau}$ with mean c and variance $1/(2\tau)$. Furthermore, $h_{c,\tau}$ is the positive, $\|\cdot\|_{L^2(\mathbb{R},\lambda)}$-normalized eigenfunction associated with the largest eigenvalue of $(1/2)\Delta - \tau^2(x-c)^2/2$. Hence, $\{\mu_{c,\tau}\}_{c\in\mathbb{R}}$ should be the set of solutions of (1.6) and the measures $\{\mu_{c,\tau/\sqrt{2}}\}_{c\in\mathbb{R}}$ should maximize $-\tau^2 \operatorname{var}(\mu)/4 - J(\mu)$.

Using this heuristic argument, one can guess the limiting measures in the mean field and polaron case using (2.33): For each $c \in \mathbb{R}$ and $\tau > 0$ let $\mathbb{Q}^{c,\tau}$ denote the path measure on $(\Omega, \mathcal{F})$ of an Ornstein-Uhlenbeck process $\{X_t\}_{t\geq 0}$ which satisfies the stochastic differential equation $dX_t = d\beta_t - \tau(X_t - c)\,dt$ with $X_0 = 0$. Note that the drift is directed towards the center c and that $\mu_{c,\tau}$ is the reversible distribution for $\mathbb{Q}^{c,\tau}$. Finally, we define the probability measure $\widehat{\mathbb{P}}^\tau$ on $(\Omega, \mathcal{F})$ by

$$\widehat{\mathbb{P}}^\tau(A) = \int_{\mathbb{R}} \mathbb{Q}^{c,\tau}(A) \left(\frac{\tau}{4\pi}\right)^{1/4} h_{c,\tau}(0)\,dc\,, \qquad A \in \mathcal{F}.$$

Theorem 4.2 *If $\tau > 0$, then with respect to the weak convergence on $(\Omega, \mathcal{F})$*

$$\lim_{T\to\infty} \widehat{\mathbb{P}}_T^\tau = \widehat{\mathbb{P}}^\tau \qquad \textit{and} \qquad \lim_{\alpha\downarrow 0} \lim_{T\to\infty} \widehat{\mathbb{P}}_{\alpha,T}^\tau = \widehat{\mathbb{P}}^{\tau/\sqrt{2}}\,.$$

Although the variational problem gives us the correct answer, we cannot apply the theory of large deviations as in the previous sections, since the Wiener measure $\mathbb{P}$ is not sufficiently mixing. However, the interaction is quadratic and the transformed path measures are Gaussian (Lemma 4.5). In order to show the above theorem it will suffice to investigate the corresponding covariances.

To unify the treatment of the mean field and polaron type case, define the weight function $W_{\alpha,T} : [0,\infty)^2 \to [0,\infty)$ by

$$W_{\alpha,T}(s,t) = \begin{cases} (1/T)1_{[0,T]^2}(s,t) & \text{for } \alpha = 0 \text{ and } T > 0, \\ \alpha \exp(-\alpha|s-t|)1_{[0,T]^2}(s,t) & \text{for } \alpha, T > 0. \end{cases}$$

and introduce the transformed probability measure $\overline{\mathbb{P}}_{\alpha,T}^\tau$ on $(\Omega, \mathcal{F})$ by

$$\overline{\mathbb{P}}_{\alpha,T}^\tau(A) = \frac{1}{\bar{Z}_{\alpha,T}^\tau} \mathbb{E}\left[1_A \exp\left(-\frac{\tau^2}{4}\int_0^T\int_0^T W_{\alpha,T}(s,t)(\beta_s-\beta_t)^2\,ds\,dt\right)\right]$$

for all $A \in \mathcal{F}$, where $\bar{Z}_{\alpha,T}^\tau$ is the normalizing constant. Since $\overline{\mathbb{P}}_{\alpha,T}^\tau = \widehat{\mathbb{P}}_{\alpha,T}^{\sqrt{2}\tau}$ for $\alpha > 0$, Theorem 4.2 is proved if we show that

$$\lim_{T\to\infty} \overline{\mathbb{P}}_{0,T}^\tau = \widehat{\mathbb{P}}^\tau \qquad \text{and} \qquad \lim_{\alpha\downarrow 0}\lim_{T\to\infty} \overline{\mathbb{P}}_{\alpha,T}^\tau = \widehat{\mathbb{P}}^\tau\,. \tag{4.3}$$

For the remaining part of this section we omit the superscript τ.

For every weight function $W_{\alpha,T}$ define $k_{\alpha,T} : [0,\infty)^2 \to [0,\infty)$ by

$$k_{\alpha,T}(u,v) = \frac{\tau^2}{2}\int_0^T\int_0^T W_{\alpha,T}(s,t)\,1_{[s\wedge t, s\vee t]^2}(u,v)\,ds\,dt \tag{4.4}$$

and let $K_{\alpha,T}$ be the symmetric integral operator with kernel $k_{\alpha,T}$, which maps $L^2([0,\infty),\lambda)$ into itself. Since $k_{\alpha,T}$ is in $L^2([0,\infty)^2,\lambda^2)$, the operator $K_{\alpha,T}$ is Hilbert-Schmidt and therefore compact [17, Theorem IV.23 and VI.22(e)]. For every $\alpha > 0$

let K_α be the integral operator on $L^2([0,\infty),\lambda)$ with kernel k_α given by $k_\alpha(u,v) = (\tau^2/2)(\mathrm{e}^{-\alpha|u-v|} - \mathrm{e}^{-\alpha(u\vee v)})$ for $u,v \geq 0$.

Lemma 4.5

(a) *For every $\alpha \geq 0$ and $T > 0$ the operator $K_{\alpha,T}$ is positive, hence $I + K_{\alpha,T}$ is invertible and $\|(I + K_{\alpha,T})^{-1}\| \leq 1$. If $\alpha > 0$ and $T > 0$, then $\|K_{\alpha,T}\| \leq 2\tau^2/\alpha^2$. The operators $\{K_\alpha\}_{\alpha>0}$ have the same properties.*

(b) *If $\alpha \geq 0$ and $T > 0$, then $\overline{\mathbb{P}}_{\alpha,T}$ is a centered Gaussian measure which satisfies $\overline{\mathbb{E}}_{\alpha,T}[\beta_s\beta_t] = \langle 1_{[0,s]}, (I + K_{\alpha,T})^{-1} 1_{[0,t]}\rangle$ for all $s,t \geq 0$.*

(c) *The set $\{\overline{\mathbb{P}}_{\alpha,T}\}_{T>0}$ is tight for every $\alpha \geq 0$.*

(d) *For every $\alpha > 0$ the measures $\{\overline{\mathbb{P}}_{\alpha,T}\}_{T>0}$ converge weakly to $\widehat{\mathbb{P}}_\alpha$ as T tends to infinity, where $\widehat{\mathbb{P}}_\alpha$ is the centered Gaussian measure on $(\Omega,\mathcal{F})$ which satisfies $\widehat{\mathbb{E}}_\alpha[\beta_s\beta_t] = \langle 1_{[0,s]}, (I + K_\alpha)^{-1} 1_{[0,t]}\rangle$ for all $s,t \geq 0$.*

(e) *The set $\{\widehat{\mathbb{P}}_\alpha\}_{\alpha>0}$ is tight.*

Proof. (a) Since $W_{\alpha,T} \geq 0$, it follows from (4.4) that, for all $f \in L^2([0,\infty),\lambda)$,

$$\langle f, K_{\alpha,T} f\rangle = \frac{\tau^2}{2}\int_0^T\int_0^T W_{\alpha,T}(s,t)\langle 1_{[s\wedge t, s\vee t]}, f\rangle^2\,ds\,dt \geq 0\ , \tag{4.6}$$

hence $K_{\alpha,T}$ is positive and $\|(I + K_{\alpha,T})f\|_2^2 \geq \|f\|_2^2$. Therefore $I + K_{\alpha,T}$ is invertible with $\|(I + K_{\alpha,T})^{-1}\| \leq 1$. If $\alpha, T > 0$, then, for all $u,v \geq 0$,

$$k_{\alpha,T}(u,v) = \frac{\tau^2}{\alpha}(\mathrm{e}^{-\alpha|u-v|} - \mathrm{e}^{-\alpha(u\vee v)} - \mathrm{e}^{-\alpha T}(\mathrm{e}^{\alpha(u\wedge v)} - 1))1_{[0,T]^2}(u,v)\ . \tag{4.7}$$

Note that $k_{\alpha,T}$ is symmetric, $k_{\alpha,T} \geq 0$, and $\int_0^\infty k_{\alpha,T}(\cdot,v)\,dv \leq 2\tau^2/\alpha^2$. By the Cauchy-Schwarz inequality,

$$|(K_{\alpha,T}f)(u)|^2 \leq \int_0^\infty k_{\alpha,T}(u,v)\,dv \int_0^\infty k_{\alpha,T}(u,v)|f(v)|^2\,dv\ , \qquad u \geq 0,$$

hence $\|K_{\alpha,T}f\|_2 \leq (2\tau^2/\alpha^2)\|f\|_2$ for all $f \in L^2([0,\infty),\lambda)$. Using

$$\langle f, K_\alpha f\rangle = \frac{\tau^2}{2}\int_0^\infty\int_0^\infty \alpha\mathrm{e}^{-\alpha|s-t|}\langle 1_{[s\wedge t, s\vee t]}, f\rangle^2\,ds\,dt \geq 0\ ,$$

the same proof applies to the operators $\{K_\alpha\}_{\alpha>0}$.

(b) Let $\Phi\colon L^2([0,\infty),\lambda) \to L^2(\Omega,\mathcal{F},\mathbb{P})$ denote the Wiener integral with respect to the Brownian motion $\{\beta_t\}_{t\geq 0}$. Then $\{\Phi(f)\}_{f\in L^2([0,\infty),\lambda)}$ is a centered Gaussian process on $(\Omega,\mathcal{F},\mathbb{P})$ with covariance $\langle\cdot,\cdot\rangle$ as defined in [21, Theorem 2.3A]. Define $\Omega_T = \{\,\omega \in C([0,T],\mathbb{R}) \mid \omega(0) = 0\,\}$ and $Q\colon \Omega_T^2 \to \mathbb{R}$ by

$$Q(\omega,\tilde\omega) = \frac{\tau^2}{2}\int_0^T\int_0^T W_{\alpha,T}(s,t)(\omega(s) - \omega(t))(\tilde\omega(s) - \tilde\omega(t))\,ds\,dt\ .$$

Furthermore, define $S\colon L^2([0,\infty),\lambda) \to \Omega_T$ by $Sf(t) = \langle f, 1_{[0,t]}\rangle$ for all $t \in [0,T]$ and note that $Q(Sf,Sg) = \langle f, K_{\alpha,T}g\rangle$ for all $f,g \in L^2([0,\infty),\lambda)$. Since $K_{\alpha,T}$ is a positive compact operator, there exists a complete orthonormal system $\{f_i\}_{i\in\mathbb{N}}$ in $L^2([0,\infty),\lambda)$ and $\{\gamma_i\}_{i\in\mathbb{N}} \subset [0,\infty)$ such that $K_{\alpha,T}f_i = \gamma_i f_i$ for all $i \in \mathbb{N}$, see [17, Theorem VI.16].

Since $\sum_{i=1}^{\infty}\langle 1_{[s\wedge t,s\vee t]}, f_i\rangle^2 = \|1_{[s\wedge t,s\vee t]}\|_2^2 = |s-t|$ for all $s,t \in [0,T]$, it follows with (4.6) that

$$\sum_{i=1}^{\infty}\gamma_i = \sum_{i=1}^{\infty}\langle f_i, K_{\alpha,T} f_i\rangle = \frac{\tau^2}{2}\int_0^T\int_0^T W_{\alpha,T}(s,t)|s-t|\,ds\,dt\ ,$$

hence $K_{\alpha,T}$ is a trace class operator. Define the projection $\pi_T\colon \Omega \to \Omega_T$ by $\pi_T(\omega) = \omega|_{[0,T]}$. Since Φ is continuous, it follows that, for each $t \in [0,T]$,

$$\beta_t = \Phi(1_{[0,t]}) = \Phi\left(\sum_{i=1}^{\infty} Sf_i(t) f_i\right) = \sum_{i=1}^{\infty} Sf_i(t)\Phi(f_i) \qquad \mathbb{P}\text{-a.s.}$$

The last series converges uniformly in $t \in [0,T]$ $\mathbb{P}$-almost surely [11, Theorem 5.1]. Since Q is continuous and bilinear, it follows that $\mathbb{P}$-a.s.

$$Q(\pi_T,\pi_T) = \sum_{i,j=1}^{\infty}\Phi(f_i)\Phi(f_j)\langle f_i, K_{\alpha,T} f_j\rangle = \sum_{i=1}^{\infty}\gamma_i(\Phi(f_i))^2\ .$$

This corresponds to [21, (3.15)], hence [21, Theorem 3.11(b)] implies that the process $\{\Phi(f)\}_{f\in L^2([0,\infty),\lambda)}$ on $(\Omega,\mathcal{F},\overline{\mathbb{P}}_{\alpha,T})$ is centered Gaussian with covariance $\langle\cdot,(1+K_{\alpha,T})^{-1}\cdot\rangle$.

(c) For each $n \in \mathbb{N}$ define the projection π_n as in part (b). It follows from parts (a) and (b) that $\overline{\mathbb{E}}_{\alpha,T}[(\beta_s-\beta_t)^2] = \langle 1_{(s,t]},(I+K_{\alpha,T})^{-1}1_{(s,t]}\rangle \le t-s$ and, since $\overline{\mathbb{P}}_{\alpha,T}$ is Gaussian, $\overline{\mathbb{E}}_{\alpha,T}[(\beta_s-\beta_t)^4] \le 3(t-s)^2$ for all $s,t \in [0,\infty)$ with $s \le t$ and all $T > 0$. Theorem 12.3 in [1] and the remark following the proof of [1, Theorem 8.2] show that $\{\overline{\mathbb{P}}_{\alpha,T}\pi_n^{-1}\}_{T>0}$ is tight for each $n \in \mathbb{N}$. Analogously to the proof of [10, Chap. 3, Proposition 2.4] it follows that $\{\overline{\mathbb{P}}_{\alpha,T}\}_{T>0}$ is tight.

(d) If we show that, for all $f,g \in L^2([0,\infty),\lambda)$,

$$\lim_{T\to\infty}\langle f,(I+K_{\alpha,T})^{-1}g\rangle = \langle f,(I+K_\alpha)^{-1}g\rangle \tag{4.8}$$

then the weak convergence of $\{\overline{\mathbb{P}}_{\alpha,T}\}_{T>0}$ to $\widehat{\mathbb{P}}_\alpha$ as $T\to\infty$ follows from part (c) and [20, Lemma 2.3.1]. By (4.7) and the definition of k_α,

$$\lim_{T\to\infty}\|(K_\alpha - K_{\alpha,T})h\|_2 = 0 \tag{4.9}$$

for all continuous $h\colon [0,\infty)\to\mathbb{R}$ with compact support. Since $\|K_{\alpha,T}\| \le 2\tau^2/\alpha^2$ for all $T>0$ by part (a), equation (4.9) holds for all $h \in L^2([0,\infty),\lambda)$. Define $h = (I+K_\alpha)^{-1}g$. Then, for all $T > 0$,

$$\begin{aligned}
&|\langle f,((I+K_{\alpha,T})^{-1}-(I+K_\alpha)^{-1})g\rangle|\\
&\qquad = |\langle f,(I+K_{\alpha,T})^{-1}((I+K_\alpha)-(I+K_{\alpha,T}))h\rangle|\\
&\qquad \le \|f\|_2\|(K_\alpha-K_{\alpha,T})h\|_2\ ,
\end{aligned}$$

because $\|(I+K_{\alpha,T})^{-1}\| \le 1$ by part (a). Thus, (4.9) implies (4.8).

(e) Using (d) instead of (b), the proof is similar to the proof of part (c). □

Let R be the symmetric integral operator on $L^2([0,\infty),\lambda)$ whose kernel is given by $r(u,v) = (\tau/2)(\mathrm{e}^{-\tau|u-v|}-\mathrm{e}^{-\tau(u+v)})$ for all $u,v \ge 0$. As in the proof of Lemma 4.5(a) it follows that $\|R\| \le 1$.

Lemma 4.10 *The measure $\widehat{\mathbb{P}}$ is centered Gaussian and, for all $s, t \geq 0$,*

$$\begin{aligned}\widehat{\mathbb{E}}[\beta_s \beta_t] &= \frac{1}{2\tau}(\mathrm{e}^{-\tau|s-t|} - \mathrm{e}^{-\tau(s+t)}) + \frac{1}{\tau}(1 - \mathrm{e}^{-\tau s})(1 - \mathrm{e}^{-\tau t}) \\ &= \langle 1_{[0,s]}, (I - R) 1_{[0,t]} \rangle \ .\end{aligned} \tag{4.11}$$

Proof. By Itô's formula, $\mathbb{Q}^{c,\tau}$ is the law of the Gaussian process

$$X_t = c(1 - \mathrm{e}^{-\tau t}) + \mathrm{e}^{-\tau t} \int_0^t \mathrm{e}^{\tau u} \, d\beta_u \ , \qquad t \geq 0.$$

If we replace c by a $\mathrm{N}(0, 1/\tau)$-distributed random variable C which is independent of $\{\beta_t\}_{t \geq 0}$, then the new process is centered Gaussian with law $\widehat{\mathbb{P}}$ and covariances given by (4.11). The second equality in (4.11) can be verified by computing the corresponding integrals. □

Proof of Theorem 4.2. Due to Lemma 4.5(d), we have to show that the measures $\{\overline{\mathbb{P}}_{0,T}\}_{T>0}$ converge weakly to $\widehat{\mathbb{P}}$ as $T \to \infty$ and that the measures $\{\widehat{\mathbb{P}}_\alpha\}_{\alpha>0}$ converge weakly to $\widehat{\mathbb{P}}$ as $\alpha \downarrow 0$ in order to prove (4.3). In view of Lemma 4.5 and Lemma 4.10, it remains to show that $\overline{\mathbb{E}}_{0,T}[\beta_s \beta_t]$ as well as $\widehat{\mathbb{E}}_\alpha[\beta_s \beta_t]$ converge to $\widehat{\mathbb{E}}[\beta_s \beta_t]$ for all $s, t \geq 0$ as $T \to \infty$ and $\alpha \downarrow 0$, respectively [20, Lemma 2.3.1]. Hence it suffices to prove that, for all $f, g \in L^2([0, \infty), \lambda)$,

$$\lim_{T \to \infty} \langle f, (I + K_{0,T})^{-1} g \rangle = \langle f, (I - R) g \rangle \tag{4.12}$$

and

$$\lim_{\alpha \downarrow 0} \langle f, (I + K_\alpha)^{-1} g \rangle = \langle f, (I - R) g \rangle \ . \tag{4.13}$$

Choose $\mu > \tau$, define $g_\mu \in L^2([0, \infty), \lambda)$ by $g_\mu(v) = \exp(-\mu v)$, and define g_τ accordingly. Elementary calculations show that

$$R g_\mu = \frac{\tau^2}{\mu^2 - \tau^2}(g_\tau - g_\mu) \ , \tag{4.14}$$

hence

$$(I - R) g_\mu = \frac{1}{\mu^2 - \tau^2}(\mu^2 g_\mu - \tau^2 g_\tau) \ . \tag{4.15}$$

Explicit computations yield

$$(K_{0,T}(I - R) g_\mu)(u) = \frac{\tau^2}{\mu^2 - \tau^2}\left(g_\tau(u \wedge T) - g_\mu(u \wedge T) - \frac{u \wedge T}{T}(g_\tau(T) - g_\mu(T)) \right)$$

for all $u \geq 0$ and all $T > 0$. Using (4.15), further computations yield

$$\begin{aligned}(K_\alpha(I - R) g_\mu)(u) = \frac{\tau^2}{\mu^2 - \tau^2}\Bigg(&\frac{\tau}{\tau - \alpha} g_\tau(u) - \frac{\mu}{\mu - \alpha} g_\mu(u) - \frac{\alpha(\mu - \tau)\mathrm{e}^{-\alpha u}}{(\mu - \alpha)(\tau - \alpha)} \\ &+ \left(1 - \mathrm{e}^{-\alpha u}\right)\left(\frac{\tau}{\tau + \alpha} g_\tau(u) - \frac{\mu}{\mu + \alpha} g_\mu(u) \right) \Bigg)\end{aligned}$$

for all $u \geq 0$ and $\alpha \in (0,\tau)$. Note that $\alpha e^{-\alpha u} \leq (u \vee (1/\tau))^{-1}$ for these u and α. It now follows with (4.14) and the dominated convergence theorem, that

$$\lim_{T\to\infty} \|(I - (I + K_{0,T})(I - R))g_\mu\|_2 = 0 \tag{4.16}$$

and

$$\lim_{\alpha\downarrow 0} \|(I - (I + K_\alpha)(I - R))g_\mu\|_2 = 0 \; . \tag{4.17}$$

If $f \in L^2([0,\infty),\lambda)$, then, for all $T > 0$,

$$\begin{aligned} &|\langle f, (I + K_{0,T})^{-1}g_\mu - (I - R)g_\mu\rangle| \\ &\qquad \leq \|f\|_2 \|(I + K_{0,T})^{-1}\| \, \|(I - (I + K_{0,T})(I - R))g_\mu\|_2 \; . \end{aligned}$$

A similar estimate holds for every K_α. It follows from (4.16), (4.17), and Lemma 4.5(a) that

$$\lim_{T\to\infty} \langle f, (I + K_{0,T})^{-1}g_\mu - (I - R)g_\mu\rangle = 0 \tag{4.18}$$

and

$$\lim_{\alpha\downarrow 0} \langle f, (I + K_\alpha)^{-1}g_\mu - (I - R)g_\mu\rangle = 0 \; . \tag{4.19}$$

Let $U \subset L^2([0,\infty),\lambda)$ be the vector space of all finite linear combinations of the functions $\{g_\mu\}_{\mu>\tau}$. If $g \in L^2([0,\infty),\lambda)$ is orthogonal to U, then [24, Chap. I, Corollary 6.2b] implies that $g = 0$. Hence, $L^2([0,\infty),\lambda)$ coincides with the closure of U. By Lemma 4.5(a), $\|(I + K_{0,T})^{-1}\| \leq 1$ and $\|(I + K_\alpha)^{-1}\| \leq 1$ for all $\alpha, T > 0$. Therefore, (4.12) and (4.13) follow from (4.18) and (4.19), respectively. □

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