(A2) $$\mathbb{E}\Big[\sup_{t\le s\le T} X_s^{*2}\Big] = x_0^2\, \mathbb{E}\Big[\sup e^{2\int_t^s g_r\,dr + 2\int_t^s k_r\,dW_r}\Big]$$

$$\le \text{const} \cdot \mathbb{E}\Big[\sup e^{2\int_t^s k_r\,dW_r}\Big] =$$

$$= \text{const.} \cdot \mathbb{E}\Big[\sup e^{2\int_t^s k_r\,dW_r - 2\int_t^s k_r^2\,dr + 2\int_t^s k_r^2\,dr}\Big]$$

$$\le \text{const.}\ \mathbb{E}\Big[\sup_{t\le s\le T} e^{2\int_t^s k_r\,dW_r - 2\int_t^s k_r^2\,dr}\Big]$$

$$\le \text{const.}\ \mathbb{E}\Big[e^{2\int_t^T k_r\,dW_r - 2\int_t^T k_r^2\,dr}\Big] < \infty$$

~~Doob~~ Doob's inequality

lognormal distr.!

(A3) everything is positive ✓

$$|\varphi(t,x,u)| = \frac{c^\alpha}{\alpha} e^{-\beta t} \le \text{const.}\,(1+|c|^2) \checkmark$$

$$\|\sigma(t,x,u)\|^2 = |\pi x \sigma|^2 = \text{const.}\,\pi^2 x^2 \underline{\not\le} \text{const.}\,(1+\pi^2+x^2)$$

also: $\Phi \in C^{1,2}([0,T]\times\mathbb{R}_0^+)$ is <u>not fulfilled</u> !

BUT: one needs in the proof: $\mathbb{E}\left[\int_t^{\tau_n} \|D\Phi \cdot \sigma(s,X_s,u_s)\|^2 ds\right] < \infty$

replace in the proof: $\tau_n = \tau_n^{old} \wedge \{s>t \mid X_t = \frac{1}{n}\} \Rightarrow$

$$\le \text{const.}\,\mathbb{E}\left[\int_t^{\tau_n} \|\sigma\|^2 ds\right] = \text{const.}\,\mathbb{E}\left[\int_t^{\tau_n} \pi^2 x^2\, ds\right]$$

$$\le \text{const.}(n)\,\mathbb{E}\left[\int_t^{\tau_n} \pi^2 ds\right] \overset{(A1)}{<} \infty \Rightarrow$$

V is the value function and u^* optimal !

III.5) Linear regulator problem

$$dX_s = u_s\,ds + \sigma\,dW_s$$

$$J(t,x,u) = \mathbb{E}_{t,x}\left[\int_t^T a u_s^2 + b(X_s - x_1)^2\,ds\right] \longrightarrow \min, \qquad a,b>0; \quad x_1 \in \mathbb{R}$$

meaning: goal: X_s should stay near x_1, and we have to pay quadratic costs ("$a u_s^2$")

weighting by $a \lesseqgtr b$!

$$V(t,x) = \min_{u \in \mathcal{A}} J(t,x,u)$$

HJB:
$$\begin{cases} \min\limits_{u \in \mathbb{R}} \left\{ V_t + \frac{1}{2}\sigma^2 V_{xx} + u V_x + a u^2 + b(x - x_1)^2 \right\} = 0 \\ V(T, x) = 0 \qquad \text{(only running costs!)} \end{cases}$$

minimizer: $V_x + 2a\hat{u} = 0 \Rightarrow \hat{u} = -\frac{V_x}{2a}$

$$\Rightarrow V_t + \frac{1}{2}\sigma^2 V_{xx} - \frac{V_x^2}{2a} + a\frac{V_x^2}{4a^2} + b(x-x_1)^2 = 0$$

$$\Rightarrow \boxed{V_t + \frac{1}{2}\sigma^2 V_{xx} - \frac{V_x^2}{4a} + b(x-x_1)^2 = 0}$$

Ansatz: $V(t,x) = P(t)(x-x_1)^2 + Q(t) \Rightarrow$

$$P'(t)(x-x_1)^2 + Q'(t) + \frac{\sigma^2}{\cancel{2}} \cdot \cancel{2} P(t) - \frac{P^2(t)\cancel{4}(x-x_1)^2}{\cancel{4}a} + b(x-x_1)^2 = 0$$

comparing coefficients: $(x-x_1)^2$: $P'(t) - \frac{P^2(t)}{a} + b = 0$

1 : $Q'(t) + \sigma^2 P(t) = 0$

Boundary condition : $V(T,x) = 0 \Rightarrow P(T) = Q(T) = 0$

1st equation: Riccati type (known solution) $\Rightarrow$

$$P(t) = \sqrt{ab}\,\tanh\left((T-t)\sqrt{\tfrac{b}{a}}\right) \Rightarrow$$

$$Q(t) = \sigma^2 b\,\ln\left[\cosh\left((T-t)\sqrt{\tfrac{b}{a}}\right)\right]$$

control : $\underline{u(t,x)} = -\frac{1}{2a} V_x = -\frac{1}{2a}\left(2(x-x_1)P(t)\right) = \underline{-\frac{(x-x_1)}{a} \cdot P(t)}$

i.e. „power" proportional to the distance to x_1 (dependent on t!)

III.6) types of control

in general: $u_t(\omega)$

special:

- <u>deterministic control</u> : $u_t(\omega) = f(t)$ — det. function
- <u>feedback control</u> u_t adapted to the filtration of the controlled process, i.e. u_t is $\sigma(X_s^u, s \le t)$ mb.
- special case : <u>Markov control</u>

  $u_t(\omega) = f(t, X_t(\omega))$

  (sometimes: $f(X_t(\omega))$)

IV) Extensions

IV.1) Infinite time horizon

Assumption : $b(x,u)$, $\sigma(x,u)$, $\varphi(x,u)$ do not depend on the time

i.e.: $dX_t = b(X_t, u_t)dt + \sigma(X_t, u_t)\, dW_t$

$$\mathcal{L}^u f(x) = D_x f \cdot b(x,u) + \tfrac{1}{2} tr\left(D_{xx} f \cdot a(x,u)\right)$$

target functional : $J(x,u) = \mathbb{E}_x \left[\int_0^\infty e^{-\beta s} \varphi(X_s, u_s)\, ds \right] \longrightarrow \max$

β ... discounting rate !

(A1) $\rightarrow$ $\mathbb{E}\left[\int_0^\infty e^{-\beta s} \|U_s\|^2 \, ds\right] < \infty$

(A2) $\rightarrow$ $\mathbb{E}\left[\int_0^\infty e^{-\beta s} \|X_s\|^2 \, ds\right] < \infty$

(A3) J is well defined (as before?)

$$V(x) = \sup_{u \in \mathcal{A}} J(x, u)$$

Theorem (Verification): assume: $\|\sigma(x,u)\|^2 \leq C_\sigma (1 + \|x\|^2 + \|u\|^2)$

$|\varphi| \leq C_\varphi (1 + \|x\|^2 + \|u\|^2), \quad x \in \mathbb{R}^n, u \in \mathcal{U}$

$\Rightarrow$

(i) if $\Phi \in C^2(\mathbb{R}^n)$ with $|\Phi(x)| \leq C_\Phi (1 + \|x\|^2)$; $\lim\limits_{\substack{\tau \to \infty \\ a.s.}} \mathbb{E}[e^{-\beta\tau} \Phi(X_\tau)] = 0$ (*)

and HJB holds: $\sup\limits_{u \in \mathcal{U}} \{ \varphi(x,u) + \mathcal{L}^u \Phi - \beta \Phi \} = 0$

$\Rightarrow \Phi(x) \geq V(x)$

ii) if maximizer $\hat{u}$ $\exists$ (of $u \longmapsto \varphi(x,u) + \mathcal{L}^u \Phi - \beta \Phi$)

s.t. $u_t^* = \hat{u}(X_t^*)$ is admissible $\Rightarrow$

$\Phi(x) = V(x)$

and u^* is optimal, i.e. $J(x, u^*) = V(x)$

idea of proof

- replace in the proof of the „$T<\infty$" case:

  $\bar{\Phi} \to \tilde{\bar{\Phi}}$ and set: $\tilde{\bar{\Phi}}(t,x) = e^{-\beta t}\,\bar{\Phi}(x)$

- for $\bar{\Phi}(x)$ one gets the equation above

  (the term $\beta\bar{\Phi}$ is generated by the explicit derivative w.r.t. the time and cancellation of $e^{-\beta t}$ [the running costs include this term!])

- rest: analogously to the case „$T<\infty$"

  (Use, instead of the terminal condition, the equation (*) above)

IV.2) Deflated consumption – infinite time horizon

state process (as above)

$$\begin{cases} dX_t = [(r + \pi_t(\mu - r))X_t - c_t]\,dt + \pi_t X_t \sigma\, dW_t \\ X_0 = x \end{cases}$$

target functional : $J(x,u) = \mathbb{E}_x\left[\int_0^\infty e^{-\beta t} \frac{1}{\alpha} c_t^\alpha \, dt\right] \longrightarrow \max$

control : $u_t = (\pi_t, c_t)$

$\beta > 0$, $\alpha \in (0,1)$, $x > 0$

$$V(t,x) = \sup_{u \in \mathcal{A}} J(x,u)$$

$\rightarrow$

HJB:

$$\sup_{u \in \mathbb{R} \times [0,\infty)} \left\{ \left([r + \pi(\mu - r)]x - c \right) V_x + \frac{1}{2}\pi^2 x^2 \sigma^2 V_{xx} - \beta V + \frac{c^\alpha}{\alpha} \right\} = 0$$

Step 1 : Assumption: $V_x > 0$, $V_{xx} < 0$ (verification a-posteriori)

$$\begin{cases} \hat{\pi} = -\gamma \dfrac{V_x}{x V_{xx}} & \text{with } \gamma = \dfrac{(\mu - r)}{\sigma^2} \\ \hat{c} = (V_x)^{\frac{1}{\alpha-1}} \end{cases}$$

Step 2: Plug into HJB $\Rightarrow \dots \Rightarrow$

$$-\frac{1}{2}y^2\sigma^2\frac{V_x^2}{V_{xx}} + rxV_x - \beta V + \frac{(1-\alpha)}{\alpha}(V_x)^{\frac{\alpha}{\alpha-1}} = 0$$

Ansatz: $V(x) = A\dfrac{x^\alpha}{\alpha}$ for some $A>0 \Rightarrow V_x = Ax^{\alpha-1}$

$$V_{xx} = A(\alpha-1)x^{\alpha-2}$$

$$\Rightarrow -\frac{1}{2}y^2\sigma^2\frac{A^2x^{2\alpha-2}}{-(1-\alpha)Ax^{\alpha-2}} + rxAx^{\alpha-1} - \beta A\frac{x^\alpha}{\alpha} + \frac{(1-\alpha)}{\alpha}(Ax^{\alpha-1})^{\frac{\alpha}{\alpha-1}} = 0 \;\Big|\; : Ax^\alpha$$

$$\Rightarrow \frac{1}{2(1-\alpha)}y^2\sigma^2 + r - \frac{\beta}{\alpha} + \frac{(1-\alpha)}{\alpha}A^{\frac{1}{\alpha-1}} = 0$$

Assumption: $\beta > \frac{\alpha}{2(1-\alpha)} \gamma^2 \sigma^2 + \alpha r =: \beta_0$

$$\Rightarrow \quad A = \left[\frac{\alpha}{(1-\alpha)} \cdot \left(\frac{\beta}{\alpha} - r - \frac{1}{2(1-\alpha)} \gamma^2 \sigma^2 \right) \right]^{\alpha - 1}$$

Remark: if the assumption is wrong: one can show: $V(x) \equiv \infty$!

Step 3: $\underline{\hat{\pi}} = -\eta \frac{V_x}{x V_{xx}} = -\eta \frac{A x^{\alpha-1}}{-(1-\alpha) A x^{\alpha-1}} = \underline{\frac{\eta}{1-\alpha}}$

$\underline{\hat{c}} = (V_x)^{\frac{1}{\alpha-1}} = (A x^{\alpha-1})^{\frac{1}{\alpha-1}} = \underline{A^{\frac{1}{\alpha-1}} \cdot x}$

$$\Rightarrow \left.\begin{aligned} \pi_t^* &= \frac{1}{1-\alpha} \frac{(\mu-r)}{\sigma^2} \\ c_t^* &= A^{\frac{1}{\alpha-1}} \cdot X_t^* \end{aligned}\right\}$$

With .... $\Rightarrow$

$$\underline{dX_t^* = \frac{1}{1-\alpha} X_t^* \left[\left((1-\alpha) r + \eta(\mu - r) - (1-\alpha) A^{\frac{1}{\alpha-1}} \right) dt + \eta \sigma dW_t \right]}$$

this has a unique strong solution (explicitly known by Ito's formula)

$$X_t^* = x \cdot \exp\left\{ \left[\left(1 - \frac{\eta}{1-\alpha}\right) r + \frac{(1-2\alpha)}{2(1-\alpha)^2}\eta^2\sigma^2 - A^{\frac{1}{\alpha-1}} \right] t + \frac{1}{1-\alpha}\eta\sigma W_t \right\}$$

is strict positive $\Rightarrow V_x(\ ) > 0, \quad V_{xx}(\ ) < 0$ ✓

Remark to the conditions of the verification theorem:

$$\left.\begin{aligned} &\mathbb{E}\left[\int_0^\infty e^{-\beta t} c_t^2 \, dt\right] < \infty \\ &\mathbb{E}\left[\int_0^\infty e^{-\beta t} X_t^2 \, dt\right] < \infty \\ &\lim_{\tau\to\infty} \mathbb{E}\left[e^{-\beta\tau}\Phi(X_\tau)\right] = 0 \end{aligned}\right\}$$

are fulfilled for c_t^*, X_t^*, if β is large enough

say $\beta > \beta_1$ (dep. on model parameters)

$\Rightarrow$ for $\beta > \max(\beta_0, \beta_1)$: V value function, u^* optimal strategy $\square$

IV.3) Stopping the state process

Let Q be an open set in $[0, T] \times \mathbb{R}^n$ with boundary ∂Q

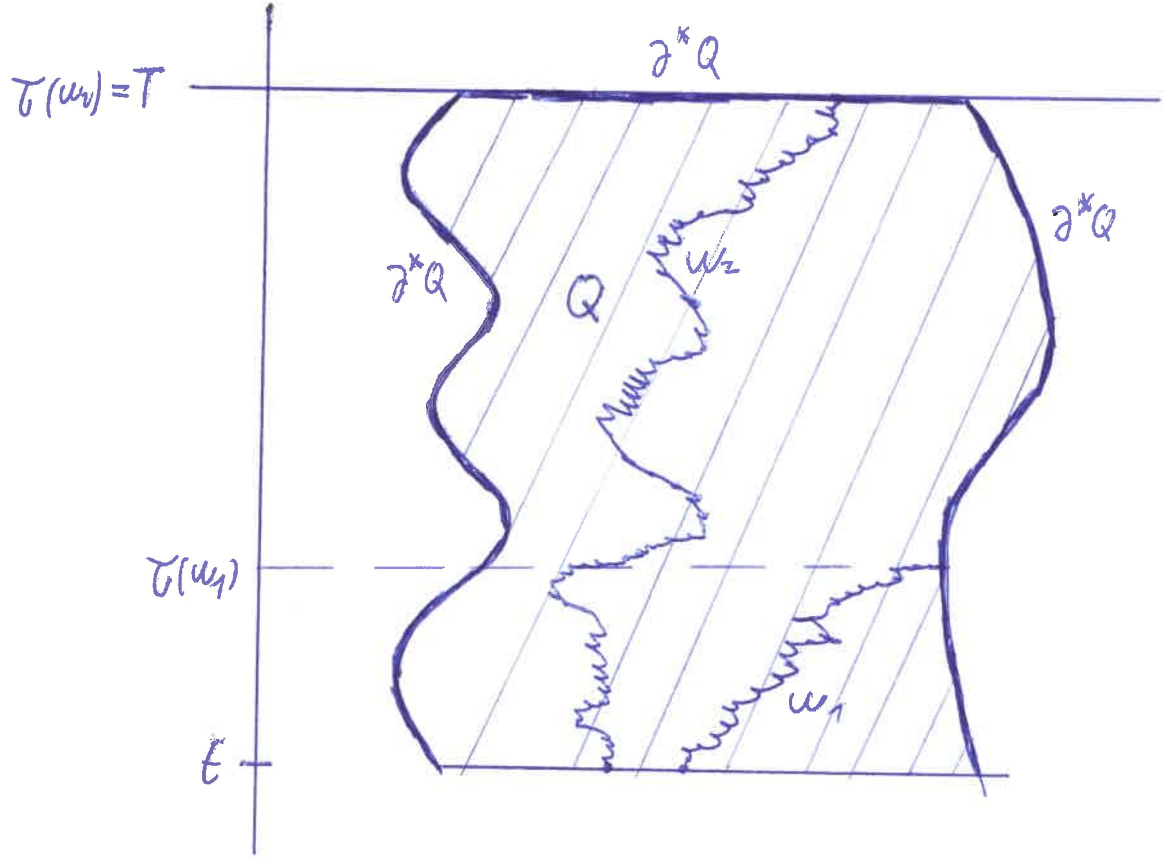
to avoid trivialities : $\bar{Q} \cap \{(t, x) \mid x \in \mathbb{R}^n\} \neq \emptyset$

We stop the process as soon as it leaves Q !

i.e.: $\tau = \inf\{s > t \mid (s, X_s) \in Q\}$

clearly: $\mathbb{P}(\tau \leq T) = 1$

Let $\partial^* Q := \partial Q \setminus (\partial Q \cap \{(t, x) \mid x \in \mathbb{R}^n\})$ .... parabolic boundary



very often : Q is a rectangle !

We have: $\mathbb{P}\left((\tau, X_\tau) \in \partial^* Q \right) = 1$

target functional: $J(t,x,u) = \mathbb{E}_{t,x}\left[\int_t^\tau \varphi(s, X_s, u_s) + \psi(\tau, X_\tau) \right] \to \max$

$\Rightarrow$ ψ („terminal costs") must be specified also for $t < T$ on $\partial^* Q$

$$V(t,x) = \sup_{u \in \mathcal{U}} J(t,x,u)$$

(A1)–(A3) as above for „$T < \infty$" case!

Theorem (Verification): assume: $\|\sigma(s,x,u)\|^2 \le C_\sigma(1+\|x\|^2+\|u\|^2)$

$|\varphi(s,x,u)| \le C_\varphi(1+\|x\|^2+\|u\|^2)$, $C_\sigma, C_\varphi > 0$

$(s,x) \in Q$

$u \in \mathcal{U}$

(i) If $\Phi \in C^{1,2}(Q) \cap C(\bar{Q})$ („classical solution")

$|\Phi(t,x)| \le C_\Phi(1+\|x\|^2)$

$$\left.\begin{aligned} \text{HJB:}\quad \sup_{u\in\mathcal{U}} \{\varphi(t,x,u) + \mathcal{L}^u \Phi(t,x)\} &= 0, && (t,x) \in Q \\ \Phi(t,x) &= \Psi(t,x) && \forall (t,x) \in \partial^* Q \end{aligned}\right\}$$

$\Rightarrow \Phi(t,x) \ge V(t,x)$

(ii) if $\hat{u}$ $\exists$ (maximizer of $u \mapsto \varphi(t,x,u) + \mathcal{L}^u \Phi$ on Q)

and $u^* = \hat{u}(t, X_t^*)$ is admissable $\Rightarrow$

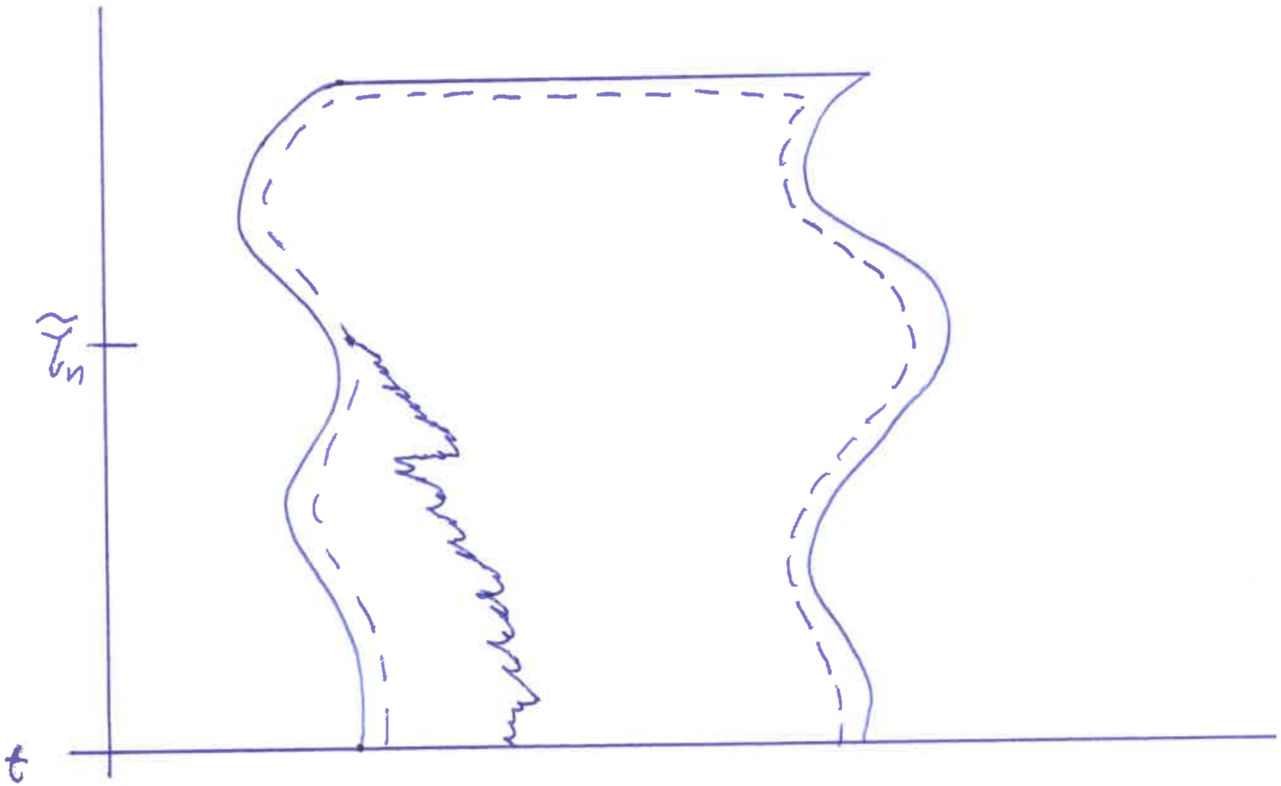
$$V(t,x) = \Phi(t,x)$$

and u^* is the optimal strategy, $J(t,x,u^*) = V(t,x)$

Proof: • analogue proof as in the case „$T < \infty$"

• replace τ_n by $\tau_n \wedge \inf\{s > t \mid \|X_s - \partial^* Q\| = \frac{1}{n}\} =: \tilde{\tau}_n$

(since Φ is not $C^{1,2}$ until the boundary)



$\tilde{\tau}_n$
t

Remark: in applications one usually has one of the cases:

- maximization of a linear combination of running and terminal costs
- minimization of ruin probabilities
- maximization of dividend payments

IV.4) Optimization of dividend payments – „Bang-Bang" strategies

Cramer Lundberg model for the endowment of an insurance company (IC)

i.e.: $R_t = x + ct - \sum_{i=1}^{N_t} Z_i$

c.... premium rate

Z_i.... individual claims (iid)

N_t... number of claims until t (usually Poisson distr.)

frequent, small claims: $\Rightarrow$ diffusion approximation, i.e.

$R_t \to X_t$ with

$dX_t = \mu dt + \sigma dW_t$, $\mu, \sigma > 0$ const.

control: U_t ... dividend rate

$$\Rightarrow dX_t = (\mu - U_t)\,dt + \sigma\,dW_t, \quad X_0 = x$$

Assumption: $0 \le U_t \le M$ ... finite rate!

Let τ ... time of ruin, i.e. $\tau := \inf\{t \mid X_t < 0\}$

Goal: Find: $V(x) = \sup_U \mathbb{E}_x\left[\int_0^\tau e^{-\beta s} U_s\,ds\right]$, $\beta > 0$ ... deflation

$$\Rightarrow \text{HJB:} \quad \sup_{u \in [0,M]} \left(\frac{1}{2}\sigma^2 V'' + (\mu - u)V' - \beta V + u\right) = 0$$

Remark: • We are looking for a $V(x)$ with quadratic growth! (Verif. Th.!)

• But: We know that $\underline{V(x)}$ has actually to $\underline{\text{bounded}}$, since:

$$\underline{\underline{J(x,u)}} = \mathbb{E}_x\left[\int_0^{\tau} e^{-\beta s} u_s \, ds\right] \leq \mathbb{E}\left[\int_0^{\tau} e^{-\beta s} M \, ds\right] \leq$$

$$\mathbb{E}\left[\int_0^{\infty} e^{-\beta s} M \, ds\right] = \int_0^{\infty} e^{-\beta s} M \, ds = \underline{\underline{\frac{M}{\beta}}}$$

Boundary condition: $\underline{V(0) = 0}$ (for $x = 0 \Rightarrow \tilde{\tau} = 0$ a.s., because of the oscillation of B.M.

$\Rightarrow$ no dividend payments $\Rightarrow V(0) = 0$

maximizer in the HJB:

$$\arg\max \left[u - u \cdot V'(x) \right] = \begin{cases} 0 & V'(x) > 1 \\ M & V'(x) \leq 1 \end{cases}$$

Assume that V is strictly concave: (verification a-posteriori)

$\Rightarrow \exists\, x_1 \in [0, \infty]$, s.t. $V'(x) = \begin{cases} > 1 & x < x_1 \\ < 1 & x > x_1 \end{cases}$

$$\Rightarrow u(x) = \begin{cases} 0 & x < x_1 \\ M & x > x_1 \end{cases}$$

„Bang-Bang" control

$$\Rightarrow \frac{1}{2}\sigma^2 V'' + \mu V' - \beta V = 0 \qquad x < x_1 \qquad (1)$$

$$\frac{1}{2}\sigma^2 V'' + (\mu - M)V' - \beta V + M = 0 \qquad x > x_1 \qquad (2)$$

solution of (1): characteristic equation: $\frac{1}{2}\sigma^2\theta^2 + \mu\theta - \beta = 0$

$$\Rightarrow \theta_\pm = \frac{-\mu \pm \sqrt{\mu^2 + 2\beta\sigma^2}}{\sigma^2}$$

$$\Rightarrow V(x) = C_1 e^{\theta_+ x} + C_2 e^{\theta_- x} \underset{V(0)=0}{\Rightarrow} C_2 = -C_1$$

$$\Rightarrow \underline{V(x) = C\left(e^{\theta_+ x} - e^{\theta_- x}\right)}$$

solution of (2) : charac. equation : $\frac{1}{2}\sigma^2 \gamma^2 + (\mu - M)\gamma - \beta = 0$

$$\Rightarrow \gamma_\pm = \frac{M-\mu \pm \sqrt{(M-\mu)^2 + 2\beta\sigma^2}}{\sigma^2}$$

particular solution : (inhom. equation!)

$$V_{part}(x) = \frac{M}{\beta}$$

Moreover: V is bounded $\Rightarrow$ solution part $\cancel{e^{\gamma_+ x}}$

$$\Rightarrow V(x) = \frac{M}{\beta} + D e^{\gamma_- x}$$

to determine: C, D, x_1

- we have: $V'(x_1) = 1$
- smooth fit: $V \in C^2$! because of the Verification Theorem (Ito Lemma!)

$\Rightarrow$ we demand: $\left.\begin{array}{l} V'(x_1-) = V'(x_1+) = 1 \\ V(x_1-) = V(x_1+) \end{array}\right\} (3)$

Claim $(3) \Rightarrow C^2$

Proof: $\frac{1}{2}\sigma^2 V''(x_1-) = \beta V(x_1-) - \mu V'(x_1-) \overset{(3)}{=}$

$$\beta V(x_1+) - \mu = \frac{1}{2}\sigma^2 V''(x_1+) \quad ✓$$

(2) because of $V'(x_1+) = 1$

$$(3)_1 \Rightarrow D \cdot \gamma_- e^{\gamma_- x_1} = C\left(\theta_+ e^{\theta_+ x_1} - \theta_- e^{\theta_- x_1}\right) = 1 \qquad (4)$$

$$(3)_2 \Rightarrow \frac{M}{\beta} + D \cdot e^{\gamma_- x_1} = C\left(e^{\theta_+ x_1} - e^{\theta_- x_1}\right) \qquad (5)$$

$$(4) \Rightarrow D = \frac{1}{\gamma_-} e^{-\gamma_- x_1}$$

$$\overset{(5)}{\Rightarrow} \alpha := \frac{M}{\beta} + \frac{1}{\gamma_-} = C\left(e^{\theta_+ x_1} - e^{\theta_- x_1}\right) \qquad (6)$$

(6) has only for $\alpha > 0$ positive solutions x_1 !

$(6) \,|\, \cdot \,$ „$V'(x_1) = 1$" $\Rightarrow$

$$\alpha \cdot C\left(\theta_+ e^{\theta_+ x_1} - \theta_- e^{\theta_- x_1}\right) = C\left(e^{\theta_+ x_1} - e^{\theta_- x_1}\right) \Rightarrow$$

$$e^{\theta_+ x_1}(\alpha\theta_+ - 1) = e^{\theta_- x_1}(\alpha\theta_- - 1) \Rightarrow$$

$$e^{(\theta_+ - \theta_-)x_1} = \frac{\alpha\theta_- - 1}{\alpha\theta_+ - 1} \qquad (7)$$

Question: Is the r.h.s. of (7) positive?

numerator $< 0 \Rightarrow$ denominator < 0 ???

<u>Claim:</u> $\underline{\alpha\theta_+ < 1}$ (8)

<u>Proof:</u> One has: $\sqrt{a^2+b} - a < \frac{b}{2a} \quad \forall a,b > 0$ (*)

$$\Rightarrow \theta_+ = \frac{-\mu + \sqrt{\mu^2 + 2\beta\sigma^2}}{\sigma^2} \underset{(*)}{<} \frac{\frac{2\beta\sigma^2}{2\mu}}{\sigma^2} = \frac{\beta}{\mu} \qquad (9)$$

$\Rightarrow$ to prove: $\frac{M}{\beta} + \frac{1}{\gamma_-} < \frac{1}{\theta_+} \Leftrightarrow \boxed{\frac{1}{\theta_+} - \frac{1}{\gamma_-} > \frac{M}{\beta}}$ (10)

Case 1: $M \le \mu$: $(9) \Rightarrow \frac{1}{\theta_+} > \frac{\mu}{\beta} \ge \frac{M}{\beta} \Rightarrow (10)$ ✓ $(\gamma_- < 0\,!)$

Case 2: $M > \mu$: $-\gamma_- = \frac{\mu - M + \sqrt{(\mu-M)^2 + 2\beta\sigma^2}}{\sigma^2} \overset{(*)}{<} \frac{\frac{2\beta\sigma^2}{2(M-\mu)}}{\sigma^2} = \frac{\beta}{M-\mu}$ (11)

$\Rightarrow \frac{1}{\theta_+} - \frac{1}{\gamma_-} \underset{(9)(11)}{>} \frac{\mu}{\beta} + \frac{M-\mu}{\beta} = \frac{M}{\beta} \Rightarrow (10)$ ✓

$\overset{(7)}{\Rightarrow} \boxed{x_1 = \frac{1}{\theta_+ - \theta_-} \ln\left(\frac{1-\alpha\theta_-}{1-\alpha\theta_+}\right)}$

C, D .... determine by plugging in!

if $\alpha \leq 0$ $\Rightarrow$ $x_1 \nexists$ in $(0,\infty) \Rightarrow$

$\exists$ only one interval $(\mathbb{R}^+)$: there $V(x) = \frac{M}{\beta} \cdot (1 - e^{r_- x})$

(since $V(0) = 0$!)

i.e.: $\alpha > 0$: $U_t^* = \begin{cases} M & X_t^* > x_1 \\ 0 & X_t^* \leq x_1 \end{cases}$

$\alpha \leq 0$: $U_t^* \equiv M$, we pay always the full dividend rate!

Remarks: 1.) the conditions of the Verificationtheorem are ok.;

BUT: for the existence of a strong solution

we can not use a standard existence result!

($U(x)$ is not even continuous, hence not Lipschitz cont.)

$\Rightarrow$ • use a result by ZVONKIN: $\left.\begin{array}{l}\text{const. volatility}\\ \text{bounded drift}\end{array}\right\} \Rightarrow \exists$ unique strong solution

2.) lump sum payments are not possible, since we have asumed a rate!

$\Rightarrow$ for a vigorous treatment we need the concept of the local time of Brownian motion

V) Local time of Browian motion

V.1) Introduction

Goal: Tanaka formula : decomposition of the submartingale $|W_t|$ (Jensen!)
into another Brownian motion $\hat{W}_t$
and a continuous, nondecreasing process L_t

in general: decomposition : $|W_t - x| = x + \hat{W}_t + L_t(x)$

$L_t(x)$... local time of W_t at the level x

(introduced by P. Levy (1948))

different notation : $L(t,x) = L_t(x) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{(x-\varepsilon, x+\varepsilon)}(W_s)\, ds =$

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \underbrace{\lambda}_{\text{Lebesgue measure}} \left(\left\{ s \in [0,t] \mid W_s \in (x-\varepsilon, x+\varepsilon) \right\}\right) \tag{5.1}$$

i.e. $L(t,x) \ldots$ measure for the time, which W_t spends "at" the level x.

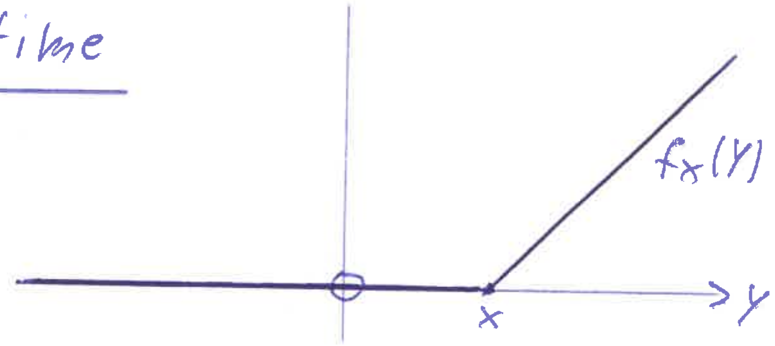
Remarks: 1.) known: $\{ t \mid B_t = x \}$: closed set with Lebesgue measure 0 (hence the def. with the neighbourhood)

2.) Question: Does the limit exist and in which sense?

3.) one can show: $L(t,x)$ is continuous in (t,x)

V.2) Existence of the local time

Def: $f_x(y) := (y-x)^+$

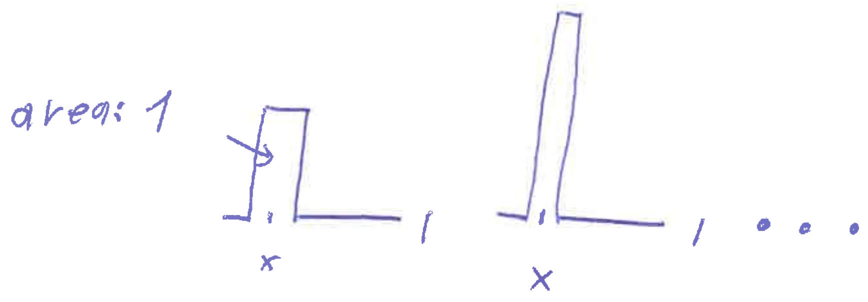


not differentiable in the classical sense, but in the distributional sense!

$f_x'(y) = \mathbb{1}_{[x,\infty)}(y)$

$f_x''(y) = \delta_x(y) \ldots$ Dirac Delta: $\langle \delta_x, \varphi \rangle = \varphi(x)$, $\varphi \in C_K^\infty$

is a limit of „needle functions"



formal application of Ito's formula $\Rightarrow$

$$|W_t - x|^+ - |W_0 - x|^+ = \int_0^t \mathbb{1}_{[x,\infty)}(W_s)\, dW_s + \frac{1}{2}\int_0^t \delta_x(W_s)\, ds$$

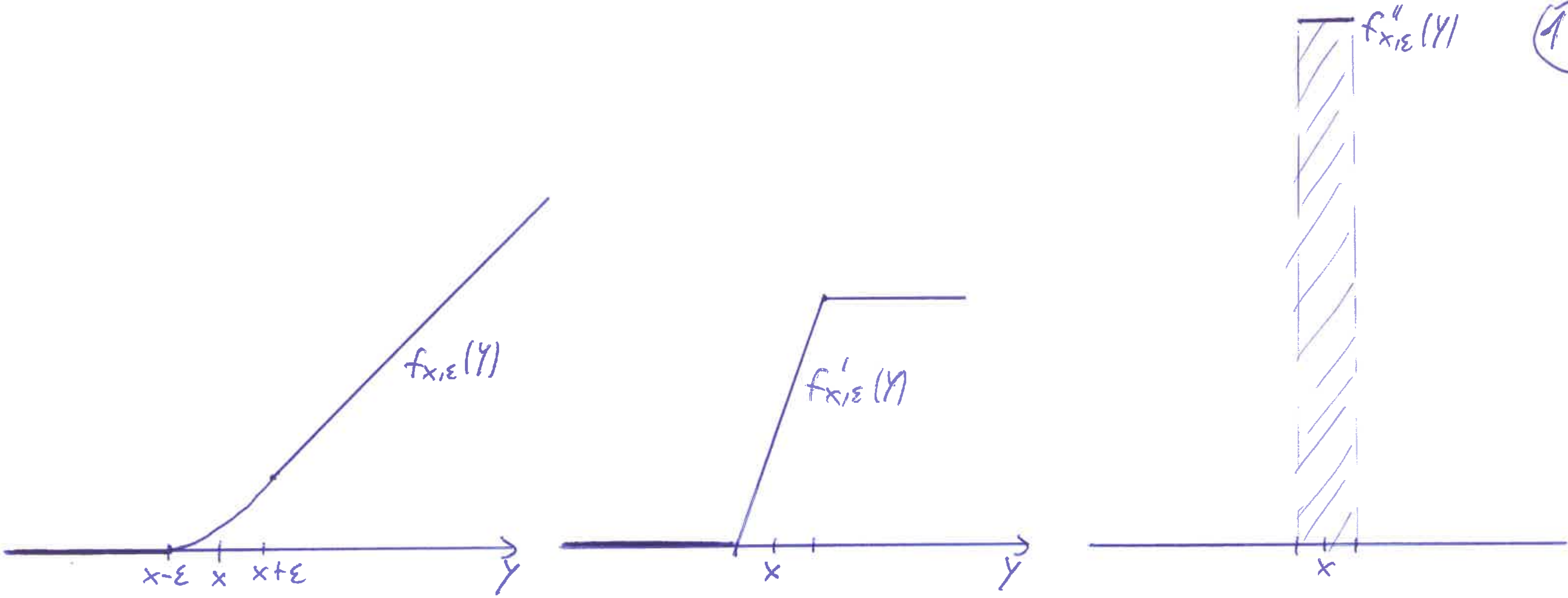
we shall see: equation is correct, if we interpret the last integral in the limit sense of (5.1)

Approximation:

$$f_{x,\varepsilon}(y) = \begin{cases} 0 & y \le x-\varepsilon \\ \frac{(y-x+\varepsilon)^2}{4\varepsilon} & y \in (x-\varepsilon, x+\varepsilon) \\ y-x & x \ge x+\varepsilon \end{cases}$$

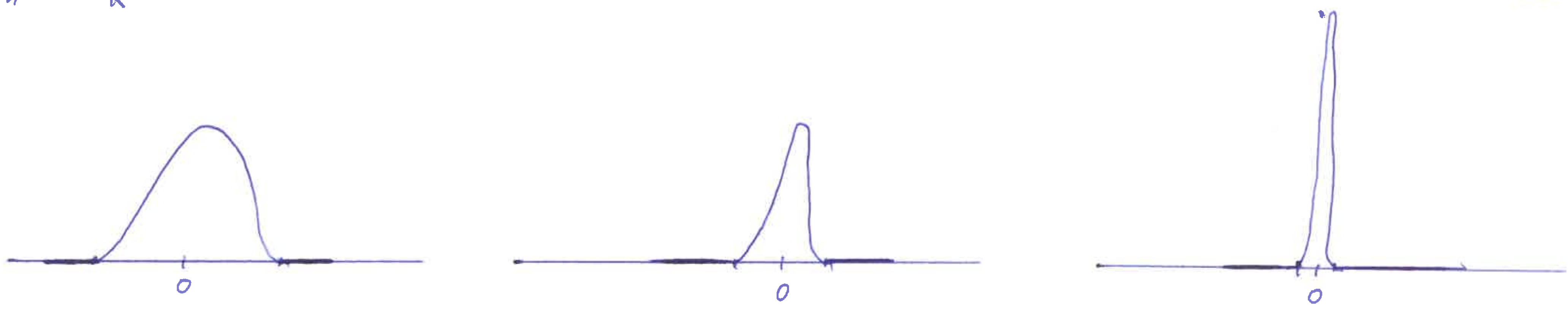
$$f_{x,\varepsilon}'(y) = \begin{cases} 0 & y \leq x - \varepsilon \\ \frac{(y - x + \varepsilon)}{2\varepsilon} & y \in (x-\varepsilon, x+\varepsilon) \\ 1 & y \geq x + \varepsilon \end{cases}$$

$$f_{x,\varepsilon}''(y) = \begin{cases} 0 & y < x - \varepsilon \\ \frac{1}{2\varepsilon} & y \in (x-\varepsilon, x+\varepsilon) \\ 0 & y > x + \varepsilon \end{cases}$$



$f''_{x,\varepsilon} \in C^2$, therefore mollifying by a standard procedure from analysis

$\exists\, \phi_n \in C_K^\infty$:



with: support $\to \{0\}$

s.t.: $g_n := \phi_n * f_{x,\varepsilon}$, i.e.: $g_n(y) := \int_{\mathbb{R}} f_{x,\varepsilon}(y-z)\, \phi_n(z)\, dz$

$\Rightarrow g_n \in C^\infty$, $g_n \to f_{x,\varepsilon}$, $g_n' \to f_{x,\varepsilon}'$ unif. in $\mathbb{R}$

$g_n'' \to f_{x,\varepsilon}''$ pointwise except at $x \pm \varepsilon$

e.g. $\phi_n(y) = n \cdot \phi(ny)$ with $\phi(y) = \begin{cases} c \cdot \exp\left(-\frac{1}{1-y^2}\right), & |y| < 1 \\ 0 & |y| \geq 1 \end{cases}$

$c > 0$, s.t. $\int_{\mathbb{R}} \phi(y)\,dy = 1$

Remark: ϕ is C^∞ at $x = \pm 1$, because

$\phi^{(k)}$ have the structure $Q_k(y) \cdot \exp\left(-\frac{1}{1-y^2}\right)$

with a rational function Q_k with poles at ± 1

$$\Rightarrow \lim_{y \to \pm 1} \phi^{(k)}(y) = 0 \quad !$$

Ito's formula:

$$g_n(W_t) - g_n(W_0) = \int_0^t g_n'(W_s)\,dW_s + \frac{1}{2}\int_0^t g_n''(W_s)\,ds \qquad (*)$$

lim $n \to \infty$:

We have: $g_n'(W_s) \to f_{x,\varepsilon}'(W_s)$ unif. on $[0,T] \times \Omega \Rightarrow$

convergence also in $L^2(\lambda \otimes \mathbb{P})$

$$\Rightarrow \text{Ito isometry} \Rightarrow \int_0^t g_n'(W_s)\,dW_s \xrightarrow{L^2(\mathbb{P})} \int_0^t f_{x,\varepsilon}'(W_s)\,dW_s \qquad (*_1)$$

Moreover: for fixed s: $\mathbb{P}(W_s = x \pm \varepsilon) = 0$

$$\Rightarrow \lim_{n\to\infty} g_n''(W_s) = f_{x,\varepsilon}''(W_s) \quad \text{a.s.} \quad \text{for each fixed } s$$

$$\Rightarrow \lim_{n\to\infty} g_n''(W_s) = f_{x,\varepsilon}''(W_s) \quad \lambda\text{-a.e.}, \quad \text{a.s.}$$

$|g_n''| \leq \frac{1}{2\varepsilon}$ $\Rightarrow$ dom. conv.

$$\int_0^t g_n''(W_s)\,ds \xrightarrow{n\to\infty} \int_0^t f_{x,\varepsilon}''(W_s)\,ds \quad \text{a.s. and in } L^2 \qquad (*_2')$$

Clearly: $g_n(W_t) - g_n(W_0) \xrightarrow{L^2} f_{x,\varepsilon}(W_t) - f_{x,\varepsilon}(W_0)$ $\qquad (*_3)$

$(*)(*_1)(*_2)(*_3) \Rightarrow$

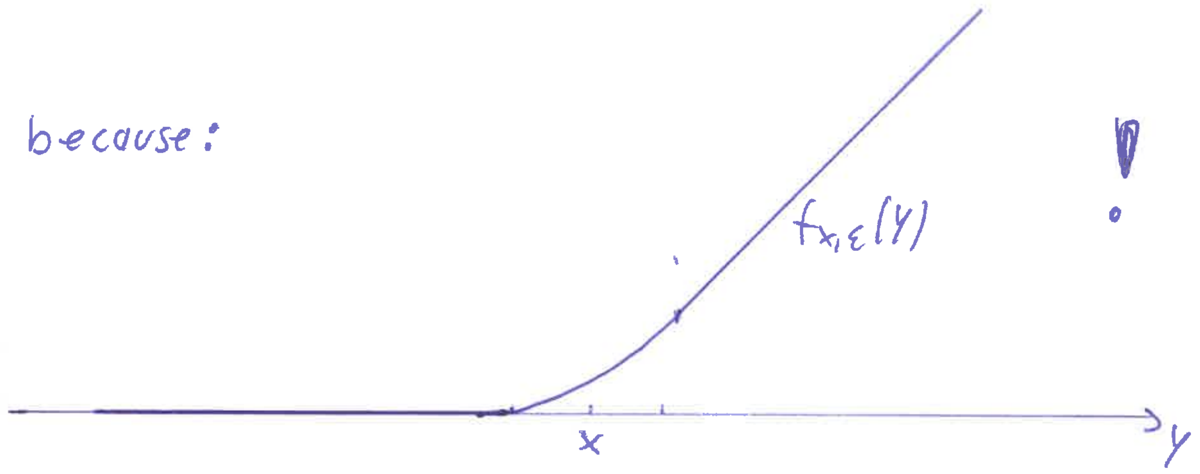
$$\boxed{f_{x,\varepsilon}(W_t) - f_{x,\varepsilon}(W_0) = \int_0^t f_{x,\varepsilon}'(W_s)\,dW_s + \frac{1}{2}\int_0^t \frac{1}{2\varepsilon}\mathbb{1}_{(x-\varepsilon,\,x+\varepsilon)}(W_s)\,ds} \qquad (+)$$

a.s.

$\varepsilon \to 0$: $f_{x,\varepsilon}(W_t) - f_{x,\varepsilon}(W_0) \xrightarrow[\varepsilon \to 0]{} (W_t - x)^+ - (W_0 - x)^+$ a.s.

one has: $|f_{x,\varepsilon}(W_t) - f_{x,\varepsilon}(W_0)| \leq |W_t - W_0|$

because:



$\Rightarrow f_{x,\varepsilon}(W_t) - f_{x,\varepsilon}(W_0) \xrightarrow[\varepsilon \to 0]{} (W_t - x)^+ - (W_0 - x)^+$ in $L^2(\mathbb{P})$

(t_1)

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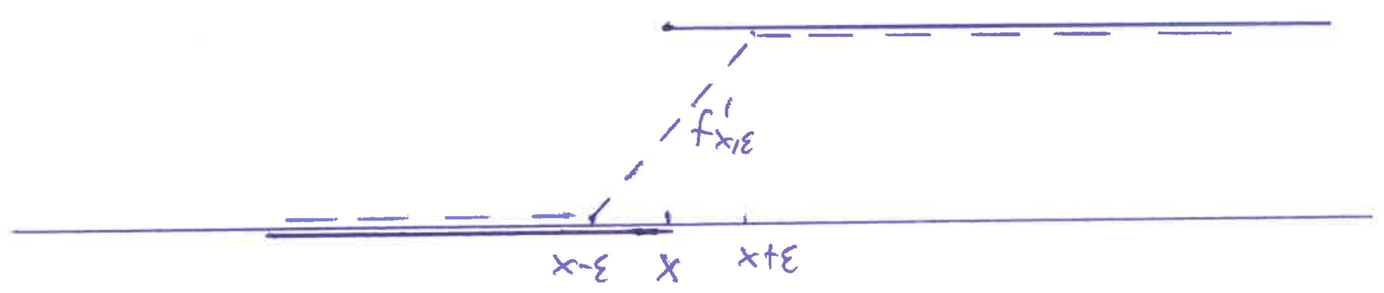
$\underset{\sim}{\text{Moreover:}}$ $\mathbb{E}\left[\int_0^t \left(f'_{x,\varepsilon}(W_s) - \mathbb{1}_{[x,\infty)}(W_s)\right)^2 ds\right] \overset{!}{\leq} \mathbb{E}\left[\int_0^t \mathbb{1}_{[x-\varepsilon, x+\varepsilon]}(W_s)\, ds\right]$

$$\underset{\text{Fubini}}{=} \int_0^t \mathbb{P}(x-\varepsilon \leq W_s \leq x+\varepsilon)\, ds \leq \int_0^t \frac{2\varepsilon}{\sqrt{2\pi s}}\, ds \xrightarrow{\varepsilon \to 0} 0$$



Ito isometry $\Rightarrow$ $\int_0^t f'_{x,\varepsilon}(W_s)\, dW_s \xrightarrow[L^2(\mathbb{P})]{} \int_0^t \mathbb{1}_{[x,\infty)}(W_s)\, dW_s$ $\quad (t_2)$

for $\varepsilon \to 0$

$\Rightarrow$ $(t_1),(t_2),(t)$

Theorem: $\forall t \in \mathbb{R}_+$ and $x \in \mathbb{R}$ we have a.s.:

$$(W_t - x)^+ - (W_0 - x)^+ = \int_0^t \mathbb{1}_{[x,\infty)}(W_s)\, dW_s + \frac{1}{2} L_t(x)$$

where $L_t(x)$ is defined in (S.1) and the limit there is in $L^2(\mathbb{P})$

<u>Remarks:</u> • one can show:

- lim in (5.1) holds also a.s.

- $\exists$ a continuous version (in (t,x)) of $L(t,x)$

• the "<u>occupation time" formula</u> holds

$\forall t \geq 0$, $a < b$:

$$\int_a^b L(t,x)\,dx = \int_0^t \mathbb{1}_{(a,b)}(W_s)\,ds$$

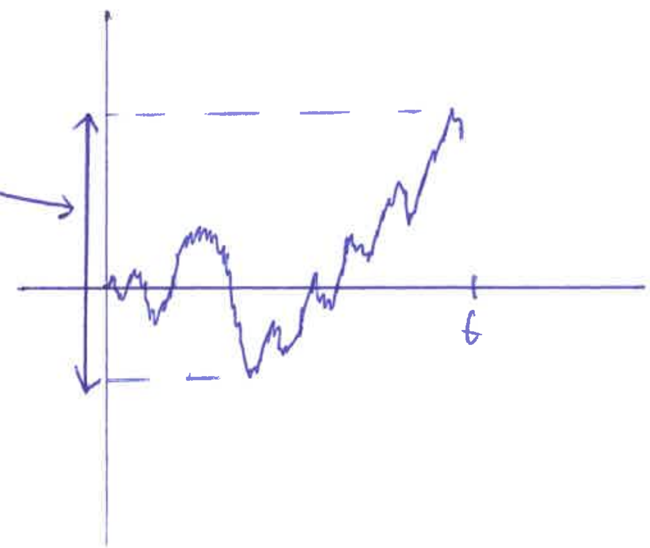
„replace the infinitesimal interval by (a,b)" !

- generalization f Borel mb., locally integrabel

$$\Rightarrow \int_{-\infty}^{\infty} L(t,x) f(x)\,dx = \int_0^t f(W_s)\,ds \qquad \text{a.s.}$$

Rem: for fixed t, ω: the $\int$ on the l.h.s. runs only over a bounded set:

Ex:



5.3) Tanaka formula

Theorem: $\forall (t,x)$ we have a.s.:

$$|W_t - x| - |W_0 - x| = \int_0^t \mathrm{sgn}(W_s - x)\, dW_s + L(t,x)$$

with $\mathrm{sgn}(y) = \begin{cases} 1 & y > 0 \\ 0 & y = 0 \\ -1 & y < 0 \end{cases}$

Proof: $-W_t$ is a Brownian motion

$\Rightarrow$ has local time at $-x$: $L^-(t,-x)$

symmetry $\Rightarrow$ $L^-(t,-x) = L(t,x)$ a.s. $\quad (*)$

apply the theorem above (from 5.2) to $-W_t, -x$:

$$\Rightarrow (-W_t+x)^+ - (-W_0+x)^+ = \int_0^t \mathbb{1}_{[-x,\infty)}(-W_s)\, d(-W_s) + \tfrac{1}{2} L^-(t,-x)$$

$$\overset{(*)}{\Rightarrow} (W_t-x)^- - (W_0-x)^- = -\int_0^t \mathbb{1}_{(-\infty,x]}(W_s)\, dW_s + \tfrac{1}{2} L(t,x)$$

add this to the original formula $\Rightarrow$

$$|W_t-x| - |W_0-x| = \int_0^t \operatorname{sgn}(W_s-x)\, dW_s + L(t,x) \qquad \square$$

Remark: • formula holds also if we define: $\operatorname{sgn}(y) = \begin{cases} 1 & y \geq 0 \\ 0 & y < 0 \end{cases}$

because: $\int_0^t \mathbb{1}_{\{x\}}(W_s)\, dW_s = 0$ a.s.

(since: $\mathbb{E}\left[\left(\int_0^t \mathbb{1}_{\{x\}}(W_s)\, dW_s\right)^2\right] \underset{\text{Ito isometry}}{=} \mathbb{E}\left[\int_0^t \mathbb{1}^2_{\{x\}}(W_s)\, ds\right] = \mathbb{E}\left[\int_0^t \mathbb{1}_{\{x\}}(W_s)\, ds\right] = 0$

• formula above: special case of the <u>Doob-Meyer decomposition</u>:

$$\underset{\substack{\uparrow \\ \text{+ techn. conditions}}}{\text{submartingale}} = \text{martingale} + \text{nondecreasing process}$$

Def: $\hat{W}(t,x) := |W_0 - x| + \int_0^t \operatorname{sgn}(W_s - x)\, dW_s \quad \Rightarrow$

Theorem: $\forall x$ we have a.s.:

$$P\left(|W_t - x| = \hat{W}(t,x) + L(t,x) \quad \forall t\right) = 1,$$ i.e. Tanaka holds pathwise!

where for fixed x:

- $\hat{W}(t,x)$ is a Brownian motion, started at x
- $L(t,x)$ is a continuous non decreasing process ($L(0,x) = 0$)
- $L(t,x)$ increases only if $W_t = x$, i.e.

$$\int_0^\infty \mathbb{1}_{\{t \mid W_t \neq x\}}\, dL(t,x) = 0 \quad \text{a.s.}$$

idea of proof: 1.) both sides are continuous $\Rightarrow$ formula holds path wise

(if X_t is a modification of Y_t and both are e.g. right cont.

$\Rightarrow$ X_t and Y_t are indistinguishable)

2.) $\langle \hat{W} \rangle_t = \int_0^t (\operatorname{sgn}(W_s - x))^2 \, d\langle W \rangle_s = t$ a.s.

$\Rightarrow$

Theorem of Levy: continuous local martingale M with $\langle M \rangle_t = t$

is a Brownian motion

$\Rightarrow$ $\hat{W}$ is a ✓

3.) $L_t(x)$ continuous (sh Remark above; without proof)

4.) $L_t(x)$ non decreasing : $L_t(x)$ as a.s. limit of the r.h.s of (5.1)

$$\underbrace{\frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{(x-\varepsilon,\, x+\varepsilon)}(W_s)\, ds}_{\text{non decreasing !}}$$

5.) $\exists\, \Omega_1$ with $\mathbb{P}(\Omega_1) = 1$: (S.1) holds on Ω_1 $\forall t \in \mathbb{Q}$ (denumerable!)

Let $\omega \in \Omega_1$ fixed; let t a point of increase, i.e.:

$\exists\, r, r' \in \mathbb{Q}$: $r < t < r'$ arbitrary close to t, s.t.

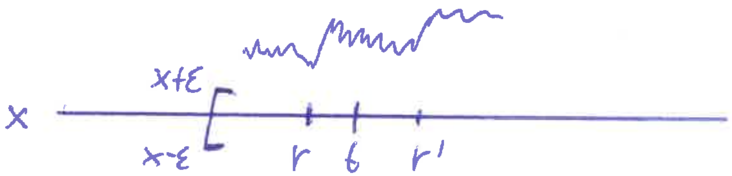
$$L(r, \omega) < L(r', \omega)$$

$\overset{(S.1)}{\Rightarrow}$ $\lambda(\{s \in [r, r'] \mid W_s(\omega) \in (x-\varepsilon, x+\varepsilon)\}) > 0 \quad \forall \varepsilon > 0 \qquad (*)$

$W_t(\omega)$ is contin.

$\Rightarrow$ $W_s(\omega) = x$ for an $s \in [r, r']$

otherwise



$\nearrow$ to $(*)$

r, r' arbitr. close to t

$\Rightarrow$ $W_t(\omega) = x \qquad \square$

Remark: formula can be written as (for convenience we set $x = 0$!)

$$\hat{W}_t = |W_t| - L_t$$

this is called a Skorohod decomposition:

cont. function = (non negative − non decreasing) functions

(with the additional property: $\int_0^\infty 1\!\!1_{\{t \,|\, W_t \neq 0\}} \, dL_t = 0$)

one can show:

- Skorohod decomposition is unique
- $L_t = \max_{0 \le s \le t} \hat{W}_s^-$

VI.) Singular stochastic control theory

Motivation from insurance math.: lump sum payments in dividend problem!

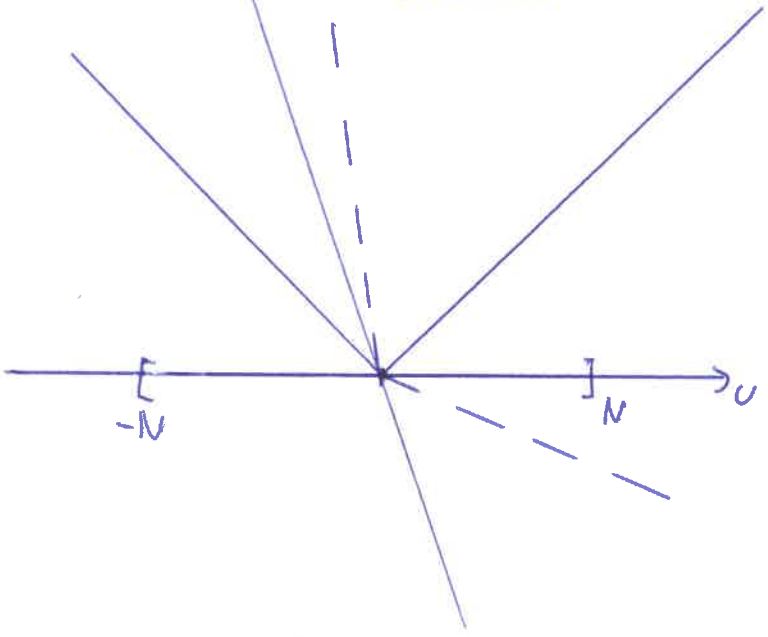
VI.1) Heuristics

Ex: Linear regulator problem with linear costs:

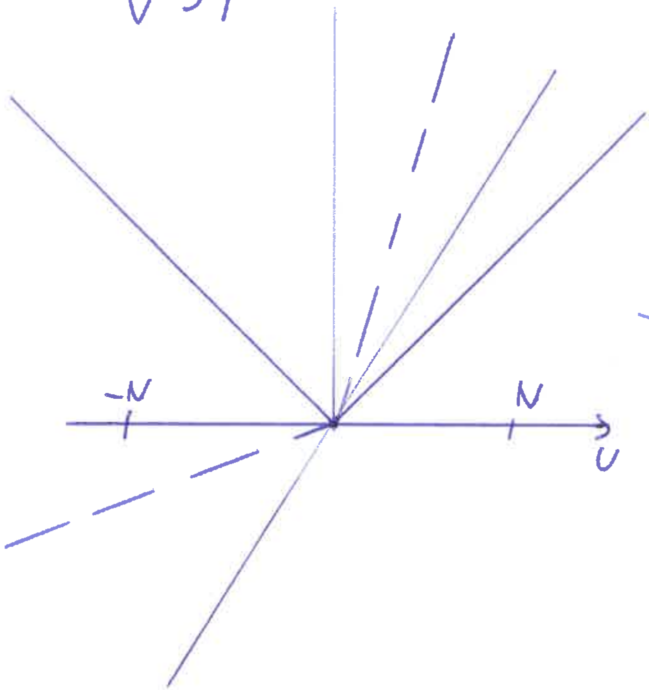
$$\begin{cases} dX_t = u_t\,dt + dW_t, & u \in \mathcal{U} := [-N, N] \\ \mathbb{E}\left[\int_0^\infty e^{-\beta t} \left(|u_t| + X_t^2 \right) dt \right] \longrightarrow \min \end{cases}$$

HJB: $$\inf_{u \in [-N,N]} \left[\frac{1}{2} V''(x) + u V' - \beta V + |u| + x^2 \right] = 0$$

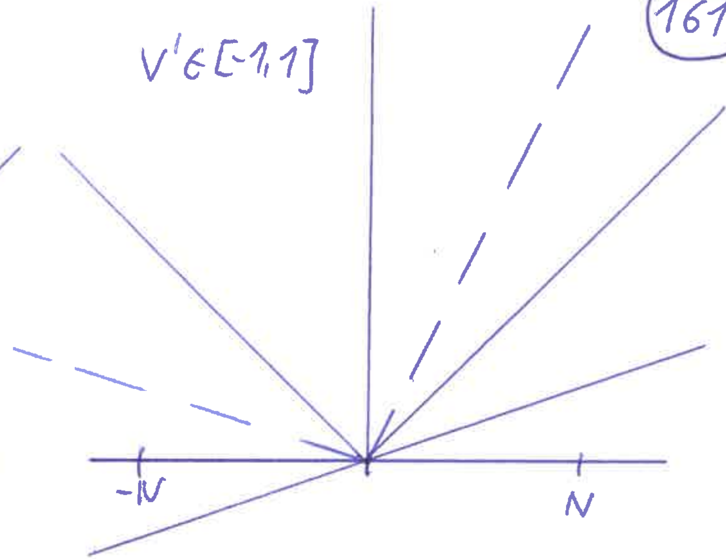
$V' < -1$



$V' > 1$



$V' \in [-1,1]$



$$\Rightarrow \hat{u} = \begin{cases} N & V' < -1 \\ 0 & V' \in [-1,1] \\ N & V' > 1 \end{cases}$$

$\lim N \to \infty$??? $\Rightarrow$

<u>different formulation of the problem:</u>

Let $L_t := \int_0^t v_s \, ds \Rightarrow |L|_t = \int_0^t |v_s| \, ds$

$\uparrow$ total variation of L_t

new formulation: (is also possible, if L_t is <u>not</u> of the form $\int_0^t v_s \, ds$!)

$$dX_t = dW_t + dL_t$$

$$J(x, L_t) = \mathbb{E}_x \left[\int_0^\infty e^{-\beta t} X_t^2 \, dt + \int_0^\infty e^{-\beta t} d|L|_t \right] \longrightarrow \min$$

with: L_t (i) adapted

(ii) of bounded total variation on compact intervals

(iii) cadlag paths

VI.2) Motivation of the HJB equation

Problem:
$$\begin{cases} dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t + dL_t \\ X_0 = x \end{cases}$$

L admissible if: (i) adapted

(ii) bounded total variation on compact intervals

(iii) cadlag

$$J(x,L) = \mathbb{E}_x\left[\int_0^\infty e^{-\delta t} h(X_t)\,dt + c\int_0^\infty e^{-\delta t}\,d|L|_t\right]$$

$$V(x) = \inf_L J(x,L), \quad h(x) \geq 0, \quad c > 0 \ldots \text{const.}$$

We are looking for: $V(x)$, L^* s.t.: $J(x, L^*) = V(x)$

Theorem (HJB-eq.): Let $V \in C^2 \Rightarrow V$ fulfills the HJB-equation:

$$\min\{\mathcal{L}V(x) - \beta V(x) + h(x),\ c - |V'(x)|\} = 0$$

with $\mathcal{L}V = \frac{\sigma^2(x)}{2} V'' + \mu V'(x)$

idea of proof: assume $\exists$ optimal control $L_x^*(t)$ s.t. $V(x) = J(x, L_x^*)$

$$\text{set} \quad L(s) = \begin{cases} 0 & s < \varepsilon \\ L^*_{X(\varepsilon-)}(s-\varepsilon) & s \geq \varepsilon \end{cases} \Rightarrow$$

$$J(x, L) = \mathbb{E}_x\Big[\int_0^\varepsilon e^{-\beta s} h(X_s)\,ds + e^{-\beta\varepsilon}\int_\varepsilon^\infty e^{-\beta(s-\varepsilon)} h(X^*_{X(\varepsilon-)})(s-\varepsilon)\,ds$$

$$+\, e^{-\beta\varepsilon} c \cdot \int_\varepsilon^\infty e^{-\beta(s-\varepsilon)}\,d|L^*_{X(\varepsilon-)}|(s-\varepsilon)\Big] \Longrightarrow$$

L suboptimal

$$V(x) \leq J(x, L) = \mathbb{E}_x \left[\int_0^{\varepsilon} e^{-\beta s} h(X_s)\, ds + e^{-\beta \varepsilon} V(X_{\varepsilon-}) \right] \qquad (*)$$

Ito's formula $\Rightarrow$
$L(s) \equiv 0,\ s < \varepsilon$!

$$e^{-\beta \varepsilon} V(X_{\varepsilon-}) = V(x) + \int_0^{\varepsilon} e^{-\beta s} \left[\mathcal{L} V(X_s) - \beta V(X_s) \right] ds$$
$$+ \int_0^{\varepsilon} e^{-\beta s} \sigma(X_s) V'(X_s)\, dW_s$$

Plug into $(*)$: reduce by $V(x)$, $\mathbb{E}$ ($\Rightarrow \mathbb{E}[\int \ldots dW_s] = 0$!)

$$\Rightarrow 0 \leq \mathbb{E}_x \left[\int_0^{\varepsilon} e^{-\beta s} \left[\mathcal{L} V(X_s) - \beta V(X_s) + h(X_s) \right] ds \right. \quad \Big| : \varepsilon,\ \varepsilon \to 0 \Rightarrow$$

$$\boxed{0 \le \mathcal{L}V(x) - \delta V(x) + h(x)} \qquad (t_1)$$

Let now: $L_s = \varepsilon + L^*_{x+\varepsilon}$ („$L(0-) = 0$", $\Delta L(0) = \varepsilon$)

i.e. jump by $+\varepsilon$ at $t=0$, then behave optimally!

$$\Rightarrow J(x,L) = c\varepsilon + V(x+\varepsilon) \overset{L \text{ suboptimal}}{\Rightarrow}$$

$$V(x) \le c\varepsilon + V(x+\varepsilon) \quad \big| -V(x), \; -c\varepsilon$$

$$-c\varepsilon \le V(x+\varepsilon) - V(x) \quad \big| :\varepsilon, \; \varepsilon \to 0$$

$$-c \le V'(x)$$

Now: $L(s) = -\varepsilon + L^*_{x-\varepsilon}$

$$\Rightarrow J(x,L) = c\varepsilon + V(x-\varepsilon) \overset{L \text{ suboptimal}}{\Rightarrow}$$

$$c\varepsilon + V(x-\varepsilon) \geq V(x) \quad \Big| - V(x-\varepsilon)$$

$$c\varepsilon \geq V(x) - V(x-\varepsilon) \quad \Big| : \varepsilon, \quad \varepsilon \to 0$$

$$c \geq V'(x) \qquad \Rightarrow -c \leq V'(x) \leq c$$

$$\Rightarrow \quad \boxed{c - |V'(x)| \geq 0} \qquad (t_2)$$

one can show that one of the inequalities (t_1), (t_2) is sharp, i.e. the HJB equation holds. $\square$

to prove a general verification theorem is difficult $\Rightarrow$ simplification!

„monotone follower" problem

$$dX_t = \mu(X_t) + \sigma(X_t) - dL_t, \quad X_0 = x$$

L admissible, if (i) L_t adapted

(ii) L_t monotone non decreasing, $L_{0-} = 0$

(iii) L_t cadlag

$$V(x) = \inf_L J(x,L) = \inf_L \mathbb{E}_x \left[\int_0^\infty e^{-\beta t} h(t)\, dt + c \int_0^\infty e^{-\beta t}\, dL_t \right], \quad c > 0, \; h(x) \geq 0$$

?: L^*, s.t. $J(x, L^*) = V(x)$

HJB equation: (as above!)

$$\boxed{\min\{\mathcal{L}V(x) - \beta V(x) + h(x),\ c - V'(x)\} = 0}$$

Construction of the value function and the optimal strategy

1.) value function

Construct a $V(x)$, s.t. the real axis $\mathbb{R}$ is separated into 2 intervals:

$$\mathcal{C} = \left\{ x \,\middle|\, \frac{\sigma^2}{2} V''(x) + \mu V'(x) - \beta V + h(x) = 0,\ c - V'(x) \geq 0 \right\} = (-\infty, m^*]$$

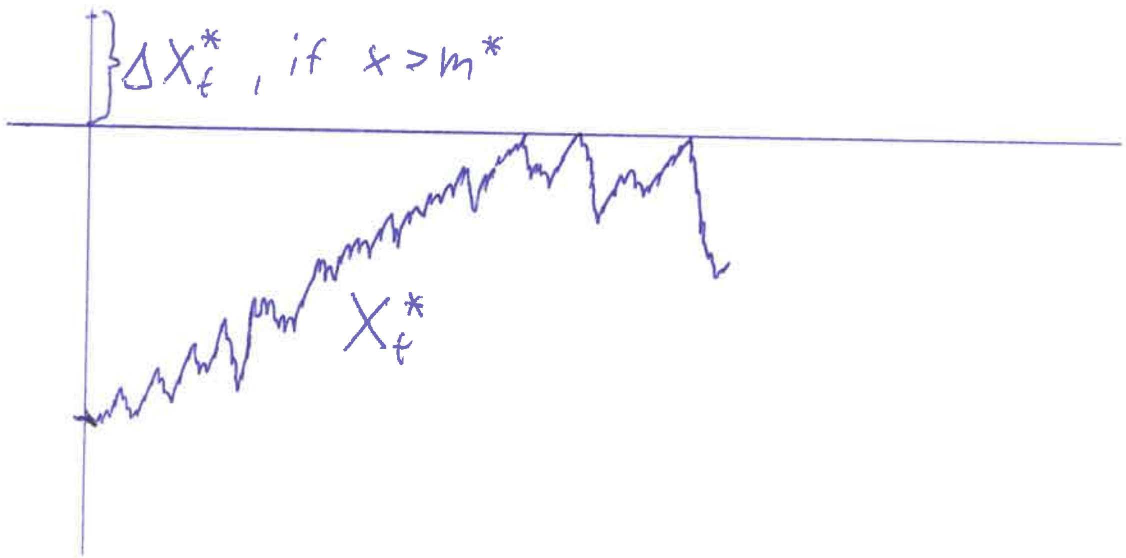
$$\mathcal{D} = \left\{ x \,\middle|\, \frac{\sigma^2}{2} V''(x) + \mu V'(x) - \beta V + h(x) \geq 0,\ c - V'(x) = 0 \right\} = [m^*, \infty)$$

2.) strategy

to find the optimal strategy, one has to solve the following <u>Skorohod problem:</u>

Find (L_s^*, X_s^*), s.t.

$$\left.\begin{aligned} x + \int_0^t \mu(X_s^*)\,ds + \int_0^t \sigma(X_s^*)\,dW_s &= X_t^* + L_t^* \\ \text{with: } \int_0^\infty \mathbb{1}_{\{X_t^* \neq m^*\}}\,dL_t^* &= 0 \\ X_t^* \in C, \quad t &> 0 \end{aligned}\right\}$$



compare : Skorohod problem for Brownian motion:

$$\hat{W} = |W| - L_t$$

$$\int_0^\infty 1\!\!1_{\{W_t \neq 0\}} \, dL_t = 0$$

$$|W_t| \geq 0 \quad , \qquad \mathbb{R}^+ \longleftrightarrow C \quad , \qquad 0 \longleftrightarrow m^*$$

general solution of this problem: Lions, P.L; Snitzman (1984)

X_t^* .... at m^* reflected diffusion X_t

(where $dX_t = \mu(X_t)dt + \sigma(X_t)\,dW_t$)

VI.3) A verification theorem

Theorem: assume (X_t^*, L_t^*) is a solution of the Skorohod problem above,

$V \in C^2$, fulfills the HJB equation and

$$\lim_{t \to \infty} \mathbb{E}\left[e^{-\beta t} V(X_t)\right] = 0 \quad (*) \qquad \forall \text{ admissuble } L$$

$\Rightarrow$ V is the value function and L_t^* the optimal strategy

Proof: Step 1: Let L be admissable with stateprocess X_t:

Ito's formula + $\mathbb{E}$ + vanishing of the $\int \cdot\cdot \, dW$ integral in expectation

$\Longrightarrow$

$$\mathbb{E}_x\left[e^{-\beta t}V(X_t)\right] - V(x) = \mathbb{E}_x\left[\int_0^t e^{-\beta s}\left(\frac{\sigma^2}{2}V''(X_s) + \mu V'(X_s) - \beta V(X_s)\right)ds\right]$$

$$- \mathbb{E}_x\left[\int_0^t e^{-\beta s}V'(X_s)\,dL_s^c\right] + \mathbb{E}_x\left[\sum_{\substack{s\in S_L = S_X \\ s\le t}} e^{-\beta s}\left(V(X_s)-V(X_{s-})\right)\right]$$

L_s^c ... continuous part of L_s

S_L .... jump times of L

We have: $X_s - X_{s-} = -(L_s - L_{s-}) \Rightarrow$

mean value theorem, $\theta \in [X_s, X_{s-}]$

$$V(X_s) - V(X_{s-}) = V'(\theta)(X_s - X_{s-}) = -V'(\theta)(L_s - L_{s-}) \overset{\text{HJB}}{\geq} -c(L_s - L_{s-}) \Rightarrow$$

$$\underline{\mathbb{E}_x\left[e^{-\beta t}V(X_t)\right] - V(x)} \geq \mathbb{E}_x\left[\int_0^t e^{-\beta s}\left(\frac{\sigma^2}{2}V''(X_s) + \mu V'(X_s) - \beta V(X_s)\right)ds\right]$$

$$- \mathbb{E}_x\left[\int_0^t e^{-\beta s}\, c \, dL_s^c\right] - \mathbb{E}_x\left[\sum_{\substack{s \in J_L \\ s \leq t}} e^{-\beta s}(L_s - L_{s-})\right]$$

$$= \mathbb{E}_x\left[\int_0^t e^{-\beta s}\left(\frac{\sigma^2}{2}V''(X_s) + \mu V'(X_s) - \beta V(X_s)\right)ds\right] - \mathbb{E}_x\left[\int_0^t e^{-\beta s} c \, dL_s\right]$$

$$\overset{HJB}{\geq} \underline{\mathbb{E}_x\left[\int_0^t e^{-\beta s}(-h(X_s))\,ds - c\int_0^t e^{-\beta s}\,dL_s\right]}$$

$$\lim_{t\to\infty}, \quad (*) \quad \Rightarrow$$

$$-V(x) \geq -\mathbb{E}_x\left[\int_0^\infty e^{-\beta s} h(X_s)\,ds\right] - c\,\mathbb{E}\left[\int_0^\infty e^{-\beta s}\,dL_s\right] = -J(x,L)$$

$$\Rightarrow \underline{\underline{V(x) \leq J(x,L)}}$$

Step 2: Let L^* be the solution of the Skorohod problem

One knows (Lions/Snitzman): Apart from a possible jump at $t=0$: L^* contin. (as the local time of Brownian motion)

Case 1: $x \leq m^*$ (Case 2: $x > m^*$: Exercise!: $L^*(0) - L^*(0-) = x - m$!)

$$\underline{\mathbb{E}_x\left[e^{-\beta t}V(X_t)\right] - V(x)} \underset{\substack{\uparrow \\ \text{no jumps}}}{=} \mathbb{E}\left[\int_0^t e^{-\beta s}\left[\frac{\sigma^2}{2}V''(X_s^*) + \mu V'(X_s^*) - \beta V(X_s^*)\right] ds\right]$$

$$- \mathbb{E}_x\left[\int_0^t e^{-\beta s} V'(X_s^*)\, dL_s^*\right] =$$

$$\left| \begin{array}{l} X_t^* \in C \\ L^* \text{ increases only for } X_t^* = m^* \end{array} \right| = \mathbb{E}_x\left[\int_0^t e^{-\beta s}(-h(X_s^*))\, ds\right] - \mathbb{E}_x\left[\int_0^t e^{-\beta s} V'(X_s^*) \mathbb{1}_{\{X_s^* = m^*\}}\, dL_s^*\right]$$

$$\underset{=}{V \ldots} \; C^2 \text{ at } m^* \quad \mathbb{E}_x\left[\int_0^t e^{-\beta s}(-h(X_s^*))\, ds\right] - \mathbb{E}_x\left[\int_0^t e^{-\beta s} c\, \mathbb{1}_{\{X_s^* = c\}}\, dL_s^*\right]$$

$$= \underline{\mathbb{E}_x\left[\int_0^t e^{-\beta s}(-h(X_s^*))\, ds\right] - \mathbb{E}_x\left[\int_0^t e^{-\beta s} c\, dL_s^*\right]}$$

$t \to \infty$ as in step 1 $\Rightarrow$

$$-V(x) = \mathbb{E}\Big[\int_0^\infty e^{-\beta s}(-h(X_s^*))\,ds\Big] - c\,\mathbb{E}_x\Big[\int_0^\infty e^{-\beta s}\,dL_s^*\Big] = -J(x, L^*) \qquad \square$$

<u>Remark:</u> One has <u>analogue theorems</u> for <u>stopped processes</u>!

VI.4) Ex: Dividend payments with lump sums

state process : $dX_t = \mu\, dt + \sigma\, dW_t - dL_t$, $X_0 = x$, $\mu, \sigma > 0 \dots$ const.

$L_t \dots$ accumulated payments until t

$$V(x) = \sup_L \mathbb{E}_x \left[\int_0^\tau e^{-\delta t}\, dL_t \right]$$

$\tau = \inf\{ t > 0 \mid X_t < 0 \} \dots$ time of ruin

additional admissibility condition: $\Delta L_t \leq X_{t-}$

„one should not pay more than the current endowment" in the setting above : $h \equiv 0, \quad c = 1 \Rightarrow$

HJB: $\max\left\{ \frac{\sigma^2}{2} V''(x) + \mu V'(x) - \beta V, \; 1 - V'(x) \right\} = 0$

Boundary cond.: $V(0) = 0$ $\quad$ ($\tau = 0$ at $x = 0$!)

Construction of $V(x)$ on $\mathbb{R}^+$

on $[0, m^*]$: $0 \geq 1 - V'(x)$, $\quad \underline{\frac{\sigma^2}{2} V''(x) + \mu V'(x) - \beta V = 0} \quad (1)$

$[m^*, \infty)$: $\underline{0 = 1 - V'(x) \quad (2)}$, $\quad \frac{\sigma^2}{2} V''(x) + \mu V'(x) - \beta V \leq 0$

Solve first (1), (2):

$$\frac{\sigma^2}{2} V''(x) + \mu V'(x) - \beta V(x) = 0$$

$$\Rightarrow V(x) = C_1 e^{\theta_+ x} + C_2 e^{\theta_- x}$$

with $\theta_\pm = \dfrac{-\mu \pm \sqrt{\mu^2 + 2\beta\sigma^2}}{\sigma^2}$

$V(0) = 0$

$$\Rightarrow \underline{V(x) = c\left(e^{\theta_+ x} - e^{\theta_- x}\right)}$$

$$(2) \Rightarrow \underline{V(x) = (x - m^*) + K}$$

smooth fit : $V'(m^*-) = V'(m^*+) = 1 \qquad (3)$

$$V''(m^*-) = V''(m^*+) = 0 \qquad (4)$$

$$(4) \Rightarrow \theta_+^2 e^{\theta_+ m^*} - \theta_-^2 e^{\theta_- m^*} = 0 \Rightarrow$$

$$e^{\theta_+ m^* - \theta_- m^*} = \frac{\theta_-^2}{\theta_+^2} \Rightarrow \underline{m^* = \frac{1}{\theta_+ - \theta_-} \ln\left(\frac{\theta_-^2}{\theta_+^2}\right)}$$

$$(|\theta_-| > |\theta_+| \Rightarrow m^* > 0 \text{ !})$$

$$(3) \Rightarrow \underline{C = \frac{1}{\theta_+ e^{\theta_+ m^*} - \theta_- e^{\theta_- m^*}}}$$

Finally: $V(m^*+) = V(m^*-) \Rightarrow \underline{K = C\cdot\left(e^{\theta_+ m^*} - e^{\theta_- m^*}\right)}$

still to show (HJB!) on $[0, m^*]$: $1 - V'(x) \leq 0$ (5)

on $[m^*, \infty)$: $\frac{\sigma^2}{2} V''(x) + \mu V'(x) - \beta V(x) \leq 0$ (6)

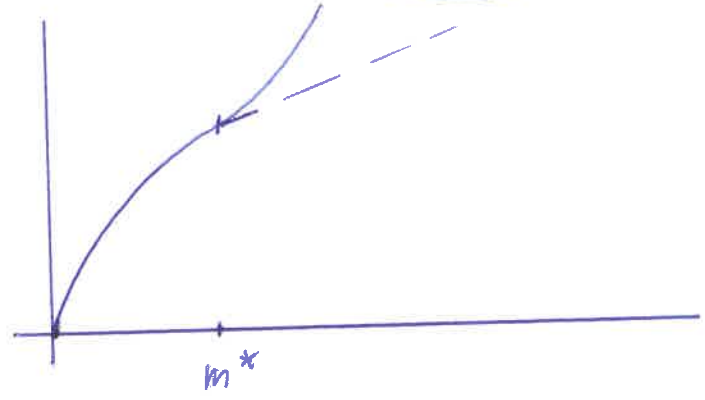
ad (5): m^* is turning point of $C\left(e^{\theta_+ x} - e^{\theta_- x}\right)$, ex: $(\quad)''' > 0$

and $V'(m^*) = 1$

$\Downarrow$

$V'(x) \geq 1$

for $x \leq m^*$ ✓



ad (6) : $\frac{\sigma^2}{2} V''(m^*) + \mu V'(m^*) - \beta V(m^*) = 0$ by construction

moreover: right of m^* : V linear, i.e.

V'', V' remain constant

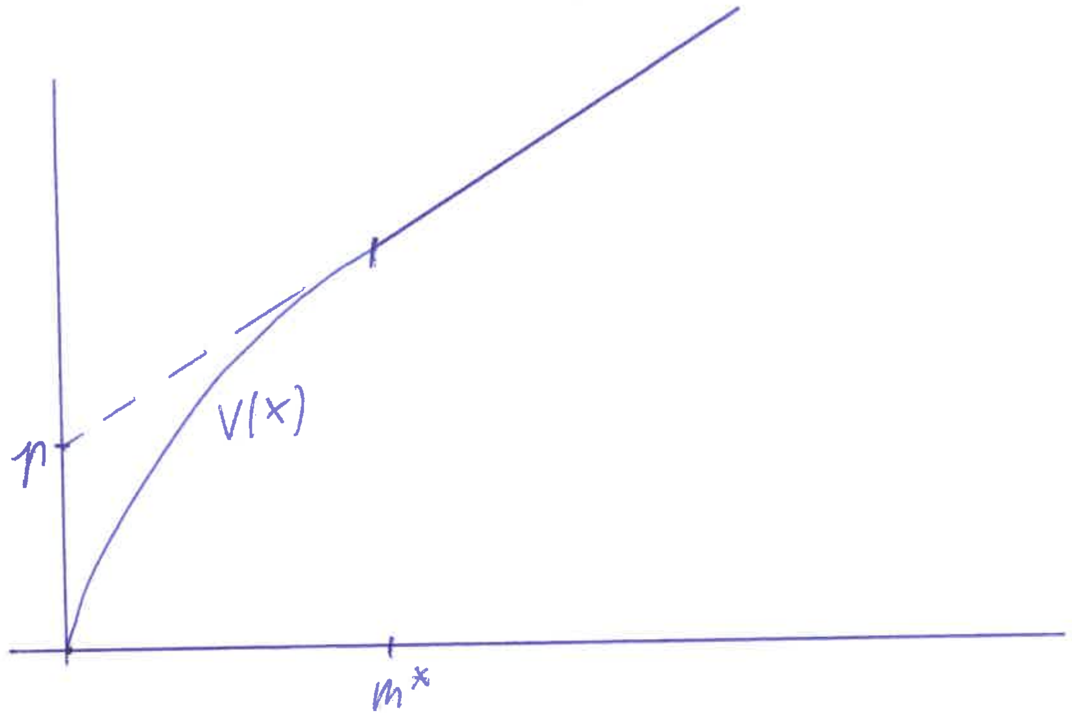
V increases $\Rightarrow \frac{\sigma^2}{2} V''(x) + \mu V'(x) - \beta V(x) \leq 0$

for $x \geq m^*$ ✓

- it remains to check: $\lim_{t\to\infty} \mathbb{E}[e^{-\beta t} V(X_t)] = 0$

$L_t \geq 0$, $\mathbb{E}[W_t] = 0 \Rightarrow \mathbb{E}[X_t] \leq \mu t$

we also have: $V(x) \leq \gamma + x$, for some positive constant γ



$$\Rightarrow \mathbb{E}\left[e^{-\beta t} V(X_t)\right] \leq \mathbb{E}\left[e^{-\beta t}(\gamma + X_t)\right] \leq \mathbb{E}\left[e^{-\beta t}(\gamma + \mu t)\right] \xrightarrow{t \to \infty} 0 \checkmark$$

$\square$

i.e.: optimal strategy : pay nothing until X_t reaches m^*,

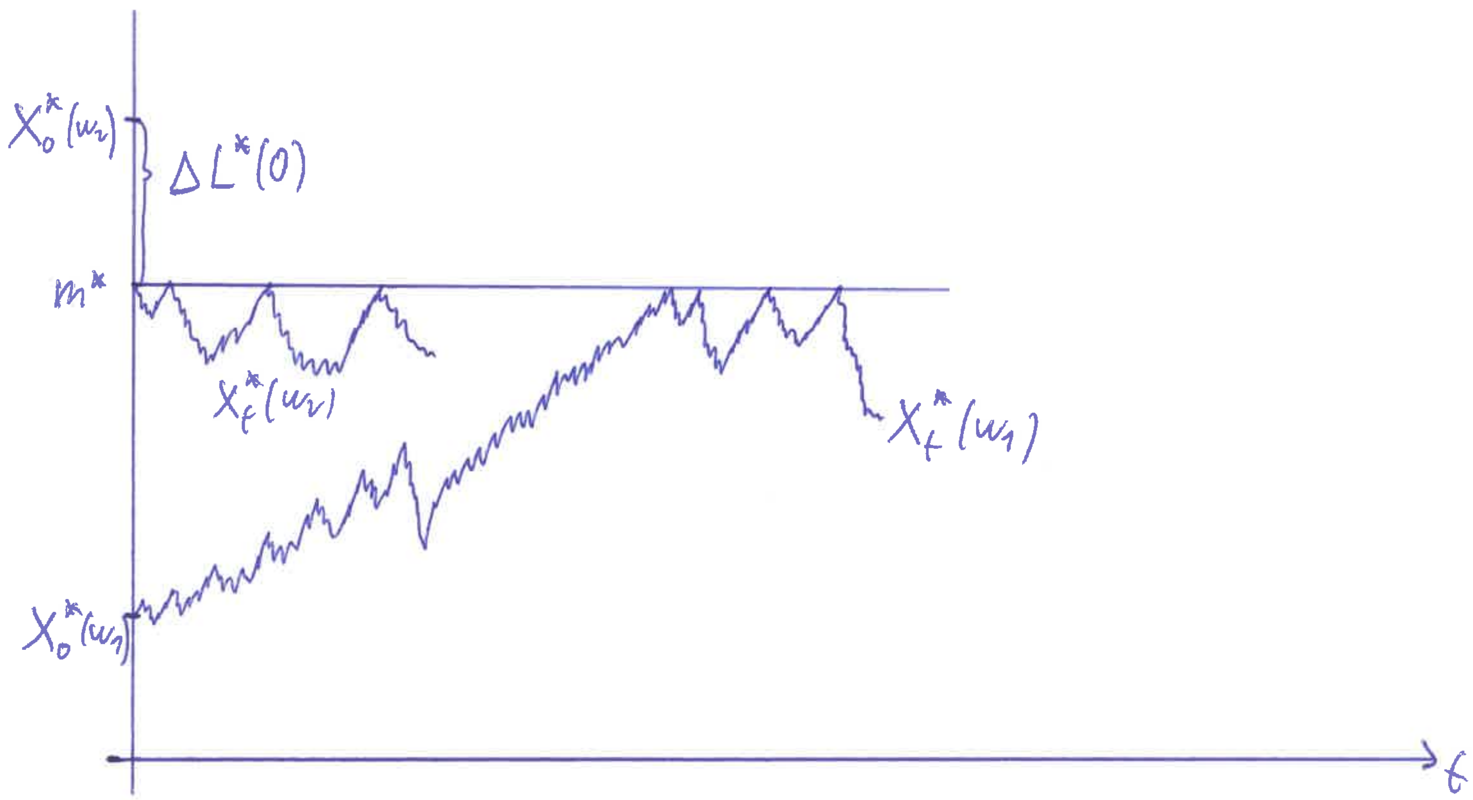
then just enough to keep X_t below m^*:

„barrier strategy"

L_t^* ... local time of the Brownian motion with drift

X_t^* .... at m^* reflected — " —

$$\begin{cases} L_t^* = \max\limits_{s \le t} \left(x + \mu s + \sigma W_t - m^* \right)^+ \\ X_t^* = x + \mu t + \sigma W_t - L_t^* \end{cases}$$



$X_0^*(w_2)$
$\Delta L^*(0)$
m^*
$X_t^*(w_2)$
$X_t^*(w_1)$
$X_0^*(w_1)$
t