Some global topological properties of a free boundary problem appearing in a two dimensional controlled ruin problem

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Abstract

In this paper a two-dimensional Brownian motion (modeling the endowment of two companies), absorbed at the boundary of the positive quadrant, with controlled drift, is considered. The volatilities of the Brownian motions are *different*. We control the drifts of these processes and allow that both drifts add up to the maximal value of one. Our target is to choose the strategy in a way, s.t. the probability that both companies survive is maximized. It turns out that the state space of the problem is divided into two sets. In one set the first company gets the full drift, and in the other set the second one. We describe some topological properties of these sets and their separating curve.

1 Introduction

In this paper we investigate the following problem: Given is a two dimensional stochastic process, "living" in the positive quadrant. The individual components of the process are independent Brownian motions, with *different* volatilities and controllable non negative drift. The total drift should add up to one. The aim of the control is to maximize the probability that the two-dimensional process stays in the positive quadrant, in the following denoted by G, forever. (At least) two economic interpretations are possible: The first one, given by McKean and Shepp in [14], is that a government can influence the drift of the wealth of the companies by a certain tax policy, but the total amount of "support" is bounded by the condition that the sum of the drifts is one. The aim is that both companies should survive. On the other hand, formulating the model a bit different (see section 2), one could also imagine two collaborating companies, again with the aim that both of them survive. Collaborating companies were considered in Actuarial Mathematics recently, also with different objectives, e.g., to maximize expected discounted dividends until ruin, see. e.g. [11] and [13]. For our problem described above, we show that G is separated into two connected sets, where on the one set

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one has to give the full support to one company, and on the complement full support is provided for the other one. The sets are separated by a C^1-curve.

Let us note that, for the case of equal volatilities, it was shown in [14] - by guessing explicitly the value function of the problem - that this separating curve is just the first median, i.e., the optimal strategy is, what they called a push-bottom strategy. This means that full support is given to the weaker company. The authors also mention that the case of different volatilities is open. It is clear that global results on the structure of the solution depend crucially on the boundary conditions of the problem. And indeed, if one replaces the homogeneous boundary data (implied by the "ruin-type problem") on the finite part of our domain by different ones, our results would not be true any more, see [14] and [9]. Let us finally mention a result, proved in [10], namely that in the case that one of the volatilities appearing in the problem is small one finds, using methods from singular perturbation theory: There exists a function approximating the value function of the problem arbitrary well, and this function is provided explicitly. It turns out that this approximating value function, resp. the corresponding ϵ-optimal strategy, has the properties, which we shall prove in the present paper for the optimal strategy (allowing also general positive volatilities, and not necessarily small ones). Finally, let us mention that one can find some results for the case of correlated Brownian motions, in [12].

Our problem can be seen as a two-phase problem, and the structure of the Hamilton-Jabobi-Bellman (HJB) equation (see section 3) is similar to the one considered in [3]; the difference being that in their case the min-operator applies to two second order elliptic operators, where in our case the max operator applies to elliptic operators with second and first order ingredients, with identical second order ones. In [3] regularity results for their problem were proved. In general it seems that there are not a lot of results in the literature where two different operators appear in the formulation of the FBP, see section 4.3 of [6]. Moreover, there exists a vast literature concerning regularity and local results for free boundary problems, see e.g. [4] or [16] and the references therein. Results on more global or topological properties seem to be not so numerous. One example would be [7]. Our paper is intended to be a contribution in this direction.

2 The model

We consider the following two-dimensional controlled ruin problem. Let us denote the wealth of two companies by $(X_t)_{t\geq 0}$, resp. $(Y_t)_{t\geq 0}$, and the corresponding two dimensional state process by $(Z_t)_{t\geq 0}$, i.e.

$$Z_t = \begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} x + \int_0^t u_s\,ds + B_t^{(1)} \\ y + \int_0^t (1-u_s)\,ds + \sigma B_t^{(2)} \end{pmatrix}. \tag{1}$$

Here $(x,y) =: z$ denotes the initial endowment of the companies, $B^{(1)}, B^{(2)}$ are independent standard Brownian motions, and u_t is our control processes. We will write G for the positive quadrant, i.e. $G := \{(x,y)|x > 0, y > 0\}$, and σ denotes a positive constant. Moreover, we define the ruin time $\tau = \inf\{t < 0|Z_t \notin G\}$, i.e. the first time at which one of the two companies is ruined. Finally, we define the set of admissible strategies u as

$$\mathcal{U}_{x,y} := \{u|u_t = \hat{u}(Z_t) \text{ for a Borel measurable function } \hat{u}(z); 0 \leq u_t \leq 1\}. \tag{2}$$

Our aim is to maximize the target functional, given by

$$J(x,y,u) = \mathbf{P}_{x,y}\left(\tau = \infty\right) \to \max, \tag{3}$$

where

$$\tau_1 := \inf\{t > 0|X_t = 0\},$$

$$\tau_2 := \inf\{t > 0 | Y_t = 0\}, \tag{4}$$
$$\tau := \tau_1 \wedge \tau_2,$$

i.e. the probability that both companies survive should be maximized. The value function of the problem is given by

$$V(x,y) := \sup_{u \in \mathcal{U}} J(x,y,u). \tag{5}$$

Note that, writing the drift vector as $\binom{1/2+\hat{u}_t}{1/2-\hat{u}_t}$, models two collaborating companies with transfer payments $\hat{u}$, with the goal that both survive. This is the second interpretation mentioned in the introduction.

3 Preliminary results

Let us start with the definition of an anisotropic Laplacian $\Delta^{(\sigma)} := \frac{\partial^2}{\partial x^2} + \sigma^2 \frac{\partial^2}{\partial y^2}$. The HJB equation corresponding to our problem then reads

$$\mathcal{L}V := \max\{V_x, V_y\} + \frac{1}{2}\Delta^{(\sigma)} V = 0.$$

As in [9], Theorem 3.1, Proposition 3.1 and Proposition 3.2, we find

Theorem 3.1 *There exists a bounded solution $V \in C(\overline{G}) \cap C^2(G)$ of the system*

$$\begin{aligned} \mathcal{L}V &= 0, \\ V(x,0) &= 0, \\ V(0,y) &= 0, \\ \lim_{x\to\infty, y\to\infty} V(x,y) &= 1. \end{aligned}$$

Proposition 3.1 *The function $V(x,y)$ constructed in Theorem 3.1 is the value function of our problem (5).*

Proposition 3.2 *The value function $V(x,y)$ fulfills $V \in C^2(\overline{G} \setminus \{(0,0)\})$.*

Indeed, the only differences, in comparison to the proofs of the paper mentioned, are: Use in the proof of Theorem 3.1 the functions

$$w(x) = \left(1 - e^{-2x}\right)\left(1 - e^{-\frac{2y}{\sigma^2}}\right)$$
$$v(x) = 1 - e^{-x} - e^{-\frac{y}{\sigma^2}},$$

and the in the proof of Proposition 3.1 the process

$$R_t^{\tau_2} = 1 - e^{-\frac{2}{\sigma^2} Y_t^{\tau_2}}.$$

The following result (originally) by Hartman and Wintner will be crucial for our paper. So, for convenience of the reader, we state it explicitly.

Theorem 3.2 *Let $u \in W^{2,2}_{loc}(\Omega)$ be a non constant solution of*

$$\sum_{i,j=1}^{2} a_{ij} u_{x_i x_j} + \sum_{i=1}^{2} b_i u_{x_i} = 0,$$

where the a_{ij} are Lipschitz, symmetric in i,j and fulfill a uniform ellipticity condition. The b_i are bounded, $i,j = 1,2$.

For every $x^0 \in \Omega$, there exists an integer $n \geq 1$ and a homogeneous harmonic polynomial H_n, of degree n, such that u satisfies, as $x \to x^0$,

$$u(x) = u(x^0) + H_n(J(x - x^0)) + O(|x - x^0|^n),$$
$$Du(x) = DH_n(J(x - x^0)) + O(|x - x^0|^{n-1}).$$

Here, D denotes the gradient, and J is the matrix

$$J = \sqrt{a(x^0)^{-1}},$$

with $a(x^0)$ the matrix a_{ij}, evaluated at x^0.

Moreover, we have
(i) The interior critical points (the zeroes of the gradient of u) are isolated.
(ii) Every interior critical point x^0 has a finite multiplicity, that is, for every x in a neighbourhood of x^0,

$$c_1 \, |x - x^0|^m \leq |Du(x)| \leq c_2 \, |x - x^0|^m,$$

where c_1, c_2 are positive constants, $m = n - 1$ and n is the integer appearing above.
(iii) If x^0 is an interior critical point of multiplicity m, then, in a neighbourhood of x^0, the level line $\{x \in \Omega | u(x) = u(x^0)\}$ is made of $m + 1$ simple arcs intersecting at x^0.

Proof. See [1], Theorem H.-W., Remark 1.1+Remark 1.2.

4 Main result

Let us start this section with the definition of some functions and sets, which we shall need.

Definition 4.1

$$D(x, y) := V_x(x, y) - V_y(x, y),$$
$$G^* := \overline{G} \setminus \{(0, 0)\},$$
$$P := \{(x, y) \in G^* | \ D(x, y) > 0\}, \ \ R := \{(x, y) \in G^* | \ D(x, y) \geq 0\},$$
$$N := \{(x, y) \in G^* | \ D(x, y) < 0\}, \ \ S := \{(x, y) \in G^* | \ D(x, y) \leq 0\},$$
$$C := R \cap \overline{N},$$

where the closure of N is taken in G^*. Note that these sets have the following interpretations: In the set R (N) full drift is given to the X-company (Y-company). $R(N)$ is chosen w.l.o.g.. One could have taken $P(S)$ as well. (The optimal strategy is not always unique in stochastic control problems.) The set C is the set, where the strategy is changed. More precisely, for each point $z \in C$ one finds both types of strategy in each neighbourhood of z. Note that the following topological notions are understood in the trace topology of G^*, w.r.t. $\mathbf{R}^2$, with the sole exception of the second assertion in point c) of the following theorem.

Our main theorem reads now as follows:

Theorem 4.1 *One has*
a) P and N are simply connected sets.
b) $\{(x, y) \in G^* | \ D(x, y) = 0\} \subset \overline{N}$, as well as $\{(x, y) \in G^* | \ D(x, y) = 0\} \subset \overline{P}$, hence $C = \{(x, y) \in G^* | \ D(x, y) = 0\}$.
c) C is a C^1-curve in G^*, and $\hat{C} := C \cup \{(0, 0)\}$ is a connected set in the topology of $\mathbf{R}^2$.

In Figure 1 one can find a plot of the typical situation, where its topological features are proved in Theorem 4.1. We start the proof of the Theorem with several helpful lemmas.

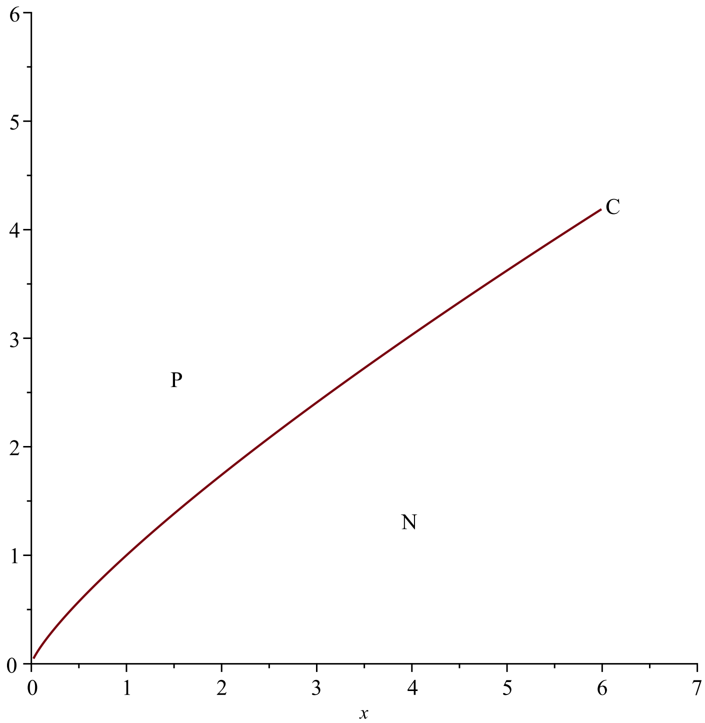

Figure 1: plot of a typical situation

Lemma 4.1 *On the boundary of G^* we have*

$$\{(0,y)|y>0\} \subset P,$$
$$\{(x,0)|x>0\} \subset N.$$

Proof. We only show the second relation, the first one works analogously. Using the strategy $u \equiv 1/2$, we get for the target functional

$$J(x,\epsilon,1/2) = \mathbf{P}(\tau_1 = \infty)\,\mathbf{P}(\tau_2 = \infty) = \left(1-e^{-x}\right)\left((1-e^{-\frac{\epsilon}{\sigma^2}}\right).$$

Moreover, we clearly have $J(x,0,u)=0$, for all admissible u, hence $V(x,0)=0$. As V is regular enough by Proposition 3.2, and since we have $V \geq J$, we conclude

$$V_y(x,0) \geq J_y(x,0,1/2),$$

hence

$$V_y(x,0) \geq (1-e^{-x})\frac{1}{\sigma^2} > 0 = V_x(x,0),$$

finishing our proof. □

The next result concerns the behavior of the function $D(x,y)$ for large values of (x,y). We defer its proof to the Appendix.

Lemma 4.2 *Let P_1, resp. N_1, be the connected component of P, resp. N, including $\{(0,y)|y>0\}$, resp. $\{(x,0)|x>0\}$. Then one has*

$$\lim_{\substack{(x,y)\to\infty \\ (x,y)\notin P_1\cup N_1}} D(x,y) = 0.$$

By $(x,y)\to\infty$ we mean $x^2+y^2\to\infty$. Let us remark that, since P, resp. N, are locally path wise connected, their connected components and connected path components are the same, see, e.g. [15], Theorem 25.5.

Before we show the simple connectedness of P and N, we need a preparatory result, the proof of which we defer to the Appendix. It is basically a consequence of forming the proper derivative of the HJB equation and known PDE regularity results.

Lemma 4.3 *On all simply connected open sets $M \subset G^*$ the function $D(x, y)$ is a distributional solution of*

$$\frac{1}{2}\Delta^{(\sigma)} D + \mathbf{1}_R D_x + \mathbf{1}_N D_y = 0, \quad \text{on } M$$

where $\mathbf{1}$ denotes the indicator function. Moreover, we have, if M is bounded, $D \in W^{2,p}_{loc}(M)$, $1 < p < \infty$.

Our next lemma shows, that the sets N and P are (pathwise) connected, i.e. we have

Lemma 4.4 *One has $G^* = P_1 \cup N_1 \cup \{(x, y) \in G^* |\ D(x, y) = 0\}$, which obviously implies $N_1 = N$ and $P_1 = P$.*

Proof. Let $z \in G^*$. We distinguish several cases.
Case A: $D(z) < 0$.
Let $\hat{N}$ be the connected component of N, with $z \in \hat{N}$.
Case A.1. $\hat{N} \cap N_1 = \emptyset$.
In this case we have $D_{/\partial\hat{N}} = 0$, where $\partial\hat{N}$ denotes the boundary of $\hat{N}$, and D fulfills $D_y + \frac{1}{2}\Delta^{(\sigma)} D = 0$ on $\hat{N}$. Now, the set $\hat{N}$ could be unbounded, but we can control the behavior of D on it by Lemma 4.2, i.e.

$$\lim_{\substack{z\to\infty \\ z\in\hat{N}}} D(x, y) = 0.$$

It allows, to apply the comparison principle in the form of [17], Theorem 10.3, resp. the Remark before Lemma 10.2. This yields $D_{/\hat{N}} \equiv 0$, hence $D(z) = 0$, a contradiction. So Case A.1 is not possible.
Case A.2. $\hat{N} \cap N_1 \neq \emptyset$.
By definition this implies $\hat{N} = N_1$, hence $z \in N_1$. Summing up we have in Case A: $z \in N_1$.
Case B: $D(z) > 0$.
Analogously one gets here $z \in P_1$, which proves the Lemma. □

Finally, we have

Proposition 4.1 *The sets N and P are simply connected.*

Proof. We give the proof only for N and argue by contradiction. So assume there exists a closed curve $\gamma \subset N$, and in the interior of γ there exists a point z_0 with $D(z_0) \geq 0$.

Now, in the interior of the curve γ the PDE of Lemma 4.3 holds, and on γ, D is strictly negative. Moreover, $D(z_0)$ is nonnegative, yielding a maximum in the interior, contradicting the maximum principle, see e.g. [8], Theorem 9.5. □

We turn now to the set C, where the strategy is changed. Here our first result is

Proposition 4.2 *We have $\{(x, y) \in G^* |\ D(x, y) = 0\} \subset \overline{N}$, as well as $\{(x, y) \in G^* |\ D(x, y) = 0\} \subset \overline{P}$, which implies*

$$C = \{(x, y) \in G^* |\ D(x, y) = 0\}.$$

Proof. We show only the first and last assertion and argue by contradiction. Assuming the first claim is false, gives the existence of a circle $B := B(z; \epsilon)$, with $D(z) = 0$ and $D_{/B} \geq 0$. On B we have by Lemma 4.3 a $W^{2,p}_{loc}$ solution of

$$\frac{1}{2}\Delta^{(\sigma)} D + D_x = 0.$$

This leaves three possibilities for the vicinity of z:
1.) a C^1-curve through z, separating the part with positive D from the part with negative D (the regular case, where the gradient of D at z does not vanish).

This follows from the regularity results in section 3 and the implicit function theorem.

2.) a finite number of curves intersecting at z, which form asymptotically neighbouring sectors, where the sign of D alternates
3.) a constant function D

The cases 2 and 3 follow from the Hartman-Wintner Theorem, see Theorem 3.2.

Obviously, the cases 1 and 2 are not possible, which leaves us with $D_{/B} \equiv 0$.

Now, let M be the connected component of $\{D = 0\}$, with $z \in M$, and consider its boundary ∂M. Let $\hat{z} \in \partial M$, and U a small neighbourhood of $\hat{z}$. In U we have again

$$\frac{1}{2}\Delta^{(\sigma)} D + \mathbf{1}_{R\cap U} D_x + \mathbf{1}_{N\cap U} D_y = 0,$$

s.t. we can again apply Theorem 3.2 from above. Now, one easily checks that each of the three possibilities above is incompatible with our construction. Indeed, D cannot be constant in the vicinity of $\hat{z}$, nor is it possible that a C^1-curve through $\hat{z}$ separates an area where D is positive from an area, where D is negative. Finally, a finite number of sectors, as described by the Hartman-Wintner result is also impossible. This yields a contradiction and concludes the proof of the first assertion.

The last assertion is easy. Indeed, obviously

$$R \cap \overline{N} \subset R \cap S = \{(x, y) \in G^* |\ D(x, y) = 0\}$$

holds. Moreover, $\{(x, y) \in G^* |\ D(x, y) = 0\} \subset R$ and (by the first assertion) $\{(x, y) \in G^* |\ D(x, y) = 0\} \subset \overline{N}$, hence $\{(x, y) \in G^* |\ D(x, y) = 0\} \subset R \cap \overline{N} = C$, finishing the proof. □

Our next result shows, that C is described by a C^1-curve.

Proposition 4.3 *The set C is given by a C^1-curve, i.e. for each point z on C, we can describe the set C locally by a C^1-functions, either $c(x)$ or $c(y)$.*

Proof. Let z be arbitrary on C, then we can again apply the "Hartman-Wintner" theorem. By the very definition of C, possibility 3.) is excluded. Now assume possibility 2.) is true. Then we would have asymptotically $2k$, $k = 2, 3, 4..., n$ sectors in the vicinity of z, where the sign of D alternates. We stick to the case $k = 2$, the other cases work analogously. So let the "sectors" S_2 and S_4 belong to N. Let k_1 be a continuous curve connecting S_2 and $\{(x, 0)|x > 0\}$, and k_2 a continuous curve connecting S_4 and $\{(x, 0)|x > 0\}$. Then either a continuous connection (lying entirely in P) from S_1 to $\{(0, y)|y > 0\}$ or from S_3 to $\{(0, y)|y > 0\}$ is prohibited by the sets k_1, k_2 and $\{(x, 0)|x > 0\}$. This contradicts the path connectedness of P, hence we have a contradiction.

Therefore we remain with possibility 1.), which proves the proposition. □

So far our results concerned the set G^*, since we do not know a better regularity result at the origin. In our last proposition we show that, if we affix the origin to the curve C, we get a connected set in the topology of $\mathbf{R}^2$.

Proposition 4.4 *Let $\hat{C} := C \cup \{0\}$, then $\hat{C}$ is connected in the topology of $\mathbf{R}^2$.*

Proof. Let $L := \{(0, y)|y > 0\}$, and $\hat{P} := P \setminus L$. The set $\hat{P}$ is open in $\mathbf{R}^2$. Moreover, it is path-connected, hence connected. Indeed, let $z_1, z_2 \in \hat{P}$. Then there exists a continuous path in P, connecting z_1 with z_2. We can easily - due to the continuity of D - deform this path to a path connecting the points and lying entirely in $\hat{P}$. Summing up we have

$$\hat{P} \text{ is an open and connected set in } \mathbf{R}^2. \qquad (6)$$

Now, by Proposition 4.2, $S = \{D \leq 0\}$ is connected. Indeed, Proposition 4.2 implies $S = \overline{N}$, and the closure of the connected set N is connected.

Let $T := G^c$. Then by, e.g., [2], Ex. 1.3, $S \cup T$ is connected, since S and T have a non empty intersection. As $\hat{P} = (S \cup T)^c$, we get by (the only) Theorem of [5],

$$\partial \hat{P} \text{ is connected .} \tag{7}$$

Our next claim is

$$\{0\} \in \overline{C}. \tag{8}$$

Indeed, if this is not the case, there will exist a circle $B := B(0; \epsilon)$ in $\mathbf{R}^2$, s.t. $B \cap C = \emptyset$. W.l.o.g. this would imply $B \cap R = \emptyset$, an obvious contradiction. Hence, (8) is true.

By the definition of C, we have $C \cap \overline{L} = \emptyset$, which gives

$$\hat{C} \cap \overline{L} = \{0\}. \tag{9}$$

Moreover, we have

$$\overline{C} \cap L = \emptyset. \tag{10}$$

Indeed, assuming that this is false, would give the existence of a $y_0 > 0$, s.t. $(0, y_0) \in \overline{C}$. As $D(0, y_0) > 0$, and D is continuous, this is not possible. Hence, we get the validity of (10).

After this preliminary considerations, we finally show the connectedness of $\hat{C}$ and argue again by contradiction. So assume we have $\hat{C} = C_1 \cup C_2$, with

$$\begin{aligned} \overline{C_1} \cap C_2 &= \emptyset, \\ \overline{C_2} \cap C_1 &= \emptyset. \end{aligned} \tag{11}$$

W.l.o.g. we assume $\{0\} \in C_1$, and $\{0\} \notin \overline{C_2}$, which provides by (9),

$$C_2 \cap \overline{L} = \emptyset. \tag{12}$$

In addition we have $\overline{\hat{C}} = \overline{C} = \overline{C_1} \cup \overline{C_2}$, which yields by (10),

$$\overline{C_2} \cap L = \emptyset. \tag{13}$$

Finally, we get for the boundary of $\hat{P}$, $\partial \hat{P} = \hat{C} \cup L = C_2 \cup (C_1 \cup L)$, as well as

$$\begin{aligned} \overline{C_2} \cap (C_1 \cup L) &= \big(\overline{C_2} \cap C_1\big) \cup \big(\overline{C_2} \cap L\big) = \emptyset, \\ C_2 \cap \overline{(C_1 \cup L)} = C_2 \cap \big(\overline{C_1} \cup \overline{L}\big) &= \big(C_2 \cap \overline{C_1}\big) \cup \big(C_2 \cap \overline{L}\big) = \emptyset, \end{aligned}$$

where we have used (11),(12) and (13). This would give a non connected set $\partial \hat{P}$, contradicting (7) and concluding the proof. □

Now we have all the requisites for the

Proof of Theorem 4.1. This is just a consequence of the Propositions 4.1,4.2,4.3 and 4.4. □

5 Appendix

The first aim of this Appendix is, to show Lemma 4.2. In order to do this, we need some auxiliary results and start with

Lemma 5.1 *One has*

$$\lim_{y \to \infty} V(x, y) = 1 - e^{-2x}, \quad \textit{unif. for } x \geq 0,$$
$$\lim_{x \to \infty} V(x, y) = 1 - e^{-\frac{2y}{\sigma^2}}, \quad \textit{unif. for } y \geq 0.$$

Proof. We show only the first relation: Let $\tilde{u} \equiv 1-\epsilon$, then we find for the corresponding stopping times, $\mathbf{P}(\tau_1 = \infty) = 1 - e^{-2(1-\epsilon)x}$, and $\mathbf{P}(\tau_2 = \infty) = 1 - e^{-\frac{2\epsilon}{\sigma^2} y}$. Moreover, we define the function $g(x,\epsilon)$ by

$$g(x,\epsilon) := (1-e^{-2x}) - (1-e^{-2(1-\epsilon)x}) = e^{-2(1-\epsilon)x} - e^{-2x},$$

and calculate its maximum over $\mathbf{R}_+$, for fixed ϵ. This gives $0 \le g(x,\epsilon) \le \frac{2\epsilon}{e}$, for ϵ small enough. Hence,

$$\begin{aligned} \left|J(x,y,\tilde{u}) - (1-e^{-2x})\right| &= \left|\mathbf{P}(\tau_1 = \infty)\mathbf{P}(\tau_2 = \infty) - (1-e^{-2x})\right| \\ &= \left|\left(1-e^{-2(1-\epsilon)x}\right)\left(1-e^{-\frac{2\epsilon}{\sigma^2}y}\right) - \left(1-e^{-2x}\right)\right| \\ &\le \frac{2\epsilon}{e} + e^{-\frac{2\epsilon}{\sigma^2}y}. \end{aligned}$$

Choosing y large enough, and noting that one clearly has $V(x,y) \le 1-e^{-2x}$, proves the Lemma. □

By the next lemma, we want to control the derivatives of V, for large (x,y), away from the boundary. We find

Lemma 5.2 *One has*

$$\begin{aligned} &\lim_{y\to\infty} \left|\left|V_x - 2e^{-2x}\right|\right|_\infty = 0, \ \textit{uniformly for } x \ge x_0, \ \lim_{x\to\infty} ||V_x||_\infty = 0, \ \textit{uniformly for } y \ge y_0, \\ &\lim_{x\to\infty} \left|\left|V_y - \frac{2}{\sigma^2}e^{-\frac{2y}{\sigma^2}}\right|\right|_\infty = 0, \ \textit{uniformly for } y \ge y_0, \ \lim_{y\to\infty} ||V_y||_\infty = 0, \ \textit{uniformly for } x \ge x_0, \\ &\lim_{y\to\infty} \left|\left|V_{xx} + 4e^{-2x}\right|\right|_\infty = 0, \ \textit{uniformly for } x \ge x_0, \ \lim_{x\to\infty} ||V_{xx}||_\infty = 0, \ \textit{uniformly for } y \ge y_0, \\ &\lim_{x\to\infty} \left|\left|V_{yy} + \frac{4}{\sigma^4}e^{-\frac{2y}{\sigma^2}}\right|\right|_\infty = 0, \ \textit{uniformly for } y \ge y_0, \ \lim_{y\to\infty} ||V_{yy}||_\infty = 0, \ \textit{uniformly for } x \ge x_0, \\ &\lim_{y\to\infty} ||V_{xy}||_\infty = 0, \ \textit{uniformly for } x \ge x_0, \quad \lim_{x\to\infty} ||V_{xy}||_\infty = 0, \ \textit{uniformly for } y \ge y_0, \end{aligned}$$

where $x_0, y_0 > 0$ are arbitrary.

Proof. Let $\tilde{V} := V - (1 - e^{-2x} - e^{-\frac{2y}{\sigma^2}})$. By Lemma 5.1 we have $\lim_{(x,y)\to\infty} \tilde{V}(x,y) = 0$. We consider now the function $\tilde{V}$ on the circle $B := B(x_0, y_0; 1/2 \wedge x_0 \wedge y_0)$, for $(x_0,y_0) \in G$, and find obviously

$$\left|\left|\tilde{V}(x,y)\right|\right|_{L^\infty(B)} \le \epsilon, \tag{14}$$

for arbitrary small ϵ, and $||(x_0,y_0)||$ large enough. A simple calculation reveals that $\tilde{V}$ fulfills

$$F(x,y,\tilde{V},D\tilde{V},D^2\tilde{V}) := \max\left(\tilde{V}_x - \frac{2}{\sigma^2}e^{-\frac{2y}{\sigma^2}}, \tilde{V}_y - 2e^{-2x}\right) + \frac{1}{2}\Delta^\sigma \tilde{V} = 0. \tag{15}$$

We now want to apply Theorem 3.1 of [19] and check the conditions. The conditions (F0)and (F1) are obviously fulfilled. For (F2), we note that $F(x,y,\tilde{V},D\tilde{V},D^2\tilde{V})$ is non increasing w.r.t. $\tilde{V}$.

We check the Lipschitz property in $\tilde{V}, D\tilde{V}$:

$$\begin{aligned} \left|F(x,y,\tilde{V},D\tilde{V},D^2\tilde{V}) - F(x,y,\hat{\tilde{V}},\widehat{D\tilde{V}},D^2\tilde{V})\right| &= \\ \left|\max\left(\tilde{V}_x - \frac{2}{\sigma^2}e^{-\frac{2y}{\sigma^2}}, \tilde{V}_y - 2e^{-2x}\right) - \max\left(\widehat{\tilde{V}_x} - \frac{2}{\sigma^2}e^{-\frac{2y}{\sigma^2}}, \widehat{\tilde{V}_y} - 2e^{-2x}\right)\right| &= \\ \left|\max\left(\tilde{V}_x, \tilde{V}_y + r(x,y)\right) - \max\left(\widehat{\tilde{V}_x}, \widehat{\tilde{V}_y} + r(x,y)\right)\right| &= \\ \left|\tilde{V}_x + \max\left(0, \tilde{V}_y - \tilde{V}_x + r(x,y)\right) - \widehat{\tilde{V}_x} - \max\left(0, \widehat{\tilde{V}_y} - \widehat{\tilde{V}_x} + r(x,y)\right)\right| &\le \\ \left|\tilde{V}_x - \widehat{\tilde{V}_x}\right| + \left|\max\left(0, \tilde{V}_y - \tilde{V}_x + r(x,y)\right) - \max\left(0, \tilde{V}_y - \tilde{V}_x + d + r(x,y)\right)\right|&, \end{aligned} \tag{16}$$

where $r(x,y) := \frac{2}{\sigma^2} e^{-\frac{2y}{\sigma^2}} - 2e^{-2x}$, and $d := \widehat{\tilde{V}_y} - \tilde{V}_y - \widehat{\tilde{V}_x} + \tilde{V}_x$ holds. In order to estimate the r.h.s. of (16) we distinguish several cases.
Case 1: $\tilde{V}_y - \tilde{V}_x + r(x,y) \geq 0, \tilde{V}_y - \tilde{V}_x + d + r(x,y) \geq 0$.
Here one finds

$$\text{r.h.s. of (16)} \leq 2\left|\tilde{V}_x - \widehat{\tilde{V}_x}\right| + \left|\tilde{V}_y - \widehat{\tilde{V}_y}\right|.$$

Case 2: $\tilde{V}_y - \tilde{V}_x + r(x,y) \leq 0, \tilde{V}_y - \tilde{V}_x + d + r(x,y) \leq 0$.
We get

$$\text{r.h.s. of (16)} \leq \left|\tilde{V}_x - \widehat{\tilde{V}_x}\right|.$$

Case 3: $\tilde{V}_y - \tilde{V}_x + r(x,y) \geq 0, \tilde{V}_y - \tilde{V}_x + d + r(x,y) \leq 0$.
We have

$$\text{r.h.s. of (16)} \leq 2\left|\tilde{V}_x - \widehat{\tilde{V}_x}\right| + \left|\tilde{V}_y - \widehat{\tilde{V}_y}\right|.$$

Case 4: $\tilde{V}_y - \tilde{V}_x + r(x,y) \leq 0, \tilde{V}_y - \tilde{V}_x + d + r(x,y) \geq 0$.
One has finally

$$\text{r.h.s. of (16)} \leq 2\left|\tilde{V}_x - \widehat{\tilde{V}_x}\right| + \left|\tilde{V}_y - \widehat{\tilde{V}_y}\right|.$$

Hence, condition (F2) is fulfilled with $K = 2$.

For (F3) we observe

$$|F(x,y,0,0,0)| = \left|\max\left(-\frac{2}{\sigma^2} e^{-\frac{2y}{\sigma^2}}, -2e^{-2x}\right)\right| \leq K_1(x,y),$$

with $\lim_{y\to\infty} K_1(x,y) = 0$, uniformly in x, $\lim_{x\to\infty} K_1(x,y) = 0$, uniformly in y, hence $\lim_{(x,y)\to\infty} K_1(x,y) = 0$. (Note that in the following K_1 is a generic function which may vary from place to place.)

Concerning (F4), we have to find an estimate for $\left|\left|\max\left(\tilde{V}_x - \frac{2}{\sigma^2} e^{-\frac{2y}{\sigma^2}}, \tilde{V}_y - 2e^{-2x}\right) + \frac{1}{2}\Delta^\sigma \tilde{V}\right|\right|_{C^\alpha(B)}$, or - since $(\tilde{V}, D\tilde{V}, D^2\tilde{V})$ is fixed - an estimate for $\left|\left|\max\left(-\frac{2}{\sigma^2} e^{-\frac{2y}{\sigma^2}}, A - 2e^{-2x}\right) + \frac{1}{2}\Delta^\sigma \tilde{V}\right|\right|_{C^\alpha(B)}$, with $A := \tilde{V}_y - \tilde{V}_x$.
We assume now a large value of y_0 and distinguish several cases. Moreover, we estimate in the following the Lipschitz norm $||\cdot||_{0,1}$, which is bigger than the Hölder norm, since the radius of our circle is smaller than 1/2.
Case 1: $A \leq 0$.
One gets

$$\left|\left|\max\left(-\frac{2}{\sigma^2} e^{-\frac{2y}{\sigma^2}}, \quad A - 2e^{-2x}\right) + \frac{1}{2}\Delta^\sigma \tilde{V}\right|\right|_{0,1} \leq K_1(y_0),$$

with $\lim_{y_0\to\infty} K_1(y_0) = 0$.
Case 2: $A > 0$.
Here we find - using the definition $s(x,y) := \max\left(-\frac{2}{\sigma^2} e^{-\frac{2y}{\sigma^2}}, A - 2e^{-2x}\right)$ -

$$\left|\left|\max\left(-\frac{2}{\sigma^2} e^{-\frac{2y}{\sigma^2}}, A - 2e^{-2x}\right) + \frac{1}{2}\Delta^\sigma \tilde{V}\right|\right|_{0,1} \leq \sup_{(x,y)\in B} \left(|s_x(x,y)| + |s_y(x,y)|\right). \tag{17}$$

One checks that the r.h.s. of (17) has the upper bound $K_1(y_0) + 2A \leq K_1(y_0) + 2\left(\left|\tilde{V}_x\right| + \left|\tilde{V}_y\right|\right)$, with $\lim_{y_0\to\infty} K_1(y_0) = 0$.

As the same considerations hold with large x_0 as well, we find that the condition (F4) is fulfilled with $K = 2$ and K_1 with $\lim_{y\to\infty} K_1(x,y) = 0$, uniformly in x, $\lim_{x\to\infty} K_1(x,y) = 0$, uniformly in y, hence $\lim_{(x,y)\to\infty} K_1(x,y) = 0$. So all the conditions are fulfilled, and we can now apply Theorem 3.1 of [19], to get

$$\left|\left|\tilde{V}\right|\right|^{(0)}_{2,\alpha;B} \leq N(n,\nu,K,\alpha,R_0)\epsilon + K_1(x,y),$$

where ϵ stems from inequality (14), and K_1 fulfills $\lim_{y\to\infty} K_1(x,y) = 0$, uniformly in x, $\lim_{x\to\infty} K_1(x,y) = 0$, uniformly in y, hence $\lim_{(x,y)\to\infty} K_1(x,y) = 0$. Because of the definition of the norm we use in [19], we get the assertion of our lemma, but only away from the boundary $\square$

Proof of Lemma 4.2: We start with the definition

$$u(x) := \sup\{y_0 > 0 | D(x,y) < 0, \ \forall y \in [0, y_0]\},$$

for $x > 0$.

Claim 1: $\lim_{x\to\infty} u(x) = \infty$.

Let us remark that one obviously has $\{(x,z)|z \in [0, u(x))\} \subset N_1$.

Now assume Claim 1 is false. Then we can find a sequence $x_n \to \infty$, s.t.

$$y_n := u(x_n) \in [0, M] \tag{18}$$

for some constant $M > 0$, and we have

$$D(x_n, y_n) = 0 \tag{19}$$

By (18) we get the existence of a subsequence x_{n_k} with $\lim_{k\to\infty} y_{n_k} = \gamma \in [0, M]$.

We now distinguish two cases:

Case A: $\gamma > 0$.

For k large enough, we find $y_{n_k} \in [\gamma/2, 3\gamma/2]$ and $D(x_{n_k}, y_{n_k}) = 0$. But Lemma 5.2 implies

$$\lim_{x\to\infty} |D + \frac{2}{\sigma^2} e^{-2y/\sigma^2}| = 0,$$

uniformly for $y \geq y_0$. Choosing $y_0 = \gamma/2$ gives a contradiction.

Case B: $\gamma = 0$.

This means we have $x_{n_k} \to \infty, y_{n_k} \to 0$, s.t. $D(x_{n_k}, y_{n_k}) = 0$.

We proceed with

Claim 2: For all $x_0 > 0, y_0 > 0$ there exists $\delta(x_0, y_0) > 0$ s.t.

$$V_y(x,y) \geq \delta > 0$$

for all $x \geq x_0, y \in [0, y_0]$

We show now Claim 2: For shorter notation we introduce the points A, B by $A := (x,y), B := (x, y+\epsilon)$ for small $\epsilon > 0$.

For the starting point A we use the optimal strategy for the problem (5), i.e. u_t^*. For B we take

$$\hat{u}_t^B := \begin{cases} u_t^*(Z_t^B - (0,\epsilon)), & t \leq \tau^A \\ u_t^*, & t > \tau^A, \end{cases}$$

where τ^A denotes the ruin time for the starting point A. This means that until the ruin time of the process started in A, we use the same strategy for B, so that the two paths move parallel with distance ϵ. After τ^A we take the optimal strategy for the path started in B.

$$V(x, y+\epsilon) \geq J(x, y+\epsilon, \hat{u}) = V(x,y) + \mathbf{P}(H)\mathbf{E}\left[\int_0^\infty V(z,\epsilon)\, dF^H(z)\right],$$

where $H := \{\tau^A < \infty, Z^A_{\tau^A} = (z, 0), z > 0\}$, i.e. the set where the ruin of the process started in A happens at the $x-$axis. Furthermore dF^H denotes the conditioned distribution of the hitting place at the $x-$axis. Finally we get

$$\lim_{\epsilon\to 0} \frac{V(x,y+\epsilon) - V(x,y)}{\epsilon} = \mathbf{P}(H) \lim_{\epsilon\to 0} \frac{1}{\epsilon} \mathbf{E}\left[\int_0^\infty V(z,\epsilon)\, dF^H(z)\right]$$

$$
\begin{aligned}
&\geq \mathbf{P}(H)\mathbf{E}\left[\int_0^\infty \liminf_{\epsilon\to 0} \frac{V(z,\epsilon)}{\epsilon}\, dF^H(z)\right] \\
&\geq \mathbf{P}(H)\mathbf{E}\left[\int_0^\infty \frac{1-e^{-z}}{\sigma^2}\, dF^H(z)\right] \\
&\geq \mathbf{P}(H)\mathbf{E}\left[\int_{x_0}^\infty dF^H(z)\right] C(\sigma,x_0) \\
&= C(\sigma,x_0)\mathbf{P}\left(H\cap\{X_{\tau^A}\geq x_0/2\}\right) = C(\sigma,x_0,y_0),
\end{aligned}
$$

where the C denote some positive constants. Here we have used Fatou's Lemma in the first inequality and the last formula in the proof of Lemma 4.1 in the second one. Finally the last probability in this chain can certainly be estimated below by a constant $C(x_0,y_0)$; take, e.g., the probability that the process with drift $(1,1)$ hits the $x-$axis at values larger than $x_0/2$. This proves Claim 2.

We proceed with

Claim 3: For all $x\geq x_0$ we have uniformly

$$V_x(x,\delta)\to 0$$

for $\delta\to 0$

We show now Claim 3: Again we introduce the points A,B by $A:=(x,\delta), B:=(x-\epsilon,\delta)$ for small $\epsilon>0$. For the starting point A we use the optimal strategy, i.e. u_t^*. For B we take

$$\hat{u}_t^B := u_t^*(Z_t^B+(\epsilon,0)),$$

i.e. the strategy which is optimal for A. Again, the two paths move parallel with distance ϵ, but since the path started in B is ruined earlier, this definition is already sufficient. Let τ^B denote the ruin time for the starting point B. This yields

$$V(x-\epsilon,\delta)\geq J(x-\epsilon,\delta,\hat{u}) = \mathbf{P}(K) := \mathbf{P}(\{\tau^B=\infty\}).$$

Let now $L:=\{\tau^B<\infty\}\cap\{X^{B,\hat{u}}=0\}$, i.e. the set where the path started in B hits the $y-$ axis first. Moreover, let dF^L be the conditional distribution for a hitting place z on the $y-$axis. We get

$$V(x,\delta) = \mathbf{P}(K)+\mathbf{P}(L)\int_0^\infty V(\epsilon,z)\, dF^L(z).$$

Summarizing, we conclude

$$\lim_{\epsilon\to 0}\frac{V(x,\delta)-V(x-\epsilon,\delta)}{\epsilon} = \mathbf{P}(L)\int_0^\infty \lim_{\epsilon\to 0}\frac{V(\epsilon,z)}{z}\, dF^L(z) \leq 2\mathbf{P}(L)\int_0^\infty dF^L(z) = 2\mathbf{P}(L)$$

Here the last inequality follows from the fact that the value function is dominated by $(1-e^{-2x})(1-e^{-2y})$, the target functional for the drift $(1,1)$.

Finally, we note that $\mathbf{P}(L)\leq\mathbf{P}(\tau_1<\tau_2)$, with $\tau_1:=\inf\{t|x/2+B_t^{(1)}=0\}$, and $\tau_2:=\inf\{t|\delta+t+B_t^{(2)}=0\}$. Clearly, $\mathbf{P}(\tau_1<\tau_2)\to 0$, for $\delta\to 0$, uniformly for $x\geq x_0$. This proves Claim 3. Claim 2 and Claim 3 provide a contradiction to $D(x_{n_k},y_{n_k})=0$ in Case B, s.t. Claim 1 is proved.

Defining

$$v(y) := \sup\{x_0>0 | D(x,y)>0,\ \forall x\in[0,x_0]\},$$

for $y>0$, analogous considerations give

$$\lim_{y\to\infty} v(y) = \infty. \tag{20}$$

Finally, one finds

$$\lim_{\substack{(x,y)\to\infty\\(x,y)\notin P_1\cup N_1}} D(x,y) = \lim_{x\to\infty,y\to\infty} D(x,y) = 0,$$

where for the first equality we have used *Claim 1*, resp. (20), and for the second Lemma 5.2. □

Proof of Lemma 4.3. We start with a slight rewriting of the basic PDE, i.e.

$$\frac{1}{2}V_{xx} + \frac{\sigma^2}{2}V_{yy} + V_y + (V_x - V_y)^+ = 0.$$

Noting that the distributional derivation of the function z^+ is $\mathbf{1}_{\{z\geq 0\}}$, and that distributional derivatives are interchangeable, see, e.g. [18], 6.12(4), we find, by differentiating first w.r.t. x, then w.r.t. y and subtracting the corresponding equations,

$$\frac{1}{2}\Delta^{(\sigma)}D + \mathbf{1}_R D_x + \mathbf{1}_N D_y = 0.$$

For the regularity statement, we note that the gradient of D on M is bounded by Proposition 3.2. As the same holds for the indicator function, we can use [16], Theorem 1.1. □

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