Stochastic control theory

0.) motivating example

given

- insurance company (IC)
- wants to invest in the stock market
- goal: minimize the probability of ruin (another poss.: maximizing dividend paym.)

math. formulation

S_t ... stock price

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t$$

$\mu, \sigma > 0$ const.

W_t ... Brownian motion

solution $S_t = S_0 \cdot \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)$, (by Itô's formula)

" Black-Scholes model

R_t ... wealth of the IC without investment

$$R_t = x - \sum_{i=1}^{N_t} X_i + ct$$

x .... initial wealth

X_i .... individual claims

c.... premium rate

N_t.... number of claims until time t (Poisson distributed)

<u>Cramer - Lundberg model</u>

U_t ... number of stocks in our portfolio

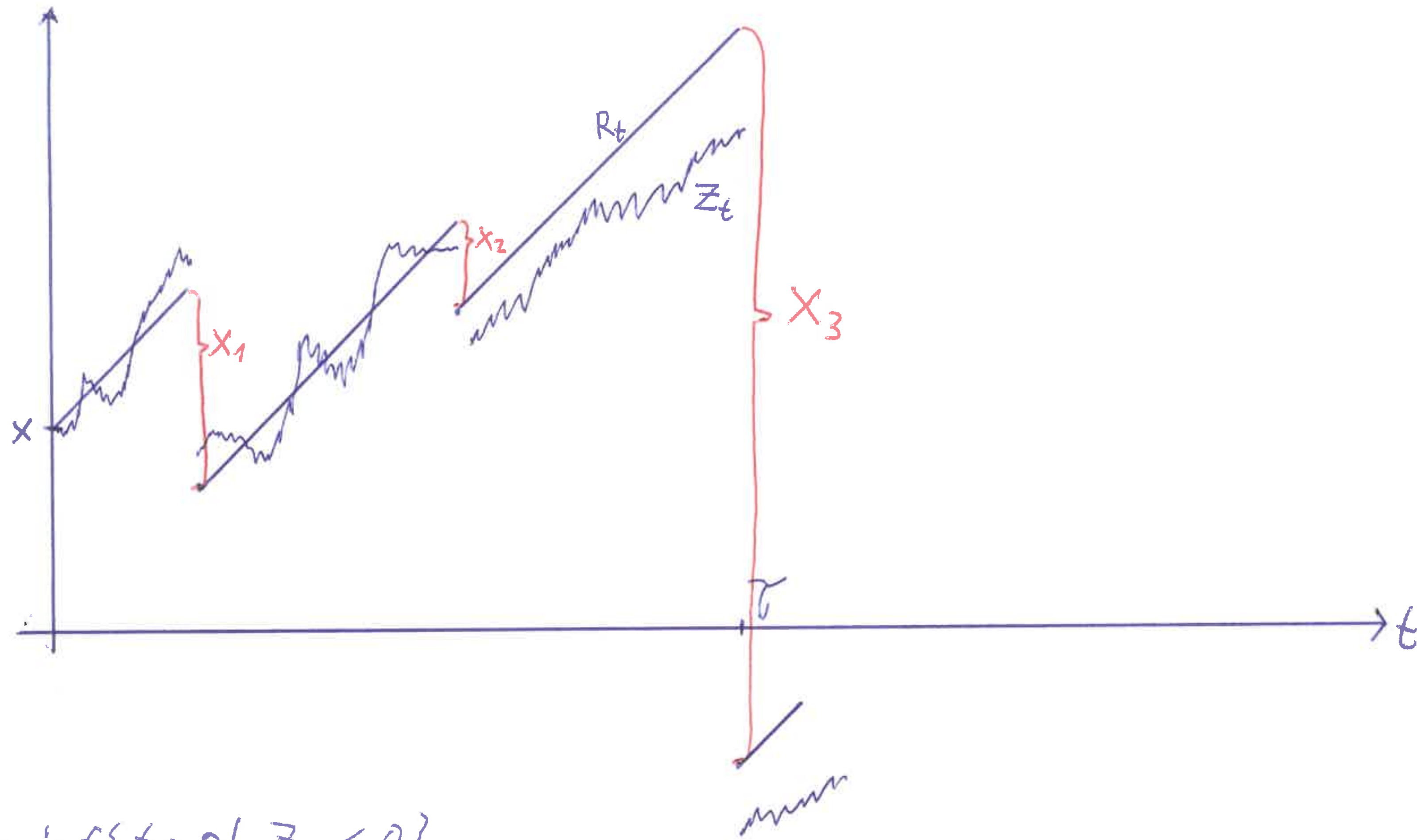
assumption: interes rate $r \equiv 0$!

$\Rightarrow$ Z_t ... wealth after investment:

$$Z_t = x - \sum_{i=1}^{N_t} X_i + ct + \underbrace{(U \bullet S)_t}$$

stochastic integral w.r.t. S_t (see later)

profit of the stock trading !



$\tilde{\tau} = \inf\{t > 0 \mid Z_t < 0\}$

goal: choose a strategy u_t s.t.

$$P(\tilde{\tau} < \infty) \longrightarrow \min$$

result: A) • <u>small claim case</u>, i.e.: $\exists\, \varepsilon_0 > 0 : \mathbb{E}[e^{\varepsilon_0 X_i}] < \infty$, ex.: Exp-distr.
Γ-distr.

asymptotic ($x \to \infty$) optimal: investment of a fixed amount of money in the stock

B) • <u>Large claim case</u> ($\mathbb{E}[e^{\varepsilon X_i}] = \infty \ \forall \varepsilon > 0$; (Pareto, lognormal, ...

more precisely: $X_i \in S$ ... subexponential claims

asymptotic optimal: investment of a fixed fraction in the stock

method in A: martingal methods

B: classical PDE methods (HJB equation)

I.) stochastic analysis and martingales (short repetition)

I.1) stochastic processes

I ... index set, e.g. $I = \mathbb{N}, \mathbb{R}$ or $[0,T]$, $T>0$ „time"

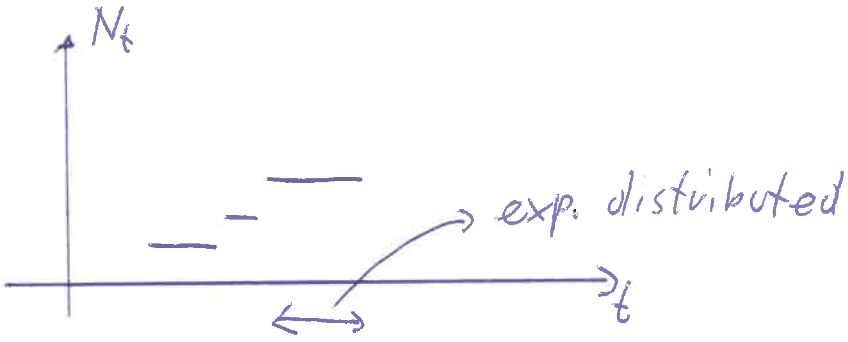
Def. a stochastic process X is a family of $\mathbb{R}$, resp. $\mathbb{R}^d$ valued random var. (RV)

$\{X_t, t \in I\}$ on the tripel

$(\Omega, \mathcal{F}, \mathbb{P})$

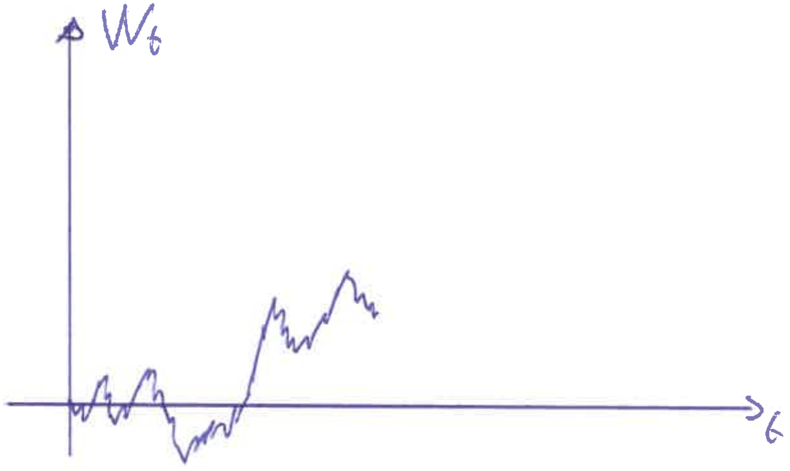
Def.: X_t is called cadlag, if almost all paths $t \longmapsto X_t(\omega)$ are continuous from the right with $\exists$ limit from the left

Ex: Poissonprocess



Def.: a process is called continous, if almost all paths are continuous

Ex: Browian motion



stationary, independent increments ~~with~~ which are normally distributed

I.2) Filtrations and stopping times

filtration

- with the time growing information is described by a filtration of the basic σ-algebra $\mathcal{F}$

- $\mathbb{F} = \{ \mathcal{F}_t , t \in I \}$, with $s \leq t :\Rightarrow \mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$

  i.e.: the later the more information

- technical assumptions: • right continuous $\mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s = \mathcal{F}_t$

  "infinitesimal look" ~~to the~~ in the future does not give additional information!

- complete: $\mathcal{F}_0$ contains all $\mathbb{P}$-zero sets of $\mathcal{F}$

  „if an event is „impossible", then this is already known at $t=0$.

Def.: a process is called adapted, if X_t is $\mathcal{F}_t$ meas. $\forall t$

i.e: observing the process does not give more information than it is contained in the filtration

Def: $\mathbb{F}^X$ .... natural filtration: smallest right continuous filtration $\mathbb{F}$, s.t. X_t is $\mathbb{F}$-adapted

(also called, the filtration generated by X_t)

Remark: bigger filtration: „insider filtration"

stopping times:

Def: a RV $\tau : \Omega \to I \cup \{+\infty\}$ is called stopping time

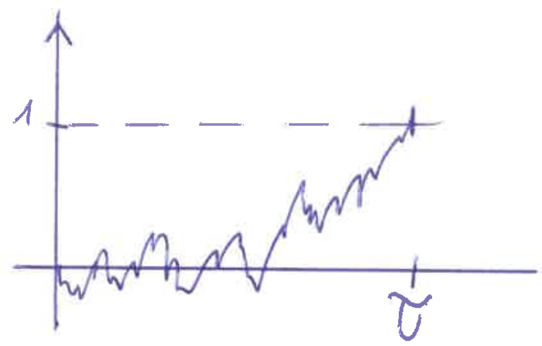
if $\{\tau \le t\} \in \mathcal{F}_t$

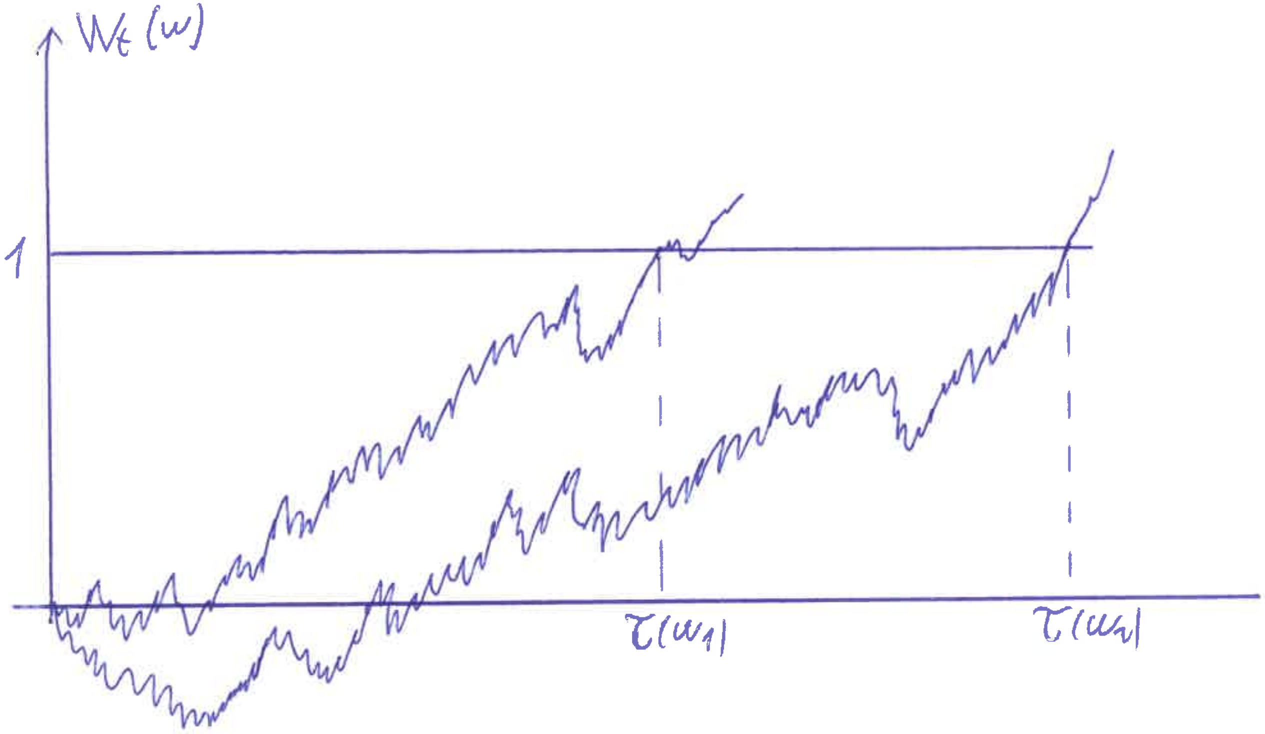
interpretation: at time t it is known, whether the process is stopped or not!

- trivial: all deterministic times „s" are stopping times
  $\{s \le t\} \in \{\emptyset, \Omega\}$
- X_t ... adapted $\Rightarrow$ $X_{t \wedge \tau}$ adapted, if τ stopping time
- typical stopping times: „hitting times"

  e.g.: W_t Brownian motion

  $\tau := \inf\{t > 0 \mid W_t = 1\}$





law of iterated log. $\Rightarrow$

$\tau < \infty \quad \mathbb{P}$-f.s

<u>Def.:</u> $\mathcal{F}_\tau = \{ A \in \mathcal{F} \mid A \cap \{\tau \le t\} \in \mathcal{F}_t \,, \ \forall t \in I \}$

contains all observable events until time τ

I.3) Martingales

Consider $\mathbb{R}$-valued stoch. process M_t

Def.: M_t is a martingale if :

- M_t is adapted
- $\mathbb{E}[|M_t|] < \infty$
- $\mathbb{E}[M_t | \mathcal{F}_s] = M_s$, $s \leq t$ (1)

expectation const. in time!

interpretation : gain history of a fair game

if in (1): $\leq$ supermartingale (in average $\searrow$ ☹)

$\geq$ submartingale (in average $\nearrow$ ☺)

important theorems:

Theorem 1: (convergence theorem), $I = \mathbb{R}_+$, X_t an L^1-bounded supermartingale

i.e. $\sup_t \mathbb{E}[|X_t|] < \infty \Rightarrow \lim_{t \to \infty} X_t = X_\infty$ $\exists$ a.s. and is a.s. finite

be careful! • in general $\lim$ $\nexists$ in L^1

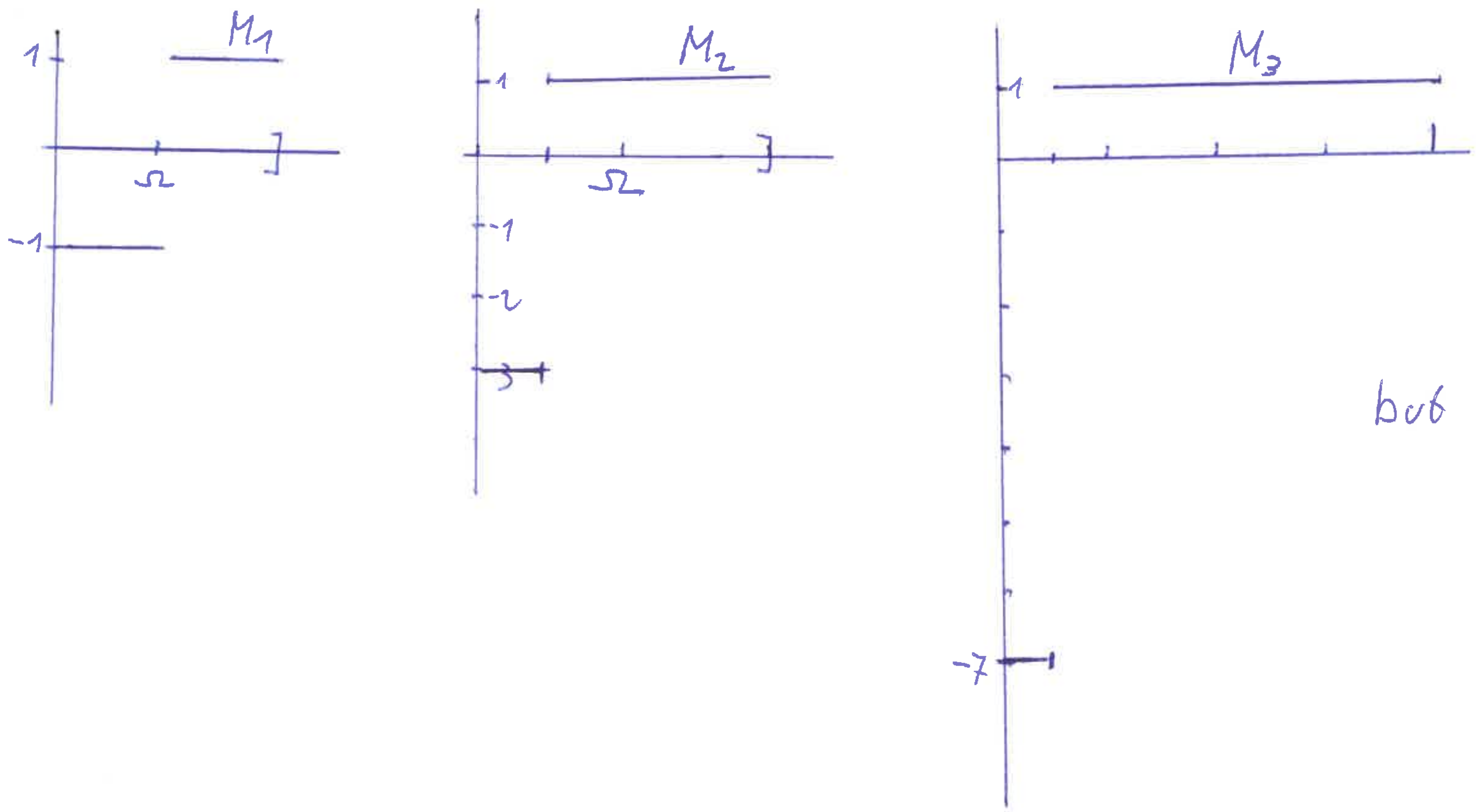
• for this one needs stronger condition: <u>uniform integrability</u> of X_t

i.e. $\lim_{\alpha \to \infty} \sup_t \int_{|X_t| \geq \alpha} |X_t| \, d\mathbb{P} = 0$

(sufficient: L^p boundedness for $p > 1$, esp. $p = 2$!)

Ex.: (non uniformly integrable martingale)

- coin tossing
- doubling strategy (stop if you win, otherwise double your bet)
- M_t .... gain at time t



$M_t \to 1$ a.s.

but $M_t \not\to 1$ in $L^1(\mathbb{P})$

Corollary to Theorem 1: If X_t is a nonnegative supermartingale $\Rightarrow$

$$\lim_{t\to\infty} X_t = X_\infty \text{ a.s. and } X_\infty \text{ finite a.s.}$$

Proof.

$$\mathbb{E}[|X_t|] \underset{X_t \geq 0}{=} \mathbb{E}[X_t] \underset{X_t \text{ superm.}}{\leq} \mathbb{E}[X_0]$$

$$\Rightarrow \sup_t \mathbb{E}[|X_t|] \leq \mathbb{E}[X_0] \Rightarrow X_t \text{ } L^1 \text{ bounded supermartingale}$$

$$\underset{\text{Th. 1}}{\Rightarrow} \blacksquare$$

Theorem 2 („stopping theorem"), simplest version:

Let M_t be a martingale, $\tau_1 \leq \tau_2$ bounded stopping times

(otherwise M_t has to be unif. integr.)

$$\Rightarrow \mathbb{E}[M_{\tau_2} \mid \mathcal{F}_{\tau_1}] = M_{\tau_1}$$

Theorem 3 $M_{t \wedge \tau}$ is martingale, if M_t is martingale and τ stopping time:

I.4) Brownian motion

Def.: $(\Omega, \mathbb{F}, \mathbb{P}, W_t)$ is called <u>(standard) Brownian motion</u>

if:

(i) W_t is $\mathbb{F}$ adopted

(ii) $W_0 = 0$ $\mathbb{P}$ a.s.

(iii) $W_t - W_s$ is indep. of $\mathcal{F}_s$, $s < t$ „independence of increments"

(iv) $W_t - W_s \sim \mathcal{N}(0, t-s)$

(v) W_t has a.s. continuous paths

Remark: (v) can be proved under certain assumptions on $\mathbb{E}[|X_t - X_s|^\alpha]$ „Kolmogorov-Čentsov" Theorem

Remarks :

- ≈ 1920 first rigorous proof of existence by N. Wiener („Wiener process")
- earlier articles: Bachelier (1900), Einstein (1905)
- often one takes the natural filtration
- W_t is martingule (exercise)
- W_t has infinite total variation on compact intervals !

  but finite quadratic variation, i.e.:

Let $\mathcal{P} = \{ 0 = t_0 < t_1 < \dots < t_n = t \}$

$$\Rightarrow \lim_{|\mathcal{P}| \to 0} \sum_{i=1}^{n} (W_{t_i} - W_{t_{i-1}})^2 = t \quad \text{in } \mathbb{P}$$

$\uparrow$

$\sup_i |t_{i+1} - t_i|$

Exponential martingale

Let $X_t := \mu t + \sigma W_t$ B. M. with drift μ, volatility σ

$M_t := \exp\left(-\frac{2\mu}{\sigma^2} X_t\right)$ ... <u>exponential martingale</u>

Claim : M_t is martingale :

Proof. 1.) M_t is adapted (since X_t is adapted)

2.) $\mathbb{E}[|M_t|] < \infty$ simple calculation using the density of W_t !

3.) $\mathbb{E}[M_t | \mathcal{F}_s] =$

$$\mathbb{E}\left[e^{-\frac{2\mu}{\sigma^2}(\mu t + \sigma W_t)} \middle| \mathcal{F}_s\right] = e^{-\frac{2\mu^2}{\sigma^2} t} \, \mathbb{E}\left[e^{-\frac{2\mu}{\sigma}(W_t - W_s)} \middle| \mathcal{F}_s\right] \cdot e^{-\frac{2\mu}{\sigma} W_s} =$$

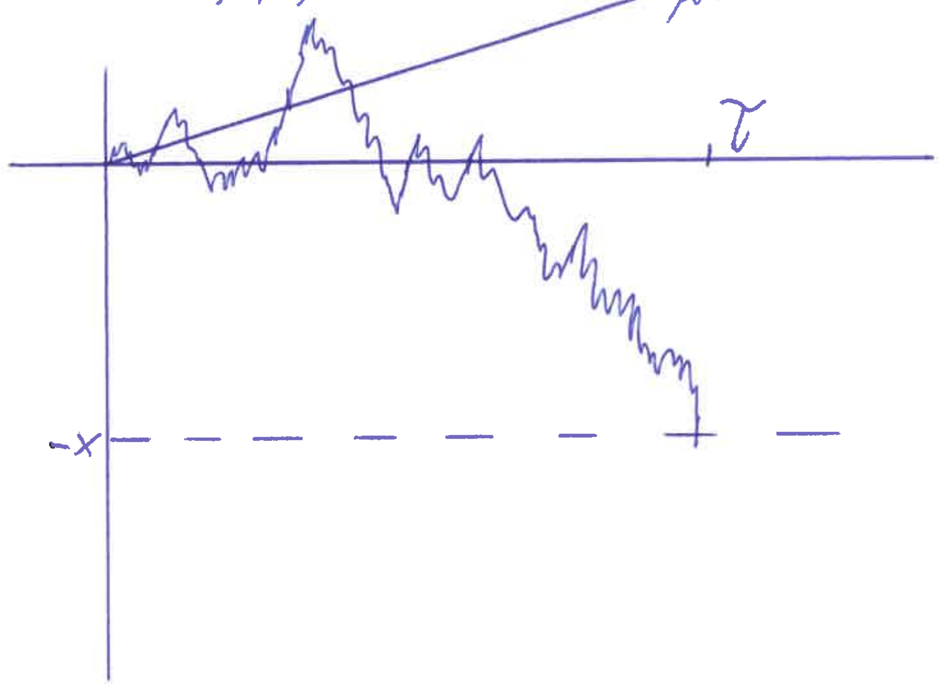
$$\underset{\text{indep. of increments}}{=} e^{-\frac{2\mu^2}{\sigma^2}t}\, \mathbb{E}\left[e^{-\frac{2\mu}{\sigma}(W_t - W_s)} \right] \cdot e^{-\frac{2\mu}{\sigma} W_s}$$

$$\underset{\text{station. of increments}}{=} e^{-\frac{2\mu^2}{\sigma^2}t}\, \mathbb{E}\left[e^{-\frac{2\mu}{\sigma}(\overbrace{W_{t-s}}^{\triangleq\, \mathcal{N}(0,1)\cdot\sqrt{t-s}})} \right] e^{-\frac{2\mu}{\sigma} W_s}$$

$$= e^{-\frac{2\mu^2}{\sigma^2}t}\, e^{\frac{2\mu^2}{\sigma^2}(t-s)} \cdot e^{-\frac{2\mu}{\sigma} W_s} = e^{-\frac{2\mu^2}{\sigma^2}s - \frac{2\mu}{\sigma} W_s}$$

$$= e^{-\frac{2\mu}{\sigma^2}(\mu s + \sigma W_s)} = M_s \quad \blacksquare$$

Application (Calculation of the ruin probability), $\mu > 0$!



Let: $\tau := \inf\{t \mid X_t = -x\}$, $x > 0$

$\psi(x) := \mathbb{P}(\tau < \infty)$

Claim: $\psi(x) = e^{-\frac{2\mu}{\sigma^2}x}$

Proof. $M_{t\wedge\tau}$ is a bounded martingale (below by 0, above by $e^{\frac{2\mu}{\sigma^2}x}$)

$\Rightarrow \lim_{t\to\infty} M_{t\wedge\tau} = M_\tau$ a.s. $\Rightarrow \lim_{t\to\infty} X_{t\wedge\tau} = X_\tau \in \{-x, \infty\}$

dominated convergence $\Rightarrow$

$$1 = \mathbb{E}\left[e^{-\frac{2\mu}{\sigma^2}X_0}\right] = \mathbb{E}\left[e^{-\frac{2\mu}{\sigma^2}X_{t\wedge\tau}}\right] =$$

$$\mathbb{E}\left[e^{-\frac{2\mu}{\sigma^2}X_\tau}\mathbb{1}_{\{\tau\le t\}}\right] + \underbrace{\mathbb{E}\left[e^{-\frac{2\mu}{\sigma^2}X_t}\mathbb{1}_{\{t>\tau\}}\right]}_{\xrightarrow{t\to\infty} 0}$$

$t\to\infty \Rightarrow$

$$1 = e^{\frac{2\mu}{\sigma^2}x}\,\mathbb{E}\left[\mathbb{1}_{\{\tau<\infty\}}\right] = e^{\frac{2\mu}{\sigma^2}x}\,\mathbb{P}(\tau<\infty) \Rightarrow \mathbb{P}(\tau<\infty) = e^{-\frac{2\mu}{\sigma^2}x} \qquad \blacksquare$$

Remark: exponential bound as in the Lundberg inequality for small claims in the Cramer Lundberg model !

I.5) Stochastic Integration w.r.t. Brownian motion

Problem: W_t has unbounded total variation, hence pathwise def. as Lebesgue-Stieltjes Integral not possible

$\Rightarrow$ Ito Integral (≈ 1940)

Step 1: Define $\int H_s \, dW_s$ for simple integrands, $H \in S_0$

$$S_0 = \{ H \mid H_t(\omega) = \phi_1 \mathbb{1}_{\{0\}}(t) + \sum_{i=1}^{n} \phi_i \mathbb{1}_{(t_{i-1}, t_i]}(t) \}, \text{ where}$$

$$0 = t_0 < t_1 < t_2 < \dots < t_n = T$$

$$\phi_i \in L^\infty(\Omega, \mathcal{F}_{t_{i-1}}, \mathbb{P})$$

Def.: $$\int_0^t H_s \, dW_s = \sum_{i=1}^{n} \phi_i \left(W_{t_i \wedge t} - W_{t_{i-1} \wedge t} \right) =: (H \bullet W)_t \quad \text{for } H \in S_0$$

Interpr.: $$\left. \begin{array}{l} W_t \dots \text{stockprice} \\ \phi_i \dots \text{trading strategy} \end{array} \right\} (H \bullet W)_t \dots \text{gain}$$

e.g. $t = t_k$: $\sum_{i=1}^{k} \phi_i (W_{t_i} - W_{t_{i-1}})$

One has:
- $(H \bullet W)_t$ is an L^2-bounded martingale
- $\mathbb{E}\left[\left(\int_0^t H_s \, dW_s\right)^2\right] = \mathbb{E}\left[\int_0^t H_s^2 \, ds\right]$ … „Ito isometry"

  i.e.: integral operator is an isometry: $L^2(ds \times \mathbb{P}) \longrightarrow L^2(\mathbb{P})$

idea of proof.:
- simple calculation using the measurability properties of ϕ_i and the martingale property of W_t (esp. the orthogonality of increments of the B.M. for disjoint intervals)

Step 2: consider L^2-adapted integrands, i.e.

$$\mathcal{H} := \{ H \mid H_t \text{ is adopted}, \ \mathbb{E}[\int_0^T H_t^2 \, dt] < \infty \}$$

use 2 facts: a) $\forall H \in \mathcal{H} \ \exists \ H^{(n)} \in S_0$, s.t.

$$\mathbb{E}[\int_0^T (H_t - H_t^{(n)})^2 \, dt] \xrightarrow[n \to \infty]{} 0$$

b) if $X^{(n)}$ are continuous martingales and $(X_T^{(n)})_{n=1}^{\infty}$ is a Cauchy sequence in L^2

$\Rightarrow \exists$ a unique continuous martingale X_t, s.t.

$$X_t^{(n)} \to X_t \quad \text{in } L^2 \quad \forall t \in [0,T]$$

a) + Ito isometry $\Rightarrow (H^{(n)} \bullet W)_T$ is Cauchy sequence in $L^2(\mathbb{P})$

$\Downarrow$

<u>Def:</u> for $H \in \mathcal{H}$: using b) def. $(H \bullet W)_t = \lim_{n\to\infty} (H_n \bullet W)_t$

<u>properties of the Ito integral:</u>

- $\mathbb{E}\left[\left(\int_0^T H_s \, dW_s\right)^2\right] = \mathbb{E}\left[\int_0^T H_s^2 \, ds\right]$ ... <u>Ito isometry</u> for $H \in \mathcal{H}$
- $\int_0^T (\lambda H_t + \mu K_t) \, dW_t = \lambda \int_0^T H_t \, dW_t + \mu \int_0^T K_t \, dW_t$; $\lambda, \mu \in \mathbb{R}$ <u>linearity</u>
- $\int_0^{t \wedge \tau} H_s \, dW_s = \int_0^t H_s \mathbb{1}_{[0,\tau]}(s) \, dW_s$

<u>idea of proof:</u> - show properties for $H \in S_0$

- then $\lim_{n\to\infty}$

step 3: localisation

$$\tilde{\mathcal{H}} := \{ H \mid H_t \text{ adapted}, \ \mathbb{P}\left(\int_0^T H_s^2 \, ds < \infty\right) = 1 \}$$

Define: $\tau_n := \inf\{ 0 \le t \le T \mid \int_0^t H_s^2 \, ds = n \}$, $\inf \emptyset = T$

set $H_t^{(n)} := H_t \mathbb{1}_{[0,\tau_n]}(t)$

Def: $\int_0^t H_s \, dW_s := \int_0^t H_s^{(n)} \, dW_s$ on $\{ t \le \tau_n \}$ $\qquad \tau_n \xrightarrow{n\to\infty} T$ a.s. !

Remark: one can show that this definition is consistent,

i.e.: it is well defined on $\{ t \le \tau_n \} \cap \{ t \le \tau_k \}$

Remark: if $H \in \tilde{\mathcal{H}}$: $(H \bullet W)_t$ is not necessarily a martingale!
(as it is the case for $H \in \mathcal{H}$), but only a local martingale

Def: M_t is a <u>local martingale</u>, if there $\exists$ a sequence of stopping times τ_n (localizing sequence), s.t.

$$M_{t \wedge \tau_n} \text{ is a martingale } \forall n \quad \text{and} \quad \tau_n \to \infty \text{ a.s.}$$

Remark: extension of the integral to L^2-bounded continuous martingales M,

i.e. $\int_0^t H_s \, dM_s$:

- bracket process $[M]_t := \lim_{|P| \to 0} \sum (M_{t_i} - M_{t_{i-1}})^2$

- $[M]_t$ is not so simple as for W_t : $[W]_t = t$!

- this process is used in the construction of the Ito integral

- one has Ito isometry : $dt \longmapsto d[M]_t$

$\Rightarrow$ restriction of the integrands to

progressively measurable processes H_t

i.e: restriction of $H_s(\omega)$ to $[0,t] \times \Omega$

is $\mathcal{B}_t \otimes \mathcal{F}_t$ meas.

$\Downarrow$

Borel sets on $[0,t]$

Remark:
- H_t progr. measurable $\Rightarrow$ H_t adapted (Fubini!)
- H_t adapted + right (left) continuous $\Rightarrow$ H progr. measurable

I.6) Ito formula and Itoprocesses

Def.: a stochastic process X_t is called Ito process $\Leftrightarrow$

$$X_t = X_0 + \int_0^t K_s \, ds + \int_0^t H_s \, dW_s \qquad \text{for } t \in [0,T] \qquad P\text{-a.s.}$$

where • X_0 ... $\mathcal{F}_0$-mb.

• K_t adopted with $P\left(\int_0^T |K_s| \, ds < \infty\right) = 1$

• H_t adopted with $\mathbb{P}\left(\int_0^T H_s^2 \, ds < \infty\right) = 1$

Remark: one can show that this representation is unique in the following sense if $X_0 + \int_0^t K_s\,ds + \int_0^t H_s\,dW_s = X_0' + \int_0^t K_s'\,ds + \int_0^t H_s'\,dW_s$ a.s. $\Rightarrow$

(1) $X_0 = X_0'$ a.s.

(2) $H_s(\omega) = H_s'(\omega)$ $ds \otimes \mathbb{P}$ a.s.

(3) $K_s(\omega) = K_s'(\omega)$ $ds \otimes \mathbb{P}$ a.s

<u>idea of proof</u>: (1): $t=0$

(2)+(3): $\int_0^t (K_s - K_s')\,ds = \int_0^t (H_s' - H_s)\,dW_s$

use now: a continuous local martingale with bounded variation on compacts is constant, i.e $\equiv 0$ (*)

$\Rightarrow$ (2)+(3) ✓

<u>Remark</u> (*) is wrong for discontinous processes,

e.g.: $N_t - t$ with N_t ... Poissonprocess

Theorem (Ito formula): Let X_t be an Ito process, let $f \in C^2(\mathbb{R})$

$\Rightarrow f(X_t)$ is also an Ito process and we have:

!

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) K_s \, ds + \int_0^t f'(X_s) H_s \, dW_s + \boxed{\frac{1}{2}\int_0^t f''(X_s) H_s^2 \, ds}$$

Remarks: Using:

$$dX_s := K_s \, ds + H_s \, dW_s$$

$\langle X \rangle_s = \int_0^t H_s^2 \, ds$ .... quadratic variation ($\langle W \rangle_t = t$)

$$\Rightarrow d\langle X \rangle_s = H_s^2 \, ds$$

$$\Rightarrow \boxed{f(X_t) = f(X_0) + \int_0^t f'(X_s) \, dX_s + \frac{1}{2}\int_0^t f''(X_s) \, d\langle X \rangle_s}$$

- M_t .... quadratic integrable continuous martingale $\Rightarrow$

  $M_t^2 - \langle M \rangle_t$ is a martingale

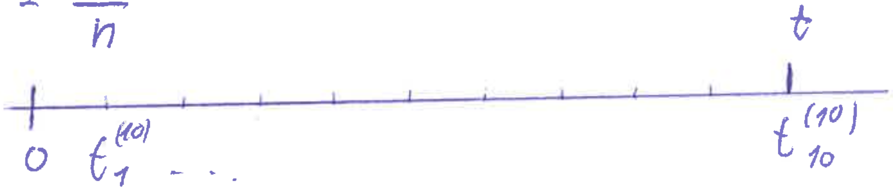
Ex: $W_t^2 - t$ is a martingale (Ex.!)

idea of proof for the Itô formula

Taylor: $f(y) - f(x) = f'(x) \cdot (y-x) + \frac{1}{2} f''(x)(y-x)^2 + R(x,y)$

$$\text{with } \lim_{y \to x} \frac{R(x,y)}{(y-x)^2} = 0$$

Let now $t_i^{(n)} := \frac{it}{n}$



$$f(X_t) - f(X_0) = \sum_{i=1}^{n} f(X_{t_i^{(n)}}) - f(X_{t_{i-1}^{(n)}}) =$$

$$\underbrace{\sum f'(X_{t_{i-1}^{(n)}})(X_{t_i^{(n)}} - X_{t_{i-1}^{(n)}})}_{\xrightarrow{n\to\infty} \int_0^t f'(X_s)\,dX_s} + \underbrace{\frac{1}{2}\sum f''(X_{t_{i-1}^{(n)}})(X_{t_i^{(n)}} - X_{t_{i-1}^{(n)}})^2}_{\xrightarrow{n\to\infty} \frac{1}{2}\int_0^t f''(X_s)\,d\langle X\rangle_s} + \underbrace{R(X_{t_i^{(n)}}, X_{t_{i-1}^{(n)}})}_{\xrightarrow{n\to\infty} 0}$$

$\square$

see construction of the stochastic integral generalization: $f: \mathbb{R}^2 \to \mathbb{R} \in C^{1,2}$, i.e. $\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}$ continuous

$\Rightarrow$

$$f(t, X_t) = f(0, X_0) + \int_0^t \underbrace{f_{,0}}_{\frac{\partial f}{\partial t}}(s, X_s)\, ds + \int_0^t \underbrace{f_{,1}}_{\frac{\partial f}{\partial x}}(s, X_s)\, dX_s + \frac{1}{2} \int_0^t \underbrace{f_{,11}}_{\frac{\partial^2 f}{\partial x^2}}(s, X_s)\, d\langle X \rangle_s$$

Remark: If $X_t \in \mathcal{O} \subset \mathbb{R}$, it is sufficient to have $f_{,0}, f_{,1}, f_{,11}$ continuous in $\mathcal{O}$

e.g.: geometric Brownian motion $\mathcal{O} = \mathbb{R}^+$

multidimensional case

Def: m-dim Brownian motion

$$W_t = \begin{pmatrix} W_t^1 \\ \vdots \\ W_t^m \end{pmatrix},$$ where W^i independent one-dimensional Brownian mot. adapted to a filtration $\mathbb{F}$

stoch. integral component by component!

Let A_t : $n \times m$ matrix consisting of adapted processes s.t. the following one-dimensional integrals make sense!

$$\Rightarrow \int_0^t A_s \, dW_s := \begin{pmatrix} \sum_{j=1}^m \int_0^t A_s^{1j} \, dW_s^j \\ \vdots \\ \sum_{j=1}^m \int_0^t A_s^{nj} \, dW_s^j \end{pmatrix}$$

Def: n-dim stoch. process is called <u>Ito process</u>, if we have:

$$X_t = X_0 + \int_0^t b_s \, ds + \int_0^t \sigma_s \, dW_s \, , \quad \text{where}$$

X_0 : $\tilde{\mathcal{F}}_0$ - mb.

b_t, σ_t $\mathbb{R}^n, \mathbb{R}^{n \times m}$ valued adapted processes, s.t. the one-dim integrals are well defined

<u>Theorem</u> (multidimensional Ito formula): Let X_t be an Itoprocess as above

$$f : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$$

$f_{,0}, f_{,i}, f_{,ij}$ $\quad i,j = 1, \dots n$ continuous

$$\Rightarrow f(X_t) = f(X_0) + \int_0^t f_{,0}(s, X_s)\, ds + \sum_{i=1}^n \int_0^t f_{,i}(s, X_s)\, dX_s^i +$$

$$\frac{1}{2} \sum_{i,j=1}^n \int_0^t f_{,ij}(s, X_s)\, d\langle X^i, X^j \rangle_s$$

where $\langle X^i, X^j \rangle_t := \int_0^t (\sigma_s \sigma_s^*)_{ij}\, ds$ .... <u>quadratic covariation</u>

σ^* ... transposed matrix

integration by parts :

Let X_t, Y_t be one-dimensional Ito processes (w.r.t. the same Brownian m.)

$$\Rightarrow X_t Y_t = X_0 \cdot Y_0 + \int_0^t X_s \, dY_s + \int_0^t Y_s \, dX_s + \langle X, Y \rangle_t$$

with $\langle X, Y \rangle_t = \int_0^t \sigma_s^X \sigma_s^Y \, ds$, $\sigma_s^X \ldots$ volatility of X

$\sigma_s^Y \ldots$ — " — Y

Proof : Ex: (set $f(x,y) = x \cdot y$ in the Ito formula above!)

I.7) stochastic differential equations (SDE)

an SDE is given by

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t \quad \text{or}$$

$$X_t = \xi + \int_0^t b(s, X_s)\,ds + \int_0^t \sigma(s, X_s)\,dW_s \qquad (*)$$

where: ξ ... $\mathcal{F}_0$ mb.

$b: [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$

$\sigma: [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ Borel mb.

W .... m-dim Brownian motion

i.e.: $(*)$ consists of n coupled equations

$$dX_t^i = b_i(t, X_t)\,dt + \sum_{j=1}^{n} \sigma_{ij}(t, X_t)\,dW_t^j$$

a continuous process X_t is called a <u>solution</u> of $(*)$, if

- $(*)$ is ~~[illegible]~~ fulfilled $\mathbb{P}$-a.s.
- X_t is $\mathbb{F}$-adapted
- $\int_0^t |b_i(s, X_s)|\,ds < \infty$, $\int_0^t |\sigma_{ij}(s, X_s)|^2\,ds < \infty$ $\forall i, j$ $\mathbb{P}$-a.s.

<u>Remark:</u>

- the notion above is called <u>strong solution</u>, more precisely for <u>given</u> $(\Omega, \mathbb{F}^W, \mathbb{P}, W_t)$ (data of the problem!) one can find an adapted solution process X_t!

- <u>weak solution</u>: W_t does <u>not</u> belong to the data, but is part of the solution!

  esp.: X_t is <u>not</u> necessarily $\mathbb{F}^W$-adapted

Ex: Tanaka equation : $dX_t = \mathrm{sgn}(X_t)\, dW_t$ $\quad (+)$

start with a Brownian m. $(X_t, \mathbb{F}^X)$

set: $dW_t = \mathrm{sgn}(X_t)\, dX_t$

- Levy Th. $\Rightarrow$ W_t is a ~~BM~~ Brownian mot. (since it is a continuous, local martingale with $\langle W \rangle_t = t$!)
- $(+)$ is fulfilled : just plug in and use the associativity of the stoch. integr.
- BUT: X_t is <u>not</u> $\mathbb{F}^W$ adapted !

  because $\sigma(W_s, s \le t) \subset \sigma(|X_s|, s \le t)$

since we have $|X_t| = \underbrace{\int_0^t \mathrm{sgn}(X_s)\, dX_s}_{W_t} + L_t^X$ .... Tanaka formula !

L_t^X: local time of X_t (see later) is $\sigma(|X_s|, s \le t)$ mb.

existence and uniqueness of SDE's

Theorem: (uniqueness): Let b, σ local Lipschitz continuous in x, i.e $\exists K_n$

s.t. $\|b(t,x)-b(t,y)\| + \|\sigma(t,x)-\sigma(t,y)\| \le K_n \|x-y\|$

unif. in t

where $\|x\|, \|y\| \le n$

$\Rightarrow$ two solutions $X_t, \tilde{X}_t$ fulfill

$$P(X_t = \tilde{X}_t, \ \forall t \in [0,T]) = 1$$

Rem: • $X_t, \tilde{X}_t$ are called <u>indistinguishable</u> (pathwise the same)

• compare with: X_t is a <u>modification</u> of $\tilde{X}_t$, if

$$P(X_t = \tilde{X}_t) = 1 \quad \forall t \in [0,T]$$

- if $X_t, \tilde{X}_t$ are regular enough (e.g. right-continuous): X_t is a modif. of $\tilde{X}_t$

$$\Downarrow$$

$X_t, \tilde{X}_t$ indistinguishable

- always: $X_t, \tilde{X}_t$ are indistinguishable $\Rightarrow$ $\tilde{X}_t$ is a modification of $\tilde{X}_t$

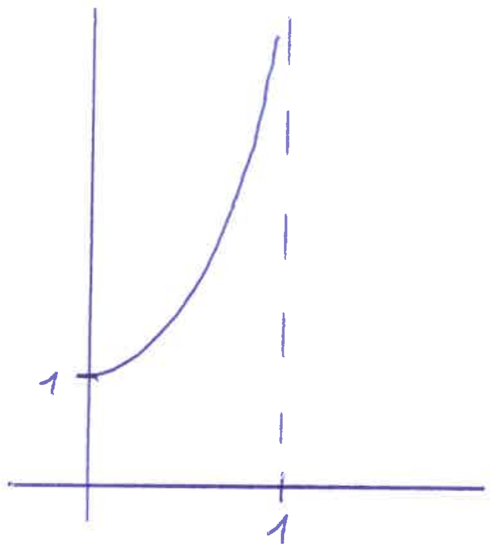
Rem: local Lipschitz continuity is not sufficient to prevent explosion of the process

deterministic example: $\quad dX_t = X_t^2\, dt \quad , \quad \|x^2 - y^2\| \leq \|x-y\| \underbrace{\|x+y\|}_{\leq 2n} \leq \underbrace{2n}_{K_n} \|x-y\|$

$$\Rightarrow \quad \frac{dX_t}{X_t^2} = dt \quad \Rightarrow$$

$$-\frac{1}{X_t} = t + C \quad \Rightarrow \quad X_t = -\frac{1}{t+C}$$

$$\Rightarrow \left| \text{e.g.: } X_0 = 1 \right| \overset{C=-1}{\Rightarrow} X_t = \frac{1}{1-t}$$



$\Rightarrow$ growth condition for coefficients necessary!

Theorem: (existence and uniqueness)

assume: $\| b(t,x) - b(t,y) \| + \| \sigma(t,x) - \sigma(t,y) \| \leq K \| x-y \|$

$\| b(t,x) \| + \| \sigma(t,x) \| \leq K \, (1 + \| x \|)$ ... linear growth condition

ξ ... $\mathcal{F}_0$ mb, $\mathbb{E}[\xi^2] < \infty$

$\Rightarrow$ $\exists$ unique, continuous strong solution X_t with

$$\mathbb{E}\Big[\sup_{0 \leq t \leq T} X_t^2\Big] < \infty$$

II.) stochastic control theory – introduction and motivating example

stoch. control problem:

given: triple $(\Omega, \mathbb{F}, \mathbb{P})$, $T>0$ … final time

W_t … m-dim Brownian motion

$$W_t = \begin{pmatrix} W_t^1 \\ \vdots \\ W_t^m \end{pmatrix}$$

notation: *controlprocess*: a progressively mb process u_t

with values in $\mathcal{U} \subset \mathbb{R}^p$

(e.g. number of stocks in a portfolio!)

- <u>state process</u> : X_t : given by

$$dX_t = b(t, X_t, u_t)\,dt + \sigma(t, X_t, u_t)\,dW_t$$

where : $b : [0,T] \times \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n$

$\sigma : [0,T] \times \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^{n \times m}$ Borel mb.

e.g.: wealth of the investor

<u>optimization criterion</u>

$$J(t,x,u) = \mathbb{E}\Big[\int_t^T \varphi(s, X_s, u_s)\,ds + \Psi(T, X_T) \,\Big|\, X_t = x\Big] \longrightarrow \begin{matrix} \max \\ \min \end{matrix}$$

φ ..... running costs

Ψ ..... terminal costs

- **admissable strategies** $\mathcal{A}(t,x)$

- set of all strategies with a unique strong solution of the SDE

- target functional is well defined $(\cancel{\infty-\infty})$

- these are minimal conditions, there are some additional ones

- **Value function** $V(t,x) := \sup_{u \in \mathcal{A}(t,x)} J(t,x,u)$

- goal: find
  - $V(t,x)$
  - optimal strategy u^*, s.t. $V(t,x) = J(t,x,u^*)$

Ex: Merton problem – simplest financial market

Bond: $dB_t = B_t r\,dt$, $B_0 = 1 \Rightarrow B_0 = e^{rt}$, $r > 0 \ldots$ const

stock: $dS_t = S_t(\mu\,dt + \sigma\,dW_t)$, $S_0 = s_0 > 0 \Rightarrow$ | $\mu, \sigma \ldots$ const.! |

$$S_t = s_0 \cdot \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right) \quad \text{by Ito's formula!}$$

$$\left.\begin{array}{ll} \text{number of stocks in the portfolio:} & N_t^S \\ \text{" — bonds " — :} & N_t^B \end{array}\right\} \xRightarrow{} \text{self financing}$$

value of the portfolio: $X_t = N_t^B B_t + N_t^S \cdot S_t$,

$$dX_t = N_t^B dB_t + N_t^S dS_t$$

u_t .... proportion of the wealth in the stock: $u_t = \frac{N_t^S \cdot S_t}{X_t}$

our control process!

$$\Rightarrow N_t^S = \frac{u_t \cdot X_t}{S_t} \qquad N_t^B = \frac{(1-u_t)X_t}{B_t}$$

$$\Rightarrow dX_t = (1-u_t) X_t r\,dt + u_t X_t (\mu\,dt + \sigma\,dW_t) =$$

$$X_t \left[r + u_t(\mu - r) \right] dt + u_t \sigma X_t \, dW_t$$

Ansatz: $X_t = x_0 e^{\int_0^t g_s\,ds + \int_0^t h_s\,dW_s} = x_0 e^{Z_t}$

Ito's formula: $\Rightarrow$ $dX_t = X_t \, dZ_t + \frac{1}{2} X_t \, d\langle Z \rangle_t \overset{*}{=}$, $\quad dZ_t = g \, dt + h \, dW_t$

$$\Downarrow$$

$$d\langle Z \rangle_t = h_t^2 \, dt$$

$$\overset{*}{=} X_t (g \, dt + h \, dW_t) + \frac{1}{2} X_t^2 h^2 \, dt =$$

$$= X_t \left(g + \frac{1}{2} h^2 \right) dt + X_t \, h \, dW_t \overset{!}{=}$$

$$X_t [r + v_t(\mu - r)] \, dt + v_t \sigma X_t \, dW_t$$

$\Rightarrow$ $h_t = v_t \sigma$

$$g_t = r + v_t(\mu - r) - \frac{1}{2} h^2 = r + v_t(\mu - r) - \frac{1}{2} v_t^2 \sigma^2$$

$$\Rightarrow \boxed{X_t = x_0 \cdot \exp\left(\int_0^t \left(r + (\mu - r) v_s - \frac{1}{2} v_s^2 \sigma^2 \right) ds + \int_0^t \sigma v_s \, dW_s \right)}$$

Remark: Question: Can X_t be zero in finite time ??

set (for convenience): $\mu = r = 0$, $\sigma = 1$, $x_0 = 1$

Claim if $\exists A$ with $\mathbb{P}(A) > 0$ s.t. $\int_0^{t_0} v_s^2 \, ds = \infty$ on A

$\Rightarrow$ on A: $X_{t_0} = 0$

we have: $X_t = \exp\left(-\int_0^t \frac{1}{2} v_s^2 \, ds + \int_0^t v_s \, dW_s\right)$

$\Rightarrow$ to show: $\lim_{t \to t_0} \left(\int_0^t -\frac{1}{2} v_s^2 \, ds + \int_0^t v_s \, dW_s\right) = -\infty$ a.s. on A ! $(*)$

Consider: $M_t := \int_0^t v_s \, dW_s \Rightarrow \langle M \rangle_t = \int_0^t v_s^2 \, ds \xrightarrow{t \to t_0} \infty$ a.s. on A

Theorem Dambis, Dubins / Schwarz $\Rightarrow$

$$M_t = B_{\langle M \rangle_t} \quad \ldots \quad \text{time change!}$$

iterated logarithm for Brownian motion $\Rightarrow$

$$\overline{\lim_{t \to t_0}} \frac{B_{\langle M \rangle_t}}{\sqrt{2 \langle M \rangle_t \ln \ln \langle M \rangle_t}} = \pm 1$$

$$\Rightarrow \overline{\lim_{t \to t_0}} \frac{M_t}{\sqrt{2 \int_0^t v_s^2 \, ds \cdot \ln \ln \int_0^t v_s^2 \, ds}} = \pm 1$$

$$\Rightarrow (*)$$

Merton problem : $\varphi \equiv 0 \qquad \Psi(x) = \ln x$

„maximization of the expected logarithmic final utility"

hence (set $t=0$!)

$$J(0, x_0, u) = \mathbb{E}[\ln X_T \mid X_0 = x_0] \longrightarrow \max$$

$$\mathcal{A}(0,x) = \Big\{ u \mid \exists! \text{ strong solution of the SDE, target functional well def,} + \mathbb{E}\Big[\int_0^T (u_s \sigma)^2 \, ds\Big] < \infty \Big\}$$

$\Rightarrow \int_0^t \sigma u_r \, dW_r$ is a martingale (see construction of the Ito integral)

for $u \in \mathcal{A}$

$$\Rightarrow J_0(0, x_0, u) = \ln x_0 + \mathbb{E}\Big[\int_0^T \big(r + (\mu - r) u_t - \tfrac{1}{2}\sigma^2 u_t^2\big) dt\Big] \longrightarrow \max$$

Determine the pathwise maximizer for fixed t:

$$\frac{\partial}{\partial u}\left(r + (\mu - r)u - \frac{1}{2}\sigma^2 u^2 \right) = (\mu - r) - \sigma^2 u \overset{!}{=} 0$$

$$\frac{\partial^2}{\partial u^2}\left(\qquad \right) = -\sigma^2 < 0 \checkmark \qquad \Rightarrow u^* = \frac{\mu - r}{\sigma^2}$$

indep. of t, ω

hence the solution of the probl.

strategy is in $\mathcal{A}$ ✓

<u>calculation of $V(0, x_0)$</u> $= \ln x_0 + T\left(r + \frac{(\mu - r)^2}{\sigma^2} - \frac{1}{2}\sigma^2 \frac{(\mu - r)^2}{\sigma^4} \right)$

$$= \underline{\ln x_0 + T\left(r + \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2} \right)}$$

discussion of the optimal strategy: $\quad u^* = \dfrac{(\mu - r)}{\sigma^2}$

$\mu \nearrow \Rightarrow u^* \nearrow$ ✓

$r \nearrow \Rightarrow u^* \searrow$ ✓

$\sigma \nearrow \Rightarrow u^* \searrow$ ✓

Ex: $\left.\begin{array}{l} \mu = 0.05 \\ r = 0.02 \\ \sigma^2 = 0.2 \end{array}\right\} \Rightarrow u^* = \dfrac{0.03}{0.2} = 0.15 \Rightarrow$ 15% in stock

„Merton strategy"

<u>Remark:</u> • strategy apparently simple, but it requires permanent trading, since the prices are varying !

<u>generalization to n stocks</u> : $S_t = \begin{pmatrix} S_t^1 \\ \vdots \\ S_t^n \end{pmatrix}$

$$dS_t = \mathrm{Diag}(S_t) \cdot (\mu dt + \sigma dW_t) \qquad S_0 = s_0$$

with: $\mathrm{Diag}(S_t) = \begin{pmatrix} S_t^1 & & & 0 \\ & S_t^2 & & \\ & & \ddots & \\ 0 & & & S_t^n \end{pmatrix},$

W ... n-dim B. M.

$\mu \in \mathbb{R}^n$

$\sigma \in \mathbb{R}^{n \times n}$ ... non singular

$\Rightarrow$

$$\Rightarrow dS_t^i = S_t^i \left(\mu_i \, dt + \sum_{j=1}^{n} \sigma_{ij} \, dW_t^j \right) \qquad i = 1, \dots n$$

control process: $\quad v = (v^i)_{i=1,\dots n} \quad$ i.e.: $p = n$!

exercise

$$\Rightarrow \quad v_t^* = (\sigma \sigma^{tr})^{-1} \big(\mu - r \underbrace{\mathbb{1}}_{(1,1,\dots 1)} \big) \qquad t \in [0,T]$$

Ex: $\mu = (0.05, 0.05)$

$r = 0.01$

$\sigma\sigma^* = \begin{pmatrix} 0.2 & 0.15 \\ 0.15 & 0.2 \end{pmatrix}$ positive correlated!

$\Rightarrow u^* = (0.1144, 0.1144)$ $\Rightarrow$ 11.44% in both stocks!

III.) Dynamic programming

III.1) Generator - Dynkin formula

Consider SDE: $dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t$

W_t ... m-dim Brownian motion

b, σ ... fulfill conditions for existence and uniqueness

$\mathbb{F}^W$ ... natural filtration

$a(t,x) = \sigma(t,x)\,\sigma^{tr}(t,x)$ .... diffusion matrix

Notation: $\mathbb{E}_{t,x}[Y] := \mathbb{E}[Y \mid X_t = x]$ for some random variable Y

$\mathbb{E}_x[Y] := \mathbb{E}_{0,x}[Y]$

__Theorem__ : Assume that the given SDE is time-homogeneous, i.e. $b = b(x)$, $\sigma = \sigma(x)$;

Let $f : \mathbb{R}^n \to \mathbb{R}$ bounded, Borel mb. $\Rightarrow$

(i) $\mathbb{E}[f(X_{t+s}) \mid \mathcal{F}_t] = \mathbb{E}_{X_t}[f(X_s)]$, $\quad t, s \geq 0$

(ii) τ stopping with $\tau < \infty$ a.s.

$$\Rightarrow \mathbb{E}[f(X_{\tau+s}) \mid \mathcal{F}_\tau] = \mathbb{E}_{X_\tau}[f(X_s)], \quad s \geq 0$$

(i) <u>Markov property</u>

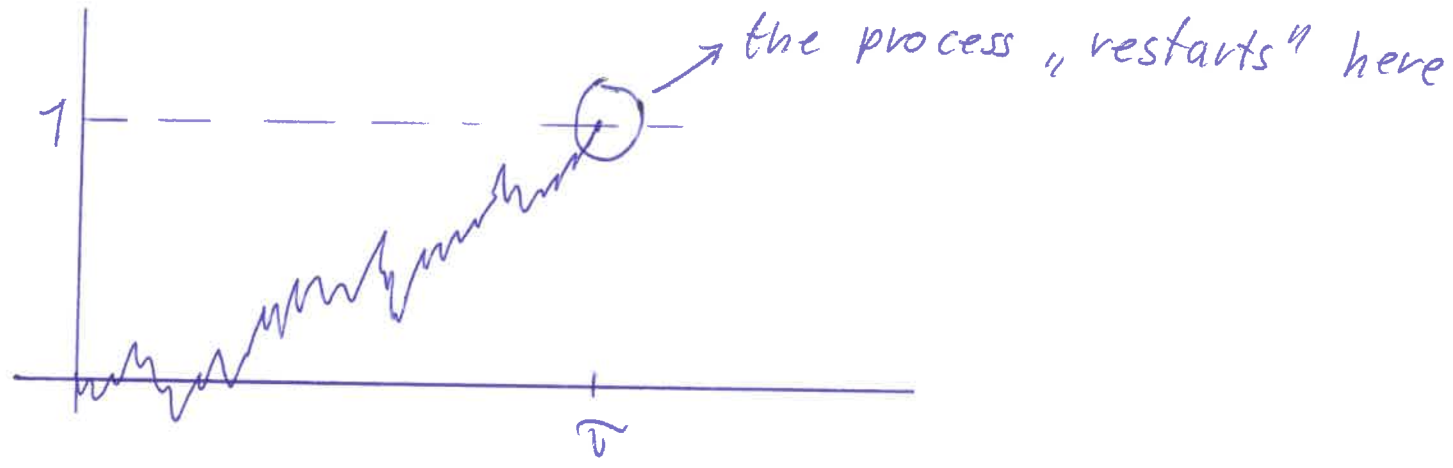
(ii) <u>strong Markov property</u>

„Meaning" (i) Evolution of the process after t depends only on the presence (i.e. the position X_t) and <u>not</u> on the whole past (described by $\mathcal{F}_t$)

(set $f = \mathbb{1}_A$ for some set A!)

(ii) (i) holds also, if we replace t by a stopping time τ!

typical application: hitting time τ



Def: <u>infinitesimal generator L of X_t</u>

$$Lf(t,x) := \lim_{s \searrow t} \frac{\mathbb{E}_{t,x}[f(s,X_s)] - f(t,x)}{s-t} \qquad t \geq 0, \quad x \in \mathbb{R}^m$$

$$f \in \mathcal{D}_L = \{ f \mid f : [0,\infty) \times \mathbb{R}^n \to \mathbb{R} \text{ for which the limit above } \exists \ \forall t,x \}$$

Let now: $\mathcal{L} := \frac{\partial}{\partial t} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$

this operator can be applied to functions, which are elements of

$$C^{1,2} = \left\{ g(t,x) \,\middle|\, \frac{\partial g}{\partial t}, \frac{\partial^2 g}{\partial x_i \partial x_j} \text{ are continuous} \right\}$$

notation: $\mathcal{L}f(t,x) = f_{,0}(t,x) + \sum_{i=1}^{n} f_{,i}\, b_i(t,x) + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(t,x)\, f_{,ij}(t,x)$

$$= f_{,0}(t,x) + \underbrace{(D_x f)}_{\text{gradient}} \cdot b + \frac{1}{2}\underbrace{\mathrm{tr}}_{\text{trace}}\big(\underbrace{D_{xx}}_{\text{Hesse matrix}} f(t,x)\, a(t,x)\big)$$

Remark:

- Ito formula: $df(t,X_t) = \mathcal{L}f(t,X_t)\,dt + D_x f(t,X_t)\,\sigma(t,X_t)\,dW_t$
- simplest case: no time dependence and 1D:

$$\mathcal{L}f(x) = b(x)f'(x) + \frac{\sigma^2}{2}(x)\, f''(x)$$

<u>Theorem</u>: Let $f \in C^{1,2}$ and we have ~~that~~ $\forall\, u \geq t \geq 0$, $x \in R^n$

$$\mathbb{E}_{t,x}\left[\int_t^u |\mathcal{L} f(s,X_s)|\, ds\right] < \infty, \quad \mathbb{E}_{t,x}\left[\int_t^u \| D_x f(s,X_s)^{tr} \sigma(s,X_s)\|^2 ds\right] < \infty$$

$$\Rightarrow f \in \mathcal{D}_L \quad \text{and} \quad Lf(t,x) = \mathcal{L} f(t,x)$$

<u>Proof</u>: by Ito's formula + vanishing of the stochastic integral in expectation !

<u>Rem.</u> Conditions of the theorem are fulfilled, if we have $f \in C^{1,2}$ with compact support !

<u>Theorem</u> (Dynkin formula): $f \in C^{1,2}$ with compact support, τ stopping time with $\mathbb{E}[\tau] < \infty$

$$\Rightarrow \boxed{\mathbb{E}_x[f(\tau, X_\tau)] = f(0,x) + \mathbb{E}_x\left[\int_0^\tau \mathcal{L} f(s,X_s)\, ds\right]}$$

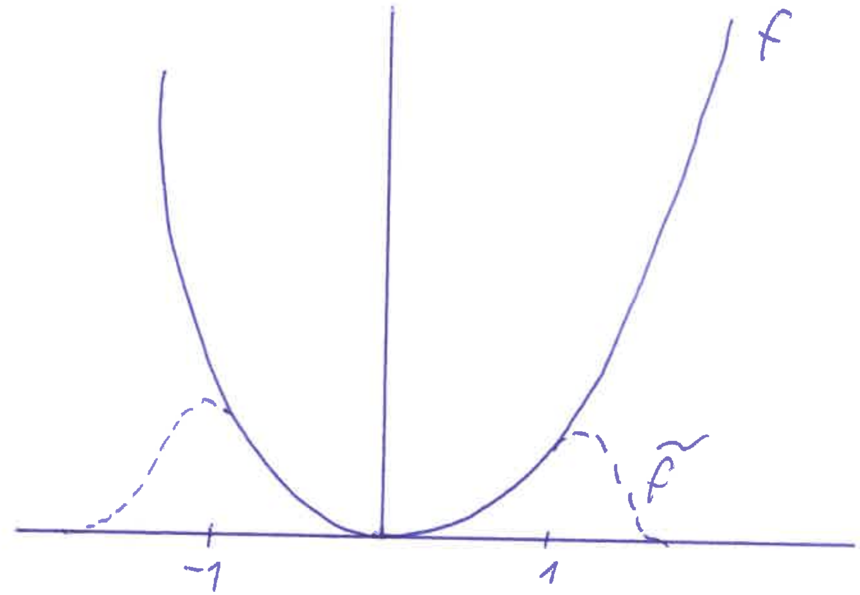
idea of proof:

- application of Itô's formula.
- to show: $\int_0^{\tau \wedge n} \mathcal{L} f(s, X_s)\, ds$ are a uniformly integrable family s.t. $\mathbb{E}$ and $\lim$ can be interchanged.
- as $f \in C^{1,2}$ + compact support: sufficient for this: $\int_0^{\tau \wedge n} 1$ unif. integrable.
- this follows by $\mathbb{E}[\tau] < \infty$ $\square$

Remark: If τ is an „exit time" of a bounded set A then Dynkins formula holds $\forall f \in C^{1,2}$!

reason: one can construct a smooth continuation of $f|_A$, say $\tilde{f}$ with compact support

Ex: $f(x) = x^2$, $A = [-1, 1]$



Ex. (Application of the Dynkin formula)

$W_t \dots$ 1D Brownian motion

$X_t = x + W_t$

$\tau = \inf\{t \geq 0 \mid X_t \notin (a,b)\}$ $\qquad a \leq x \leq b$

Calculate $p := \mathbb{P}(X_\tau = b)$

1.) We have $\mathbb{P}(\tau < \infty) = 1$

by martingale convergence theorem for $X_{t \wedge \tau}$ (a bounded martingale!)

$\Rightarrow \mathbb{E}[f(X_\tau)] = p \cdot f(b) + (1-p) f(a) \qquad (*)$

2.) Generator of X_t : $\mathcal{L} = \frac{1}{2} \frac{d^2}{dx^2}$

3.) Solve $\mathcal{L} f_0 = 0 \Rightarrow f_0(x) = cx + d$

Dynkin $\Rightarrow \mathbb{E}[f_0(X_\tau)] = f_0(x)$

$$\overset{(*)}{\Rightarrow} \quad cx + d = p(cb+d) + (1-p)(ca+d)$$

$$= c(pb + a(1-p)] + d[p + (1-p)] \qquad \forall c, d !$$

compare the coeff. of c

$\Rightarrow$

$$x = pb + a(1-p) \Rightarrow$$

$$x = p(b-a) + a \Rightarrow p = \frac{x-a}{b-a}$$

III.2) Dynamic programming principle (DPP) – motivation of the Hamilton–Jacobi–Bellman (HJB) equation

- DPP (Bellman principle)

$$V(t,x) = \sup_{u \in \mathcal{A}(t,x)} \mathbb{E}_{t,x}\left[\int_t^{t_1} \varphi(s, X_s, u_s)\, ds + V(t_1, X_{t_1})\right], \quad t < t_1 \leq T$$

meaning: if the behavior after t_1 is optimal and also between t and t_1, then it is globally optimal!

- reasonable, but it can also be proved

- our approach: Use the DPP to get an equation for V (HJB eq.!)

- <u>Assumption</u> : $V \in C^{1,2}$ and the following formal calculation is correct (interchanging lim, $\mathbb{E}$, sup; vanishing of the stochastic integral in expectation)

Ito applied to DPP $\Rightarrow$

$$V(t,x) = \sup_{u \in \mathcal{A}} \mathbb{E}_{t,x}\Big[\int_t^{t_1} \varphi(s, X_s, u_s)\,ds + V(t, \overset{=x}{X_t}) + \int_t^{t_1} V_t(s, X_s)\,ds$$

$$+ \int_t^{t_1} D_x V(s, X_s)\, b(s, X_s, u_s)\,ds + \frac{1}{2}\int_t^{t_1} \operatorname{tr}\left[D_{xx} V(s, X_s)\, a(s, X_s, u_s)\right] ds$$

$$+ \int_t^{t_1} D_x V(s, X_s)\, \sigma(s, X_s, u_s)\, dW_s \Big]$$

if dW integral is martingale $\Rightarrow$

$$0 = \sup_{u \in \mathcal{U}} \mathbb{E}_{t,x} \Big[\int_t^{t_1} \varphi(s, X_s, u_s)\, ds + \int_t^{t_1} V_{,0}(s, X_s)\, ds + \int_t^{t_1} D_x V(s, X_s)\, b(s, X_s, u_s)\, ds$$

$$+ \frac{1}{2} \int_t^{t_1} tr\left[D_{xx} V(s, X_s)\, a(s, X_s, u_s) \right] ds$$

• Divide by $(t_1 - t_0)$, $\lim_{t_1 \searrow t_0}$ + exchanges are correct $\Rightarrow$

$$0 = \sup_{u \in \mathcal{U}} \left\{ \varphi(t, x, u) + V_{,0}(t, x) + D_x V(t, x)\, b(t, x, u) + \frac{1}{2} tr\left[D_{xx} V(t, x)\, a(t, x, u) \right] \right\}$$

Def: $\mathcal{L}^u f := f_{,0}(t, x) + D_x f \cdot b + \frac{1}{2} tr\left[D_{xx} f \cdot a(t, x, u) \right]$

$$\Rightarrow \boxed{0 = \sup_{u \in \mathcal{U}} \left\{ \varphi(t, x, u) + \mathcal{L}^u V(t, x) \right\}, \quad \text{HJB-equation}}$$

Remarks • sometimes a different representation is used:

$$\sup_{u \in \mathcal{U}} \left\{ \varphi(t,x,u) + D_x V(t,x)\, b(t,x,u) + \frac{1}{2} \operatorname{tr}\left[D_{xx} V(t,x)\, a(t,x,u) \right] \right\}$$

$$=: \underbrace{H(t,x,D_xV,D_{xx}V)}_{\text{Hamiltonian}}$$

$$\Rightarrow \boxed{0 = V_{,0}(t,x) + H(t,x,D_xV,D_{xx}V)}$$ 2nd form of the HJB eq.

- the approach above shows : under certain conditions the value function solves the HJB

  - on the other hand one can ask, whether a solution of the HJB equation provides the value function, see below!

Algorithm to solve the HJB

- look for an optimizer $v = \hat{v}(t,x)$ in the HJB

  it depends on $D_x V$, $D_{xx} V$, i.e. $\hat{v}(t,x) = \tilde{v}(t,x, D_x V(t,x), D_{xx} V(t,x))$

- plug into the HJB $\Rightarrow$ (in general nonlinear) PDE for V!

- „solve" this PDE including the boundary condition $V(T,x) = \Psi(T,x)$!

„solution":

a) ansatz (see below)

b) if PDE is linear: Fourier transform

3.) numerical solution

(after existence and uniqueness proof)

- Let now X_t^* be the solution of

$$dX_t^* = b(t, X_t^*, \hat{u}(t, X_t^*)) + \sigma(t, X_t^*, \hat{u}(t, X_t^*))\, dW_t$$

and let $\hat{u}(t, X_t^*) =: U_t^* \in \mathcal{A}$

$\Rightarrow$ U_t^* is the optimal control and

$V(t,x)$ is the value function of the problem

Remark: Theorems, which guarantee that under certain conditions on $V, \varphi, \psi, b, \sigma, \mathcal{A}$ the solution of the HJB equation is indeed the value function are called

$\boxed{\text{verification theorems}}$

III.3) A verification theorem

<u>Remark:</u>

- Standard verification theorems (as below): strong assumptions
- $\Rightarrow$ in concrete applications often not useable
- $\Rightarrow$ — " — : one has to prove a verification th. (usually with similar proof!)

<u>Consider</u>: $dX_t = b(t, X_t, u_t)\,dt + \sigma(t, X_t, u_t)\,dW_t, \quad X_0 = x$

$$J(t,x,u) = \mathbb{E}_{t,x}\left[\int_t^T \varphi(s, X_s, u_s)\,ds + \psi(T, X_T)\right]$$

$$V(t,x) = \sup_{u \in \mathcal{A}(t,x)} J(t,x,u)$$

conditions on $\mathcal{A}$: (A1) u_s is progr. mb. and fulfills $\mathbb{E}\left[\int_t^T \|u_s\|^2 ds\right] < \infty$

(A2) SDE has a unique, strong solution $X_{s \geq t}$, $X_t = x$

and $\mathbb{E}_{t,x}\left[\sup_{t \leq s \leq T} \|X_s\|^2\right] < \infty$

(A3) $J(t, x, u)$ is well defined

Theorem (Verification): assume: $\|\sigma(t,x,u)\|^2 \leq C_\sigma (1+\|x\|^2+\|u\|^2)$, $C_\sigma > 0$

$|\varphi(t,x,u)| \leq C_\varphi (1+\|x\|^2+\|u\|^2)$, $C_\varphi > 0$ $\forall x \in \mathbb{R}^n$ $\forall u \in \mathcal{U}$

(i) If $\Phi \in C^{1,2}([0,T]\times\mathbb{R}^n)$, $|\Phi(t,x)| \leq C_\Phi (1+\|x\|^2)$

and $$\left.\begin{aligned} &\sup_{u\in\mathcal{U}} \{ \varphi(t,x,u) + \mathcal{L}^u \Phi(t,x)\} = 0 \\ &\Phi(T,x) = \Psi(T,x) \end{aligned}\right\}$$

$\Rightarrow \forall t \in [0,T],\ x \in \mathbb{R}^n:\ \Phi(t,x) \geq V(t,x)$

ii) if we have: the maximizer $\hat{u}(t,x)$ of

$$u \longmapsto \varphi(t,x,u) + \mathcal{A}^u \Phi(t,x) \quad \exists$$

s.t. $u_t^* = \hat{u}(t, X_t^*)$ is admissible

$\Rightarrow \Phi(t,x) = V(t,x)$

u^* is the optimal strategy, i.e.

$V(t,x) = J(t,x,u^*)$

Proof: Fix t, x; u arbitrary admissible

$$\tau_n := \inf\{s \geq t \,/\, \|X_s - x\| = n\} \wedge T, \qquad n \in \mathbb{N}$$

(guarantees a bounded process until τ_n!

Ito formula: $\Rightarrow$

$$\Phi(\tau_n, X_{\tau_n}) = \Phi(t,x) + \int_t^{\tau_n} \mathcal{L}^{u_s} \Phi(s, X_s)\, ds + \int_t^{\tau_n} D_x \Phi(s, X_s)\, \sigma(s, X_s, u_s)\, dW_s$$

One has: $\mathbb{E}_{t,x}\left[\int_t^{\tau_n} \| D_x \Phi(s, X_s)\, \sigma(s, X_s, u_s) \|^2 \right] < \infty$

uses:
- stopping of the process
- $\Phi \in C^{1,2}$
- growth condition on σ, $\mathcal{A}_1$

$$\Rightarrow \mathbb{E}_{t,x}\left[\int_t^{\tau_n} D_x \Phi(s, X_s)\, \sigma(s, X_s, u_s)\, dW_s \right] = 0$$

$\Rightarrow$

$$\underline{\mathbb{E}_{t,x}\left[\int_t^{\tau_n} \varphi(s,X_s,u_s)\,ds + \Phi(\tau_n, X_{\tau_n})\right]} =$$

$$\mathbb{E}_{t,x}\left[\int_t^{\tau_n} \varphi(s,X_s,u_s)\,ds + \Phi(t,x) + \int_t^{\tau_n} \mathcal{L}^{u_s}\Phi(s,X_s)\,ds\right] =$$

$$\Phi(t,x) + \mathbb{E}_{t,x}\Big[\int_t^{\tau_n} \underbrace{(\varphi(s,X_s,u_s) + \mathcal{L}^{u}\Phi(s,X_s)}_{\leq 0 \text{ because of HJB}}\,ds\Big] \leq \underline{\Phi(t,x)} \qquad (*)$$

$n \to \infty$: $\tau_n \to T$ f.s.

growth conditions on φ, Φ + admissibility cond. on u and X $\Rightarrow$

$$\left| \int_t^{\tau_n} \varphi(s, X_s, u_s) + \Phi(\tau_n, X_{\tau_n}) \right| \leq C_\varphi \int_t^T (1 + \|X_s\|^2 + \|u_s\|^2)\,ds + C_\Phi \left(1 + \sup_{s\in[t,T]} \|X_s\|^2\right)$$

$$\in L^1 \quad (\text{indep. of } n\,!)$$

dominated convergence + continuity of Φ $\Rightarrow$

$$\mathbb{E}_{t,x}\left[\int_t^{\tau_n} \varphi(s, X_s, u_s)\,ds + \Phi(\tau_n, X_{\tau_n}) \right] \xrightarrow{n\to\infty} J(t,x,u)$$

$$\overset{(*)}{\Rightarrow} \quad J(t,x,u) \leq \Phi(t,x,u) \qquad \Big| \sup_{u\in\mathcal{U}}$$

$$V(t,x) \leq \Phi(t,x) \quad \Rightarrow \quad (i) \checkmark$$

ad (ii) : analogue proof with one modification :

where the HJB is used: instead of arbitr. u : u^*

$$\Rightarrow \quad \leq \quad \longrightarrow \quad =$$

$$\Rightarrow V(t,x) = J(t,x,u^*) = \Phi(t,x)$$

$\square$

Remarks: (i) Bellman principle works also in the form:

$\forall$ stopping times $\tau \in [t, T]$ ($t_1 \to \tau$!)

$$V(t,x) = \sup_{u \in \mathcal{A}(t,x)} \mathbb{E}_{t,x}\left[\int_t^{\tau} \varphi(s, X_s, u_s) + V(\tau, X_\tau)\right]$$

Application: see chapter „Viscosity solutions"

(ii) by definition is the valuefunction unique $\Rightarrow$

Theorem above $\Rightarrow$ the solution of the HJB equation is unique in the class of $C^{1,2}$ functions with quadratic growth !

(iii) Existence of a solution: in general difficult to prove!

strong sufficient conditions:

- $\mathcal{U}$ ... compact
- ψ ... $\in C^3$ + bounded
- b, σ, φ ... $\in C^{1,2}$ + bounded
- uniform parabolicity, i.e.:

$$y \cdot a(t, x, u) \cdot y \geq c \|y\|^2$$

$$\forall t \in [0, T],\; x, y \in \mathbb{R}^n,\; u \in \mathcal{U}$$

$\Rightarrow$ HJB has a unique solution $\in C^{1,2}$!

Ex: consider

$$\begin{cases} dX_t = u_t + dW_t \\ J(t,x,u) = \mathbb{E}_{t,x}\left[\beta X_T - \alpha \int_t^T u_s^2 \, ds\right], \quad \alpha, \beta > 0 \\ u \ldots \text{progr. mb.}, \ \mathbb{R}\text{-valued}, \ \mathbb{E}\left[\int_t^T |u_s|^2 \, ds\right] < \infty \end{cases}$$

generator: $\mathcal{L}^u = u \frac{\partial}{\partial x} + \frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}$

HJB: $\sup_{u \in \mathbb{R}} \left\{ \mathcal{L}^u V - \alpha u^2 \right\} = 0$

$\Rightarrow \sup_{u \in \mathbb{R}} \left\{ V_t + u V_x + \frac{1}{2} V_{xx} - \alpha u^2 \right\} \Rightarrow \hat{u} = \frac{V_x}{2\alpha}$

plug in

$\Rightarrow -\alpha \frac{V_x^2}{4\alpha^2} + V_t + \frac{V_x^2}{2\alpha} + \frac{1}{2} V_{xx} = 0 \Rightarrow$

$$\Rightarrow \left.\begin{array}{l} \dfrac{V_x^2}{4\alpha} + V_t + \frac{1}{2} V_{xx} = 0 \\[2ex] \text{bound. cond.: } V(T,x) = \beta x \end{array}\right\}$$

ansatz: $V(t,x) = at + bx + \gamma$, motivated by the bound. cond.

plug in, comparing coefficients $\Rightarrow$ $\underline{V(t,x) = \beta x + \frac{\beta^2}{4\alpha}(T-t)}$

$$\Rightarrow \underline{\vartheta = \frac{V_x}{2\alpha} = \frac{\beta}{2\alpha}}$$

Conditions of the Verif. theorem:

- $dX_t^* = \frac{\beta}{2\alpha} dt + dW_t$ has strong solution (explicitly known!) with $\mathbb{E}[\sup_{0\le t\le T} X_t^{*2}] < \infty$ ✓
- ϑ... const. ✓
- J... well defined ✓

- growth conditions : $\varphi(t,x,u) = -\alpha u^2$ ✓

  $\sigma \equiv \text{const.}$ ✓

- $\Phi \in C^{1,2}$ ✓

- $\Phi(t,x) \leq C_\Phi \left(1 + \|x\|^2\right)$ ✓

$\Rightarrow$ V is really the value function and

$\hat{u} = \dfrac{\beta}{2\alpha} = u^*$ is the optimal strategy

<u>Interpretation</u> $\beta \nearrow \Rightarrow \hat{u} \nearrow$

$\alpha \nearrow \Rightarrow \hat{u} \searrow$ ✓

III.4) Optimal investment with power utility function

<u>model as above:</u> $dB_t = B_t \cdot r\,dt$, $B_0 = 1$, $r > 0$ const. $\Rightarrow B_t = e^{rt}$

$$dS_t = S_t(\mu\,dt + \sigma\,dW_t), \quad S_0 = s_0$$

<u>controls:</u> $u = (\pi, c)$

$\pi \ldots$ fraction of the wealth in stock

$c \ldots$ consumtion rate

$$\Rightarrow \underline{dX_t} = X_t \pi_t \cdot \frac{dS_t}{S_t} + X_t(1-\pi_t)\frac{dB_t}{B_t} - c_t\,dt =$$

$$\underline{\left[(r + \pi_t(\mu - r))X_t - c_t \right] dt + X_t \pi_t \sigma\,dW_t}, \qquad \text{wealth equation}$$

$$J(t,x,u) = \mathbb{E}_{t,x}\left[\int_t^T e^{-\beta s} U(c_s)\,ds + e^{-\beta T} U(X_T)\right] \longrightarrow \max$$

with $U(x) = \frac{x^\alpha}{\alpha}$, $0 < \alpha < 1$

$\Rightarrow$ generator: $\mathcal{L}^{(\pi,c)} V = V_t + [(r + \pi(\mu - r))x - c] V_x + \frac{1}{2}\pi^2 x^2 \sigma^2 V_{xx}$

$\Rightarrow$ HJB: $\sup_{\pi,c} \left\{ \underbrace{e^{-\beta t}\frac{c^\alpha}{\alpha} + \mathcal{L}^{(\pi,c)} V(t,x)} \right\} = 0$

only first order conditions considered!

Step 1: $\frac{\partial}{\partial c}(\quad) = e^{-\beta t} c^{\alpha-1} - V_x \overset{!}{=} 0$

$$\Rightarrow \hat{c} = \left(e^{\beta t} V_x\right)^{\frac{1}{\alpha-1}}$$

$$h^{(\pi,c)} = V_t + \{ rx + \pi x\mu - r\pi x - c \} V_x + \frac{1}{2}\pi^2 x^2 \sigma^2$$

$$\Rightarrow \frac{\partial h^{(\pi,c)}}{\partial \pi} = (\mu - r)x V_x + \pi x^2 \sigma^2 V_{xx} \overset{!}{=} 0$$

$$\Rightarrow \hat{\pi} = -\frac{(\mu - r)}{\sigma^2 x V_{xx}}$$

(we shall see later, that $V_{xx} < 0$ holds and that X_t stays positive $\Rightarrow$ denominator $\neq 0$

Step 2: plug into the HJB:

$$e^{-\beta t}\frac{(e^{\beta t}V_x)^{\frac{\alpha}{\alpha-1}}}{\alpha} + V_t + rxV_x - \frac{(\mu - r)}{\sigma^2 x}\frac{V_x}{V_{xx}}(\mu - r)xV_x - (e^{\beta t}V_x)^{\frac{1}{\alpha-1}}V_x$$

$$+ \frac{1}{2}\frac{(\mu - r)^2 V_x^2}{\sigma^{\cancel{4}2}\cancel{x^2}V_{xx}^{\cancel{2}}}\cancel{x^2}\cancel{\sigma^2}\cancel{V_{xx}} = 0$$

$$\Rightarrow V_x^{\frac{\alpha}{\alpha-1}} \left(\frac{e^{-\beta t + \frac{\alpha\beta}{\alpha-1} t}}{\alpha} - e^{-\frac{\beta t}{1-\alpha}} \right) + V_t + rx V_x - \frac{1}{2} \frac{(\mu-r)^2}{\sigma^2} \frac{V_x^2}{V_{xx}} = 0$$

$$\Rightarrow \boxed{V_x^{\frac{\alpha}{\alpha-1}} \left(\frac{1-\alpha}{\alpha} \right) \cdot e^{-\frac{\beta t}{1-\alpha}} + V_t + rx V_x - \underbrace{\frac{1}{2} \frac{(\mu-r)^2}{\sigma^2}}_{=\lambda} \frac{V_x^2}{V_{xx}} = 0} \qquad (1)$$

Boundary cond.: $V(T,x) = e^{-\beta T} \frac{x^\alpha}{\alpha}$

Ansatz: $V(t,x) = h(t)^{1-\alpha} \frac{x^\alpha}{\alpha} \Rightarrow h(T)^{1-\alpha} \frac{x^\alpha}{\alpha} \overset{!}{=} e^{-\beta T} \frac{x^\alpha}{\alpha}$

$$\Rightarrow \boxed{h(T) = e^{-\frac{\beta T}{1-\alpha}}} \qquad (2)$$

Plugging in $\Rightarrow$
$$\left.\begin{aligned} V_x &= h(t)^{1-\alpha} x^{\alpha-1} \\ V_{xx} &= h(t)^{1-\alpha}(\alpha-1)x^{\alpha-2} \\ V_t &= (1-\alpha)h'(t)\,h(t)^{-\alpha}\frac{x^\alpha}{\alpha} \end{aligned}\right\} \overset{(1)}{\Longrightarrow}$$

$$\left(h(t)^{1-\alpha}x^{\alpha-1}\right)^{\frac{\alpha}{\alpha-1}} e^{-\frac{\beta t}{1-\alpha}}\left(\frac{1-\alpha}{\alpha}\right) + (1-\alpha)h(t)^{-\alpha}h'(t)\frac{x^\alpha}{\alpha} + rx\,h(t)^{1-\alpha}x^{\alpha-1}$$

$$-\lambda\frac{\left(h(t)^{1-\alpha}x^{\alpha-1}\right)^2}{h(t)^{1-\alpha}(\alpha-1)x^{\alpha-2}} = 0 \quad \Big|\, : x^\alpha$$

$$h(t)^{-\alpha}e^{-\frac{\beta t}{1-\alpha}}\left(\frac{1-\alpha}{\alpha}\right) + \frac{(1-\alpha)}{\alpha}h(t)^{-\alpha}h'(t) + r\,h(t)^{1-\alpha} - \lambda\frac{h(t)^{1-\alpha}}{\alpha-1} \quad \Big|\cdot h(t)^\alpha\cdot\frac{\alpha}{1-\alpha}$$

$$\Rightarrow \boxed{h'(t) + h(t)\Big[\underbrace{r\cdot\frac{\alpha}{1-\alpha} + \lambda\frac{\alpha}{(1-\alpha)^2}}_{=:c}\Big] + e^{-\frac{\beta t}{1-\alpha}} = 0} \qquad (3)$$

ordinary linear diff. equation : standard methods $\xrightarrow[+\text{B.C.}(2)]{}$

$$h(t) = e^{-\frac{\beta t}{1-\alpha}}\, e^{-c(T-t)} + \frac{(1-\alpha)e^{-ct}}{\beta-(1-\alpha)c}\left\{ e^{-\frac{\beta-(1-\alpha)c}{1-\alpha}t} - e^{-\frac{\beta-(1-\alpha)c}{1-\alpha}T} \right\}$$

for $\beta-(1-\alpha)c \neq 0$

$$h(t) = e^{-ct}\cdot(1+T-t)\,, \quad \text{for } \beta-(1-\alpha)c = 0$$

Step 3: • $h(t)$ is strictly positive $\Rightarrow$ Value function is strictly pos. (if $x>0$)

• $V \in C^{1,2}$ and $V_{xx} < 0$

$\downarrow$

for $x>0$ (h is smooth!)

Moreover: $\hat{c} = \left(e^{\beta t} V_x\right)^{\frac{1}{\alpha-1}} = \left(e^{\beta t} h(t)^{1-\alpha} x^{\alpha-1}\right)^{\frac{1}{\alpha-1}} = \frac{e^{-\beta t}}{h(t)} \cdot x$

$$\hat{\pi}(t,x) = -\frac{(\mu-r)}{\sigma^2} \frac{V_x}{x V_{xx}} = -\frac{(\mu-r)}{\sigma^2} \cdot \frac{h(t)^{1-\alpha} x^{\alpha-1}}{h(t)^{1-\alpha}(\alpha-1) x^{\alpha-1}}$$

$$= \frac{(\mu-r)}{\sigma^2 \cdot (1-\alpha)}$$

$$\Rightarrow \boxed{\begin{aligned} c_t^*(t, X_t^*) &= \frac{e^{-\beta t}}{h(t)} X_t^* \\ \pi^*(t, X_t^*) &= \frac{1}{1-\alpha} \frac{(\mu-r)}{\sigma^2} \end{aligned}}$$

where: $dX_t^* = X_t^* f(t)\, dt + X_t^* \cdot C\, dW_t$, f... deterministic function

C... constant

this SDE has a unique strong solution: Ansatz: $e^{\int_t^s g_r\,dr + \int_t^s k_r\,dW_r}$

g_r, k_r ... deterministic, continuous

(see Merton problem)

$\Downarrow$

$X_s^* > 0$

Conditions of the Verificationth.: (A1) $\mathbb{E}\left[\int_t^T \|u_s^*\|^2 ds\right] \leq \text{const.} + \text{const.}\, \mathbb{E}\left[\int_t^T (X_s^*)^2\right] < \text{const.}$

(X_t^* has lognormal distribution!)