VII.) Viscosity solutions

VII.1) Example for a nonsmooth value function

Ex: (deterministic)

Remark: for a verification theorem in the deterministic case one needs C^1

$dX_t = u_t \, dt$, $\mathcal{U} = [-1, 1]$

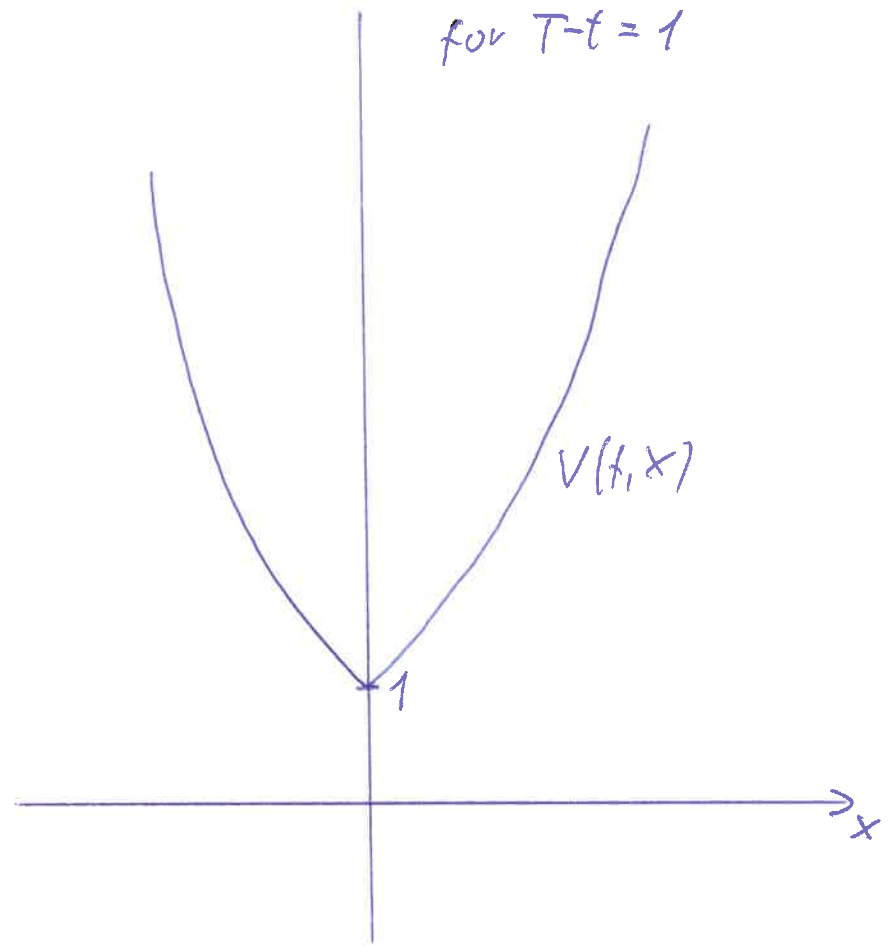
$\mathcal{A}(t,x) \ldots$ mb. functions with values in $\mathcal{U}$

$$\Rightarrow X_t = x + \int_0^t u_r \, dr$$

$$J(t,x,u) := \left(x + \int_t^T u_s \, ds\right)^2, \quad V(t,x) := \sup_{\mathcal{A}} J(t,x,u)$$

clearly: $\hat{u}(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ -1 & \text{for } x < 0 \end{cases}$

$$\Rightarrow V(t,x) = \begin{cases} (x + (T-t))^2, & x \geq 0 \\ (x + (t-T))^2, & x < 0 \end{cases}$$



hence: $V_x(t,x) = \begin{cases} 2(x + T - t), & x \geq 0 \\ 2(x + t - T), & x < 0 \end{cases}$

$$\Rightarrow V_x(t, 0+) = 2(T-t) > 0$$
$$\neq$$
$$V_x(t, 0-) = 2(t-T) < 0$$

$\Rightarrow V \notin C^1$ at $x = 0$!

VII.2) Intuition and definition of viscosity solutions

Consider a nonlinear partial diff. equation (PDE) of second order:

$$(E): \quad F(x, u(x), \underbrace{Du(x)}_{\text{gradient}}, \underbrace{D^2u(x)}_{\text{Hesse matrix}}) = 0; \quad x \in \mathcal{O} \text{ open set}, \ \mathcal{O} \subset \mathbb{R}^d$$

F continuous: $\mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \times S_d$

S_d .... symmetric matrices

Def: F is called elliptic, if

$$F(x, r, p, A) \leq F(x, r, p, B) \quad \text{for} \quad A \geq B \quad \forall (x, r, p)$$

where $A \geq B \iff A - B \in S_d^+$ ... pos. semidefinite matrices

Ex: $-\Delta u = -\sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2}$

here: $F(\;, A) = -\operatorname{trace}(A)$

Def: $u : \mathcal{O} \to \mathbb{R}$ is called classical supersolution (subsolution) of (E), if:

- $u \in C^2(\mathcal{O})$
- $F(x, u, Du, D^2u) \geq 0 \quad (\leq 0) \quad \forall x \in \mathcal{O}$

Theory of viscosity solutions is motivated by the following Lemma:

Lemma: Let u be a classical super(sub)solution of (E) $\Rightarrow$

$\forall (x_0, \varphi) \in \mathcal{O} \times C^2(\mathcal{O})$, where x_0 is a minimizer (maximizer) of $u - \varphi$ on $\mathcal{O}$ we have:

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0 \quad (\leq 0)$$

Proof for supersolution:

x_0 is minimizer $\Rightarrow$ • $D\varphi(x_0) = Du(x_0)$ (1)

• $D^2(u-\varphi)(x_0) \geq 0 \Rightarrow$

$D^2u(x_0) \geq D^2\varphi(x_0)$ (2)

$$\Rightarrow F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \underset{\substack{\uparrow \\ F \text{ elliptic } + (1),(2)}}{\geq} F(x_0, u(x_0), Du(x_0), D^2u(x_0))$$

$$\underset{\substack{\uparrow \\ u \text{ is supersolution}}}{\geq} 0$$

$\square$

Viscosity solutions can be defined for <u>discontinuous functions</u>!

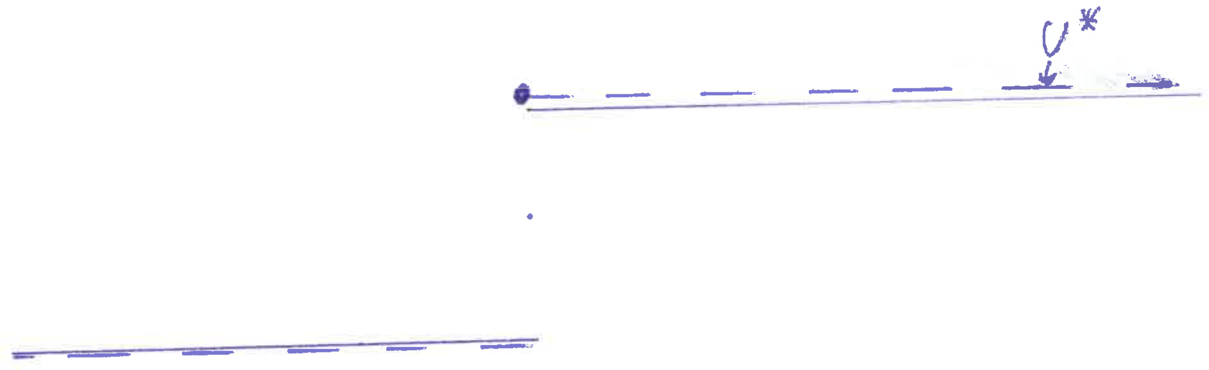
for this purpose:

Def: Let $u: \Theta \to \mathbb{R}$ locally bounded (i.e. bounded on compact sets $K \subset \Theta$)

$u_*(x) := \liminf_{x' \to x} u(x')$ ... lower semicontinuous enveloppe

$u^*(x) := \limsup_{x' \to x} u(x')$ ... upper — " —

Ex:



In the conclusio of the Lemma above we do <u>not</u> need any differentiability of u ! $\Rightarrow$

<u>Def:</u> (i) We call u a <u>viscosity supersolution</u> of (E), if

$$F(x_0, u_*(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0$$

$\forall (x_0, \varphi) \in \mathcal{O} \times C^2(\mathcal{O})$, s.t. x_0 is a minimizer of $u_* - \varphi$.

(ii) We call u a <u>viscosity subsolution</u> of (E), if

$$F(x_0, u^*(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0$$

$\forall (x_0, \varphi) \in \mathcal{O} \times C^2(\mathcal{O})$, s.t. x_0 is a maximizer of $u^* - \varphi$.

(iii) u is called <u>viscosity solution</u> of (E) if (i)+(ii) holds.

Remarks: 1.) Viscosity solutions are not necessarily continuous!

2.) Analogy to weak solutions in the Sobolev sense:

one derives for classical solution a certain relation (there by integration by parts); then this relation is used for the general concept!

3.) One can restrict to φ's, for which we have $\varphi(x_0) = u_*(x_0)$ in the supersolution case:

if this is not so: consider: $\psi(x) := \varphi(x) - (\varphi(x_0) - u_*(x_0))$

for this we have: $F(x_0, u_*(x_0), D\psi(x_0), D^2\psi(x_0)) \geq 0$

$$\Rightarrow F(x_0, u_*(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0 \checkmark$$

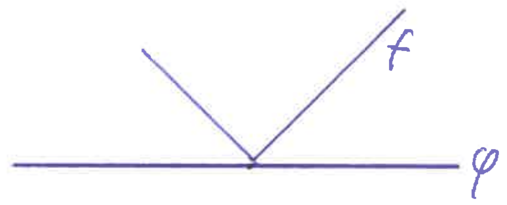
4.) Clearly (see Lemma above): classical solutions are viscosity solutions!

5.) Density argument $\Rightarrow$ we can restrict to $\varphi \in C^\infty(\mathcal{O})$!

<u>Ex:</u> (E^+) : $|U'(x)| - 1 = 0$ on $\mathbb{R}$

a) $f(x) := |x|$ is <u>not</u> a viscosity supersolution at $x = 0$!, since:

take test function: $\varphi(x) \equiv 0$



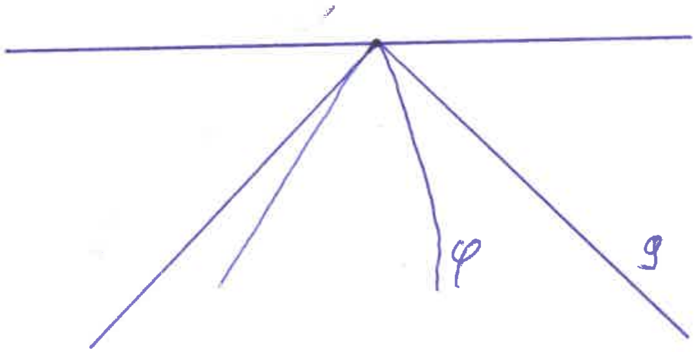
$(f_* - \varphi)(x) = |x|$ with minimum at $x_0 = 0$

BUT: $\underbrace{|\varphi'(x_0)|}_{0} - 1 \not\geq 0$

b) $g(x) = -|x|$ is a viscosity solution!

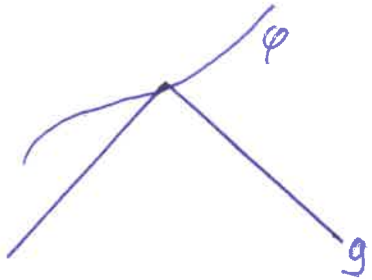
we restrict to the critical point x_0:

supersolution property: ✓



$\nexists$ C^2 function φ, s.t. $g-\varphi$ has a minimum at x_0!

subsolution property: if $\varphi \in C^1$ with $\underbrace{(g-\varphi)(0)}_{=0} = \max(g-\varphi)$



works only for $|\varphi'(0)| \leq 1$

$\Rightarrow$ this is exactly the subsolution property ✓

similarly one shows: f is a viscosity solution of

$$(E^-): \; -|u'(x)| + 1 = 0$$

$\Rightarrow$ <u>certain asymmetry in the concept</u>

VII.3) HJB-equation and viscosity solutions

Setting as in the standard control problem:

$$\begin{cases} dX_t = b(t, X_t, u_t)\,dt + \sigma(t, X_t, u_t)\,dW_t & \text{with unique strong solution} \\ J(t,x,u) = \mathbb{E}_{t,x}\left[\int_t^T f(s, X_s, u_s)\,ds + \psi(T, X_T) \right] \longrightarrow \max \\ V(t,x) = \sup_{u \in \mathcal{A}} J(t,x,u) \end{cases}$$

<u>Rem</u>: Concept of viscosity solution was formulated above for $\mathcal{O} \subset \mathbb{R}^n$!

One can show that it makes sense for HJB equations for the set

$[0, T) \times \mathbb{R}^n$.

$\boxed{\text{Assumption: } V(t,x) \text{ is continuous}}$

Proposition 1: Assume $f(t,x,u)$ is continuous in (t,x) for fixed $u \in \mathcal{U}$

$\Rightarrow$ V is a viscosity supersolution of the HJB-equation:

$$-V_t - H(t,x,DV,D^2V) = 0 \quad \text{on } [0,T) \times \mathbb{R}^n =: Q$$

$$\text{with } H(\ ,\ ,\ ,\ ) = \sup_{u \in \mathcal{U}} \left\{ b \cdot DV + \tfrac{1}{2} \operatorname{tr}(D^2 V \cdot a) + f(t,x,u) \right\}$$

... Hamiltonian

Rem: negative sign, because of ellipticity condition

Proof: Let $(\bar{t},\bar{x}) \in Q$, $\varphi \in C^2(Q)$ with

$$0 = (V-\varphi)(\bar{t},\bar{x}) = \min_Q (V-\varphi) \qquad (1)$$

Moreover: $(t_m, x_m) \xrightarrow{m\to\infty} (\bar{t},\bar{x})$

$\overset{V \text{ cont.}}{\Longrightarrow} \quad V(t_m,x_m) \xrightarrow{m\to\infty} V(\bar{t},\bar{x})$

Continuity of φ + (1) $\Rightarrow$ $\gamma_m := V(t_m,x_m) - \varphi(t_m,x_m) \to 0 \qquad (2)$

Let $a \in \mathcal{U}$ and $u_t \equiv a$

corresponding control process: $X_s^{(t_m,x_m)}$

$\tau_m := \inf\{s \geq t_m \mid |X_s^{(t_m,x_m)} - x_m| \geq \eta\}$, $\eta > 0$ fixed

h_m strictly positive, $h_m \to 0$ and $\dfrac{\gamma_m}{h_m} \to 0$, (e.g. $h_m = \sqrt{\gamma_m}$)

<u>Bellman principle $\Rightarrow$</u> with $\theta_m := \tau_m \wedge (t_m + h_m)$

$$V(t_m, x_m) \geq \mathbb{E}\left[\int_{t_m}^{\theta_m} f(s, X_s^{(t_m,x_m)}, a) + V(\theta_m, X_{\theta_m}^{(t_m,x_m)})\right]$$

$(1) \to V \geq \varphi \overset{(2)}{\Longrightarrow}$

$$\varphi(t_m, x_m) + \gamma_m \geq \mathbb{E}\left[\int_{t_m}^{\theta_m} f(s, X_s^{(t_m,x_m)}, a)\,ds + \varphi(\theta_m, X_{\theta_m}^{(t_m,x_m)})\right]$$

<u>NOW</u> Ito! (φ smooth) to the last term: $\Rightarrow$

$$\varphi(t_m, x_m) + \gamma_m \geq \mathbb{E}\left[\int_{t_m}^{\theta_m} f(s, X_s^{(t_m,x_m)}, a) + \varphi(t_m, x_m) + \int_{t_m}^{\theta_m} \mathcal{L}^a \varphi(s, X_s^{(t_m,x_m)})\,ds\right]$$

(integrand in the stochastic integral is bounded $\Rightarrow \mathbb{E}\left[\int \dots dW\right] = 0$!)

$$\Rightarrow \frac{\gamma_m}{h_m} + \mathbb{E}\left[\frac{1}{h_m}\int_{t_m}^{\theta_m}(-\mathcal{L}^a\varphi - f)(s, X_s^{(t_m, x_m)}, a)\,ds\right] \geq 0 \qquad (3)$$

$X_s^{(t_m, x_m)}$ has continuous paths $\overset{\eta \text{ fixed!}, h_m \text{ small enough}}{\Longrightarrow}$

for $m \geq N(\omega)$: $\theta_m(\omega) = t_m + h_m$ a.s.

mean value theorem
$\Longrightarrow$ the random variable in the $\mathbb{E}$ of (3)

converges a.s. $\overset{m\to\infty}{\longrightarrow} -\mathcal{L}^a\varphi(\bar{t}, \bar{x}) - f(\bar{t}, \bar{x}, a)$

Moreover: this random var. is bounded by a constant indep. of m! (φ smooth, η fixed)

dom. convergence
$\Longrightarrow$ $-\mathcal{L}^a\varphi(\bar{t}, \bar{x}) - f(\bar{t}, \bar{x}, a) \geq 0 \quad \overset{a \text{ arbitrary}}{\Longrightarrow}$

$$-\varphi_t(\bar{t}, \bar{x}) - H(\bar{t}, \bar{x}, D\varphi, D^2\varphi) \geq 0 \qquad \square$$

Proposition 2 : Assumptions as in Prop. 1 $\Rightarrow$

V is viscosity subsolution of

$$-V_t - H(t,x, DV(t,x), D^2V(t,x)) = 0 \qquad \text{on } Q$$

Proof: Let $(\bar t,\bar x) \in Q$, $\varphi \in C^2(Q)$ with

$$0 = (V-\varphi)(\bar t,\bar x) = \max_Q (V-\varphi) \qquad (1)$$

Proof by contradiction: assume $-\varphi_t(\bar t,\bar x) - H(\bar t,\bar x, D\varphi(\bar t,\bar x), D^2\varphi(\bar t,\bar x)) > 0$

Assumption: Hamiltonian is continuous $\Rightarrow$ $\forall \varepsilon > 0 \ \exists \eta > 0$:

$$-\varphi_t(s,y) - H(s,y, D\varphi(s,y), D^2\varphi(s,y)) \geq \varepsilon \quad \forall (s,y) \in \underset{\uparrow}{B}((\bar t,\bar x);\eta) \qquad (2)$$

ball with radius η

Let $(t_m, x_m) \to (\bar t,\bar x)$ with $V(t_m,x_m) \to V(\bar t,\bar x)$

φ is continuous $+ (1) \Rightarrow \gamma_m := V(t_m, x_m) - \varphi(t_m, x_m) \to 0$

let h_m strict pos.: $h_m \to 0$, $\dfrac{\gamma_m}{h_m} \to 0$

$\left.\begin{array}{l}\text{Bellman principle} \\ + (1) \to V \leq \varphi\end{array}\right\}$ $\exists\ \hat{\nu}^m \in \mathcal{A}(t_m, x_m)$, s.t.

$$\varphi(t_m, x_m) + \gamma_m - \frac{\varepsilon h_m}{2} \leq \mathbb{E}\left[\int_{t_m}^{\theta_m} f(s, X_s^{(t_m, x_m)}, \hat{\nu}_s^m)\, ds + \varphi(\theta_m, X_{\theta_m}^{(t_m, x_m)})\right] \qquad (3)$$

where: $\tau_m := \inf\{ s \geq t_m \mid |X_s^{(t_m, x_m)} - x_m| \geq \eta' \}$, $0 < \eta' < \eta$

$\theta_m := \tau_m \wedge (t_m + h_m)$

As $(t_m, x_m) \longrightarrow (\bar{t}, \bar{x})$, we can assume: $B((t_m, x_m); \eta') \subset B((\bar{t}, \bar{x}); \eta)$

s.t.: for $t_m \leq s \leq \theta_m$: $X_s^{(t_m, x_m)} \in B((\bar{t}, \bar{x}); \eta)$

where $X_s^{(t_m, x_m)}$ ... process with control $\hat{u}^m$

Ito $\Rightarrow$

$$\varphi(t_m, x_m) + \gamma_m - \frac{\varepsilon h_m}{2} \leq \mathbb{E}\Big[\int_{t_m}^{\theta_m} f(s, X_s^{(t_m, x_m)}, \hat{u}_s^m)\, ds + \varphi(t_m, x_m) + \int_{t_m}^{\theta_m} \mathcal{L}^{\hat{u}^m} \varphi(s, X_s^{(t_m, x_m)})\, ds \Big]$$

$\mathbb{E}[\text{ stoch. integral}] = 0$, because: φ smooth, δ fulfills growth cond., choose δ properly! $\quad : h_m$

$$\Rightarrow 0 \geq \frac{\gamma_m}{h_m} - \frac{\varepsilon}{2} + \mathbb{E}\Big[\frac{1}{h_m} \int_{t_m}^{\theta_m} (-\mathcal{L}^{\hat{u}^m} \varphi - f)\underbrace{(s, X_s^{(t_m, x_m)}, \hat{u}_s^m)}\, ds \Big] \qquad (4)$$

$$\left.\begin{array}{c} (2) \\ -\sup(\cdot) = \inf(-\cdot) \end{array}\right\} \; (-\mathcal{L}^{\hat{u}_s^m} \varphi - f)(\quad) \geq \varepsilon \quad \forall\, t_m \leq s \leq \theta_m \overset{(4)}{\Longrightarrow}$$

$$0 \geq \frac{\gamma_m}{h_m} - \varepsilon\Big(\frac{1}{2} - \frac{1}{h_m} \mathbb{E}[\theta_m - t_m]\Big)$$

$$0 \geq \frac{\gamma_m}{h_m} - \varepsilon \left(\frac{1}{2} - \underbrace{\frac{1}{h_m} \mathbb{E}[\theta_m - t_m]}_{\text{one can show: } \xrightarrow[m\to\infty]{} 1} \right)$$

(T_m is with very small probability smaller than $t_m + h_m$, since y' is fixed)

$$\overset{m\to\infty}{\Longrightarrow} \quad 0 \geq 0 + \frac{\varepsilon}{2} \quad \lightning \qquad \square$$

Theorem: The value function is under the assumptions above a viscosity solution of the HJB equation.

Proof: Prop. 1 + Prop. 2 $\square$

Remark: *classical approach:* construct a C^2 solution of the HJB equ. V

$\Longrightarrow$ verification theorem: V is the value function

Viscosity approach:

- we know now: value function V is a viscosity solution of the HJB-eq.
- assume: we know a viscosity solution $\tilde{V}$ of the HJB-eq.;
- in order to conclude, that $\tilde{V}$ is the value function, one needs a uniqueness result, i.e.

HJB eq. has at most one solution!

Such results can be proved using <u>Comparison Theorems</u>, e.g.

Theorem (without proof): Under certain assumptions (ASS) we have:

Let U (resp. V) a viscosity sub (super) solution with polynomial growth.

If: $U(T,x) \le V(T,x)$, $x \in \mathbb{R}^n$ $\Rightarrow$

$$U(t,x) \le V(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}^n$$

Corollary: Under the assumptions (ASS) above, we have a unique viscosity solution, fulfilling $V(T,x) = g(x)$.

Proof: assume: V_1, V_2 are two viscosity solutions, with $V_1(T,x) = V_2(T,x) = g(x)$

V_1 is subsolution, V_2 is supersolution

$$\left.\begin{array}{ll} \overset{\text{Theorem}}{\Rightarrow} \quad V_1 \leq V_2 & \text{on } [0,T] \times \mathbb{R}^n \\ \text{analogously: } V_2 \leq V_1 & \text{— " —} \end{array}\right\} V_1 = V_2 \text{ on } [0,T] \times \mathbb{R}^n$$

$\square$