A singularly perturbed ruin problem for a two dimensional Brownian motion in the positive quadrant

Peter Grandits*
Institut für Stochastik und Wirtschaftsmathematik
TU Wien
Wiedner Hauptstraße 8-10,A-1040 Wien
Austria

September 2020

Keywords: ruin probabilities, optimal control problem, free boundary problems, singular perturbation

AMS (primary) subjects classifications. 49J20,35R35
AMS (secondary) subjects classifications. 91B70

Abstract

We consider the following problem: The drift of the wealth process of two companies, modelled by a two dimensional Brownian motion, is controllable in a way, s.t. the total drift adds up to a constant. The aim is to maximize the probability that both companies survive. We assume that the volatility of one company is small w.r.t. the other one and use methods from singular perturbation theory to construct a formal approximation of the value function. Moreover, we validate this formal result by explicitly constructing a strategy, which provides a target functional, approximating the value function *uniformly* on the whole state space.

1 Introduction

In this paper we consider the following two-dimensional controlled ruin problem. Let us denote the wealth of two companies by $(X_t)_{t\geq 0}$, resp. $(Y_t)_{t\geq 0}$, and the corresponding two dimensional state process by $(Z_t)_{t\geq 0}$, i.e.

$$Z_t = \begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} x + \int_0^t u_s^{(1)}\, ds + B_t^{(1)} \\ y + \int_0^t u_s^{(2)}\, ds + \epsilon B_t^{(2)} \end{pmatrix}. \tag{1}$$

Here $(x, y) =: z$ denotes the initial endowment of the companies, $B^{(1)}, B^{(2)}$ are independent standard Brownian motions, and $u^{(1)}, u^{(2)}$ are our control processes. We will write G for the positive quadrant, i.e. $G := \{(x, y)|x > 0, y > 0\}$, and ϵ denotes a small positive constant. Moreover, we define the ruin time $\tau = \inf\{t < 0|Z_t \notin G\}$, i.e. the first time at which one of the two companies is ruined. Finally, we define the set of admissible strategies u as

$$\mathcal{U}_{x,y} := \{u|u_t = \hat{u}(Z_t) \text{ for a Borel measurable function } \hat{u}(z); 0 \leq u^{(1)}, u^{(2)} \leq 1, u^{(1)} + u^{(2)} = 1\}. \tag{2}$$

*email: pgrand@fam.tuwien.ac.at, tel. +43-1-58801-10512, fax +43-1-5880110599

Our aim is to maximize the target functional, given by

$$J(x,y,u) = \mathbf{P}_{x,y}\left(\tau = \infty\right) \to \max, \tag{3}$$

i.e. the probability that both companies survive should be maximized.

Let us mention two interpretations of this problem. The first one, given in [10], is that a government can influence the drift of the companies by a certain tax policy, but the total amount of "support" is bounded by the condition that the sum of the drifts is one.

A second interpretation would be the following. Writing the drift vector as $\binom{1/2+\hat{u}_t}{1/2-\hat{u}_t}$, one could imagine two collaborating companies with the goal that both want to survive. Collaborating companies were considered in insurance mathematics, e.g., in the paper [6], where the goal is to maximize dividends.

In [10] the problem is solved for the case $\epsilon = 1$. The authors show that it is optimal to give the whole drift to the company, which has a smaller endowment at the moment considered. It is also mentioned there that the case of different volatilities is open. For convenience, we set in our paper one of the two volatilities equal to one and consider the case, where the second company faces a small volatility in comparison to the first one.

The goal of our paper is to find an admissible strategy, which produces a target functional, which is a *uniform* approximation of the value function $V(x,y)$ in $\overline{G} := \mathbf{R}_0^+ \times \mathbf{R}_0^+$. Our method will be the following:

A) Find an approximation for $V(x,y)$, say $\tilde{V}(x,y)$ by formal methods of singular perturbation theory.

B) Show the validity of the approximation, i.e.

$$\left| V(x,y) - \tilde{V}(x,y) \right| = o(1),$$

for $\epsilon \to 0$, uniformly in $\overline{G}$.

C) for point B) we shall need a kind of Alexandrov-Bakelman-Pucci (ABP) estimate for the difference considered above. In comparison to standard results (see e.g. the monograph [5]), we have to deal with two special features: Firstly, we have an unbounded domain and secondly, we need some control over the constant on the r.h.s. of the ABP-estimate; more precisely, we have to control its dependence on ϵ. The reason for this is that we want to conclude from the smallness of the inhomogeneity of a PDE for $D := \tilde{V} - V$, that point B) above is indeed true. A result of this kind is proved in [7], and we shall use it.

D) It turns out that one can easily find a strategy $\tilde{u}$, which produces $\tilde{V}(x,y)$ as target functional. Unfortunately $\tilde{u}$ is not an admissible strategy. So, in a final step we construct an admissible strategy $\hat{u}$ with corresponding target functional $\hat{V}(x,y)$, which fulfills

$$\left| \hat{V}(x,y) - \tilde{V}(x,y) \right| = o(1),$$

for $\epsilon \to 0$, uniformly in $\overline{G}$, which, together with point B), gives the final result

$$\left| \hat{V}(x,y) - V(x,y) \right| = o(1),$$

for $\epsilon \to 0$, uniformly in $\overline{G}$.

The schedule of the paper will be the following: In section 2 we give a preliminary result, which can be taken from [8]. In that paper a similar problem is considered. More precisely, we have the same state process there as in the present paper, but the target functional is different. Namely, the goal in [8] is, to maximize the expectation of the number of surviving companies. This produces the same Hamilton-Jacobi-Bellman (HJB) equation, but with non-homogeneous boundary conditions. In section 3 we deal with the formal approximation of point A). Finally, we formulate in section 4 the ABP estimate of point C) and apply it, in order to get the points B) and D) above.

Let us finally mention two further papers considering 2-dimensional problems in risk theory, namely [1] and [2], where the first one considers an optimal dividend problem for two collaborating companies, and the second one considers a two-dimensional ruin problem in a Cramer-Lundberg setting

2 A preliminary result

Let $V(x,y) := \sup_{u\in\mathcal{U}_{x,y}} J(x,y,u)$ be the value function of our problem. Then the following proposition shows that $V(x,y)$ is a classical solution of the HJB-equation of the problem.

Proposition 2.1 *$V(x,y)$ is the unique classical - i.e. $V \in C(\overline{G})$, $V_{xx}, V_{xy}, V_{yy} \in C^2(G)$ - solution of the problem*

$$\begin{aligned} \mathcal{L}V := \max\{V_x, V_y\} + \frac{1}{2}\Delta^{(\epsilon)}V &= 0, \\ V(x,0) &= 0, \\ V(0,y) &= 0, \\ \lim_{x\to\infty, y\to\infty} V(x,y) &= 1, \end{aligned}$$

where, we have used the notation $\Delta^{(\epsilon)} := \frac{\partial^2}{\partial x^2} + \epsilon^2\frac{\partial^2}{\partial y^2}$. Additionally, we have $V_{xx}, V_{xy}, V_{yy} \in C\left(\overline{G}\setminus\{(0,0)\}\right)$.

Proof. The proof works analogously to the proof of Theorem 3.1, Proposition 3.1 and Proposition 3.2 of [8], with $w = (1-e^{-2x})(1-e^{-\frac{2y}{\epsilon^2}})$ instead of $w = 2 - e^{-2x} - e^{-2y}$, and $v = 1 - e^{-x} - e^{-y/\epsilon^2}$ instead of $v = 2 - e^{-x} - e^{-y}$ there. The fact that we have $\epsilon = 1$ there, does not cause any harm, and the homogeneous boundary conditions at $\{(x,0)|x \geq 0\}$ and $\{(0,y)|y \geq 0\}$ in our case make life sometimes even easier. □

3 Heuristics - a formal approximation

Let us first consider the case $\epsilon = 0$. In this case the strategy $u := \binom{1}{0}$ is clearly optimal. It leads to the target functional

$$V^{(0)}(x,y) := J\left(x,y,\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{cases} 1-e^{-2x}, & x\geq 0,\ y>0, \\ 0, & x \geq 0,\ y = 0. \end{cases}$$

By the Barles/Bertham procedure (see e.g. [4], Chapter VII), one can show that $V^{(0)}$ will be an approximation of $V(x,y)$ in compact subsets of G. But since $V^{(0)}(x,y)$ is discontinuous at the positive x-axis, it can never be a uniform approximation of our continuous value function on $\overline{G}$. It is one goal of this paper, to provide such a uniform approximation.

Taking a look at the case $\epsilon = 1$, where an explicit solution is given in [10], we expect that $\overline{G}$ splits into two simply connected regions P and N, with

$$\begin{aligned} P &:= \left\{(x,y)\in\overline{G}|V_x(x,y) \geq V_y(x,y)\right\}, \\ N &:= \left\{(x,y)\in\overline{G}|V_x(x,y) < V_y(x,y)\right\}. \end{aligned} \tag{4}$$

This means that we assign the full drift of one in region P to the X-company, and in region N to the Y-company. Since ϵ is small - hence, the risk for the second company to be ruined is small - we expect that the region N is very thin. The separation curve, starting from the origin, is denoted by C, i.e.

$$C := \left\{(x,y)\in\overline{G}|V_x(x,y) = V_y(x,y)\right\}. \tag{5}$$

Moreover, we expect that V has in region N a layer behavior, i.e. we expect the existence of a "fast variable". To find this variable, we use methods from singular perturbation theory (see e.g. [3]). Starting from the ansatz $z := \frac{y}{\epsilon^\alpha}$, we want to find the significant degeneration (see again [3] for this concept) of the PDE in region N. Denoting V in region N by $V^{(N)}$, we get the following PDE in region N

$$V_y^{(N)} + \frac{1}{2}\Delta^{(\epsilon)} V^{(N)} = 0.$$

Transforming to the variables (x, z), yields

$$\epsilon^{-\alpha} V_z^{(N)} + \frac{1}{2} V_{xx}^{(N)} + \frac{\epsilon^{2-2\alpha}}{2} V_{zz}^{(N)} = 0,$$

which gives the significant degeneration for $\alpha = 2$, hence

$$V_z^{(N)} + \frac{1}{2} V_{zz}^{(N)} = 0.$$

Solving this equation, we arrive at $V^{(N)}(x, z) = C(x) + D(x)e^{-2z}$, therefore - using the boundary condition $V^{(N)}(x, 0) = 0$ -

$$V^{(N)}(x, z) = D(x)\left(1 - e^{-2z}\right) = D(x)\left(1 - e^{-\frac{2y}{\epsilon^2}}\right). \tag{6}$$

Employing the "boundary condition" at $x = \infty$, we find additionally

$$D(\infty) = 1. \tag{7}$$

We now assume that the curve C is described by the function $\phi(x)$, $x \in [0, \infty)$.

Since we have to change the strategy at this curve, we find the condition

$$V_x^{(N)}(x, \phi(x)) = V_y^{(N)}(x, \phi(x)). \tag{8}$$

Using (6) and the scaled function $\tilde{\phi}(x) = \frac{\phi(x)}{\epsilon^2}$, gives finally

$$\frac{D'(x)}{D(x)} = \frac{2}{\epsilon^2} \frac{e^{-2\tilde{\phi}(x)}}{1 - e^{-2\tilde{\phi}(x)}}. \tag{9}$$

We now try to construct an approximation in region P and consider the so called "reduced equation" (i.e. setting $\epsilon = 0$)

$$V_x^{(P)} + \frac{1}{2} V_{xx}^{(P)} = 0,$$

which gives, using the boundary condition $V^{(P)}(0, y) = 0$,

$$V^{(P)}(x, y) = F(y)\left(1 - e^{-2x}\right). \tag{10}$$

The boundary condition $V^{(P)}(x, \infty) = 1 - e^{-2x}$ additionally provides

$$F(\infty) = 1. \tag{11}$$

Analogously as in (8), we impose

$$V_x^{(P)}(x, \phi(x)) = V_y^{(P)}(x, \phi(x)). \tag{12}$$

Hence, we find, using (10),

$$\frac{F'(\phi(x))}{F(\phi(x))} = \frac{2e^{-2x}}{1 - e^{-2x}}. \tag{13}$$

Clearly, we should have $V^{(P)}(x, \phi(x)) = V^{(N)}(x, \phi(x))$, giving

$$D(x)\left(1 - e^{-2\tilde{\phi}(x)}\right) = F(\phi(x))\left(1 - e^{-2x}\right). \tag{14}$$

Our final "matching condition" is $V_x^{(P)}(x,\phi(x)) = V_x^{(N)}(x,\phi(x))$, which means together with (8) and (12), that we want continuous partial derivatives over the curve C. This gives finally

$$\frac{D'(x)}{F(\phi(x))} = \frac{2e^{-2x}}{1-e^{-2\tilde{\phi}(x)}}. \tag{15}$$

We now want to calculate the functions $\phi(x), D(x), F(y)$ from the matching conditions (9),(13),(14) and (15).

We start with the derivation of (14) w.r.t. x, giving

$$D'(x)\left(1-e^{-2\tilde{\phi}(x)}\right) + 2D(x)e^{-2\tilde{\phi}(x)}\tilde{\phi}'(x) = F'(\phi(x))\phi'(x)\left(1-e^{-2x}\right) + 2F(\phi(x))e^{-2x}.$$

Using (15), (14) and (13), and dividing this equation by $F(\phi(x))$, provides - after some elementary calculations -

$$\frac{e^{-2x}\epsilon^2}{1-e^{-2x}} = \frac{e^{-2\tilde{\phi}(x)}}{1-e^{-2\tilde{\phi}(x)}}, \tag{16}$$

and finally the formula for the separation curve

$$\tilde{\phi}(x) = \frac{1}{2}\ln\left(1+\frac{e^{2x}-1}{\epsilon^2}\right). \tag{17}$$

It remains to determine the functions $D(x)$ and $F(y)$. Plugging (17) (resp. (16)) into (9), yields

$$\frac{D'(x)}{D(x)} = \frac{2e^{-2x}}{1-e^{-2x}}.$$

Integrating this ODE, using the condition (7), gives

$$D(x) = 1-e^{-2x}. \tag{18}$$

Finally, we have by (14) and (18)

$$F(\phi(x)) = \frac{D(x)\left(1-e^{-2\tilde{\phi}(x)}\right)}{1-e^{-2x}} = 1-e^{-2\tilde{\phi}(x)}.$$

Since $\phi(x)$ is bijective from $\mathbf{R}_0^+$ to $\mathbf{R}_0^+$, we get

$$F(y) = 1-e^{-\frac{2y}{\epsilon^2}}. \tag{19}$$

So, summarizing the results of our heuristic procedure, we find the approximations

$$\begin{aligned} \tilde{V}(x,y) = V^{(P)}(x,y) = V^{(N)}(x,y) &= \left(1-e^{-2x}\right)\left(1-e^{-\frac{2y}{\epsilon^2}}\right), \\ \phi(x) &= \frac{\epsilon^2}{2}\ln\left(1+\frac{e^{2x}-1}{\epsilon^2}\right), \end{aligned} \tag{20}$$

and we note again that on the separation curve $(x,\phi(x)$ we have $\tilde{V}_x = \tilde{V}_y$. In figure 1 one can find a plot of the separation curve.

As we want to show finally that $\tilde{V}$ is a uniform approximation of the value function V in $\overline{G}$, we prove as a preparatory result that $\tilde{V}$ produces a small residuum in the sense of L^p, $p=1,2$, if we plug it into the operator $\mathcal{L}$.

Lemma 3.1 *We have*

$$\mathcal{L}\tilde{V}(x,y) = \max\left\{\tilde{V}_x, \tilde{V}_y\right\} + \frac{1}{2}\Delta^{(\epsilon)}\tilde{V} =: R(x,y),$$

with

$$||R(x,y)||_{L^p(G)} \le C\epsilon^2\left(-\ln\epsilon\right),\ p=1,2,$$

for small ϵ, and a positive constant C, not depending on ϵ, and $R \in L^\infty(G)$.

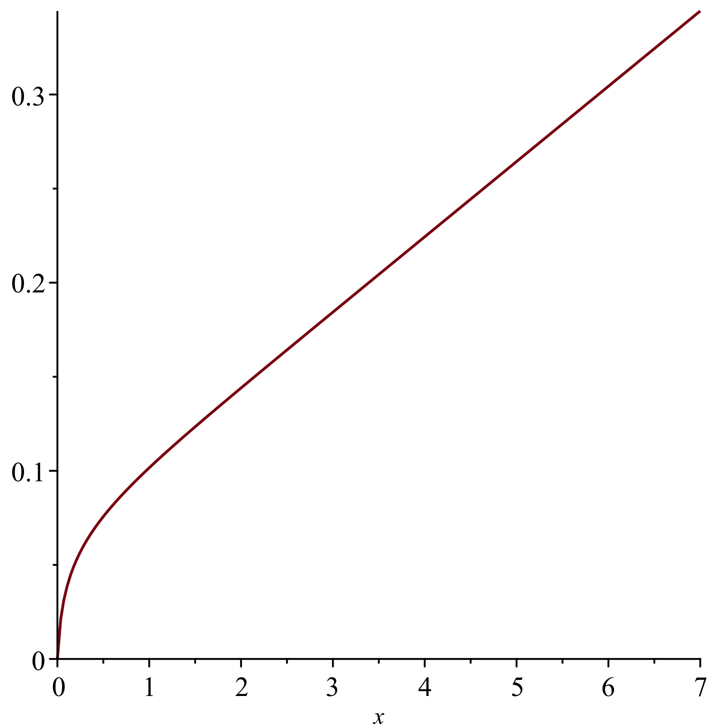

Figure 1: plot of the separation curve $(x, \phi(x))$

Proof. An elementary calculation provides

$$\mathcal{L}\tilde{V}(x,y) = \mathbf{1}_{\{y \geq \phi(x)\}} \frac{2}{\epsilon^2} \left(e^{-2x} - 1\right) e^{-\frac{2y}{\epsilon^2}} + \mathbf{1}_{\{y<\phi(x)\}} 2e^{-2x} \left(e^{-\frac{2y}{\epsilon^2}} - 1\right), \tag{21}$$

where $\mathbf{1}$ denotes the indicator function. As $R \in L^\infty(G)$ is obvious, let us now calculate an upper estimate for the L^p-norm in question. Fixing x first, one gets

$$\int_0^\infty |R(x,y)|^p \, dy = 2^p e^{-2px} \int_0^{\phi(x)} \left|1 - e^{-\frac{2y}{\epsilon^2}}\right|^p dy + \frac{2^p}{\epsilon^{2p}} \left(1 - e^{-2x}\right)^p \int_{\phi(x)}^\infty e^{-\frac{2py}{\epsilon^2}} \, dy. \tag{22}$$

We denote the first integral by J_1 and the second one by J_2 and start with J_1: Changing to the integration variable $w = 1 - e^{-\frac{2y}{\epsilon^2}}$, we find

$$J_1 = \frac{\epsilon^2}{2} \int_0^{1-e^{-2\tilde{\phi}(x)}} \frac{w^p}{1-w} \, dw \leq \frac{\epsilon^2}{2} \int_0^{1-e^{-2\tilde{\phi}(x)}} \frac{dw}{1-w} = \epsilon^2 \tilde{\phi}(x). \tag{23}$$

The second integral J_2 can be calculated explicitly, giving, if we use (17),

$$J_2 = \frac{\epsilon^2}{2p} \left(\frac{\epsilon^2}{\epsilon^2 + e^{2x} - 1}\right)^p. \tag{24}$$

So by (22),(23) and (24) one gets

$$\int_0^\infty \int_0^\infty |R(x,y)|^p \, dx \, dy \leq 2^p \epsilon^2 \int_0^\infty e^{-2px} \tilde{\phi}(x) \, dx + \epsilon^2 \frac{2^p}{2p} \int_0^\infty \frac{\left(1 - e^{-2x}\right)^p}{\left(\epsilon^2 + e^{2x} - 1\right)^p} \, dx. \tag{25}$$

Denote the integrals above by K_1, respectively K_2.

We deal with K_1 first and start with the following upper bound for $\tilde{\phi}$

$$\tilde{\phi}(x) = \frac{1}{2} \ln\left(\frac{\epsilon^2 + e^{2x} - 1}{\epsilon^2}\right) = \frac{1}{2} \ln\left(\epsilon^2 + e^{2x} - 1\right) - \ln \epsilon \leq \frac{1}{2} \ln\left(e^{2x}\right) - \ln \epsilon = x - \ln \epsilon,$$

where we have used $\epsilon < 1$. This gives for K_1 the following estimate

$$K_1 \leq \int_0^\infty e^{-2px}(x - \ln\epsilon)\,dx \leq -\text{const.}\ln\epsilon, \tag{26}$$

for some positive constant. For K_2, which can be explicitly calculated for $p = 1, 2$, one easily finds

$$K_2 \leq \frac{1}{2}. \tag{27}$$

By (25), (26) and (27) we end up with $||R(x,y)||_{L^p(G)} \leq \text{const.}\epsilon^2\,(-\ln\epsilon)$, $p = 1, 2$, which concludes the proof. □

4 Validation of the formal approximation

We start this section with the formulation of the ABP-result, we have mentioned in the introduction. Its proof is a direct consequence of Theorem 2.1 of [7], if we set $G = \mathbf{R}^+ \times \mathbf{R}^+$, $\rho = k = 1$ there.

Theorem 4.1 *Consider the following inhomogeneous linear elliptic PDE*

$$\mathcal{K}D(x,y) := a_1(x,y)D_x + a_2(x,y)D_y + \frac{1}{2}\Delta^{(\epsilon)}D + f(x,y) = 0,$$

on G, with

$$D(x,0) = D(0,y) = 0, \qquad x, y \geq 0,$$
$$\lim_{x\to\infty, y\to\infty} D(x,y) = 0.$$

Moreover, we assume that the a_i are Borel measurable and

$$a_1 + a_2 = 1, 0 \leq a_i \leq 1, i = 1, 2,$$
$$f(x,y) \in L^\infty(G) \cap L^1(G) \cap L^1(G).$$

Then the boundary value problem for D above has a unique solution in $W^{2,2}_{loc}(G) \cap C(G)$, which fulfills

$$||D||_{L^\infty(G)} \leq \frac{C}{\epsilon}\left(\sqrt{\epsilon}(-\ln\epsilon)||f||_{L^2(G)} + ||f||_{L^1(G)}\right),$$

for some positive constant C, not depending on ϵ.

The aim of the rest of this section is twofold. Firstly, we want to prove that the formal approximation $\tilde{V}$, provided in (20), is indeed a valid approximation of the value function $V(x,y)$ in $\overline{G}$. One can easily give a strategy, which produces $\tilde{V}(x,y)$ as target functional. Unfortunately, this strategy is not admissible. The second aim of this section is, to provide an *admissible* strategy, which gives a target functional, approximating the value function $V(x,y)$ uniformly in $\overline{G}$.

Proposition 4.1 *Let $V(x,y)$ be the value function of problem (1)-(3), and let $\tilde{V}(x,y)$ the formal approximation given in (20). Then we have*

$$||\tilde{V}(x,y) - V(x,y)||_{L^\infty(G)} \leq C(-\epsilon\ln\epsilon),$$

for some positive constant C, not depending on ϵ.

Proof. By Proposition 2.1 we have

$$\mathcal{L}V = \max\{V_x, V_y\} + \frac{1}{2}\Delta^{(\epsilon)}V = 0.$$

Hence, using the vector $\nu := (-1, 1)$, we can write the full BVP as

$$
\begin{aligned}
V_x + (V_\nu)^+ + \frac{1}{2}\Delta^{(\epsilon)} V &= 0, \\
V(x,0) &= 0, \\
V(0,y) &= 0, \\
\lim_{x\to\infty, y\to\infty} V(x,y) &= 1,
\end{aligned} \tag{28}
$$

where $V_\nu := V_y - V_x$. On the other hand, we have by Lemma 3.1

$$
\begin{aligned}
\tilde{V}_x + \left(\tilde{V}_\nu\right)^+ + \frac{1}{2}\Delta^{(\epsilon)} \tilde{V} + f(x,y) &= 0, \\
\tilde{V}(x,0) &= 0, \\
\tilde{V}(0,y) &= 0, \\
\lim_{x\to\infty, y\to\infty} \tilde{V}(x,y) &= 1,
\end{aligned} \tag{29}
$$

with $f(x,y) = -R(x,y)$. Moreover, we know by Lemma 3.1 that

$$
\begin{aligned}
f &\in L^\infty(G) \cap L^p(G), \\
||f||_{L^p(G)} &\leq C(-\epsilon^2 \ln \epsilon),
\end{aligned} \tag{30}
$$

with $p = 1, 2$.

Let now $D(x,y) := \tilde{V}(x,y) - V(x,y)$. Subtracting (29) from (28), we get

$$
\begin{aligned}
D_x + \left(\tilde{V}_\nu\right)^+ - (V_\nu)^+ + \frac{1}{2}\Delta^{(\epsilon)} D + f(x,y) &= 0, \\
D(x,0) = D(0,y) &= 0, \\
\lim_{x\to\infty, y\to\infty} D(x,y) &= 0.
\end{aligned} \tag{31}
$$

We now consider 4 cases:
Case 1: $(x,y) \in \{\tilde{V}_\nu \geq 0, V_\nu \geq 0\}$. We have $\left(\tilde{V}_\nu\right)^+ - (V_\nu)^+ = D_\nu$.
Case 2: $(x,y) \in \{\tilde{V}_\nu < 0, V_\nu < 0\}$. This gives $\left(\tilde{V}_\nu\right)^+ - (V_\nu)^+ = 0$.
Case 3: $(x,y) \in \{\tilde{V}_\nu \geq 0, V_\nu < 0\}$. We find

$$
\left(\tilde{V}_\nu\right)^+ - (V_\nu)^+ = \left(\tilde{V}_\nu\right) = \frac{\tilde{V}_\nu}{\tilde{V}_\nu - V_\nu}\left(\tilde{V}_\nu - V_\nu\right) =: b(x,y) D_\nu.
$$

Case 4: $(x,y) \in \{\tilde{V}_\nu < 0, V_\nu \geq 0\}$. This finally yields

$$
\left(\tilde{V}_\nu\right)^+ - (V_\nu)^+ = -V_\nu = \frac{-V_\nu}{\tilde{V}_\nu - V_\nu}\left(\tilde{V}_\nu - V_\nu\right) =: b(x,y) D_\nu.
$$

Hence, if we define

$$
b(x,y) := \begin{cases} 1 & \text{in Case 1,} \\ 0 & \text{in Case 2,} \\ \frac{\tilde{V}_\nu}{\tilde{V}_\nu - V_\nu} & \text{in Case 3,} \\ \frac{V_\nu}{V_\nu - \tilde{V}_\nu} & \text{in Case 4,} \end{cases}
$$

we can write the PDE in (31) as

$$
D_x + b(x,y) D_\nu + \frac{1}{2}\Delta^{(\epsilon)} D + f(x,y) = 0. \tag{32}
$$

One easily checks that one has $b(x,y) \in [0,1]$, for all (x,y). Finally, defining $a_1(x,y) := 1 - b(x,y)$ and $a_2(x,y) := b(x,y)$, yields

$$\begin{aligned} a_1(x,y)D_x + a_2(x,y)D_y + \frac{1}{2}\Delta^{(\epsilon)}D + f(x,y) &= 0, \\ D(x,0) = D(0,y) &= 0, \\ \lim_{x\to\infty,y\to\infty} D(x,y) &= 0, \end{aligned} \tag{33}$$

with

$$a_1(x,y) + a_2(x,y) \equiv 1, a_1(x,y) \in [0,1], a_1(x,y) \in [0,1].$$

Hence, together with (30), an application of Theorem 4.1 proves our proposition. □

Proposition 4.1 shows that we can approximate the value function V by our formal approximation $\tilde{V}$. It is easy to see that $\tilde{V}(x,y)$ can be seen as the target functional, if we use the strategy $u = \binom{1}{1}$, which is unfortunately not admissible, since the total drift exceeds one. Of course, it would be interesting to find an admissible strategy, which produces a target functional, approximating the value function.

We consider the following strategy

$$\hat{u} := \begin{pmatrix} \mathbf{1}_{II}(x,y) \\ \mathbf{1}_{I}(x,y) \end{pmatrix}, \tag{34}$$

with

$$\begin{aligned} I &:= \{(x,y) \in G \,|y < \phi(x), x > 0\}, \\ II &:= \{(x,y) \in G \,|y \geq \phi(x), x > 0\}, \end{aligned} \tag{35}$$

where $\phi(x)$ is the separation curve, defined in (20). Let now $H(x,y)$ be the solution of the BVP

$$\begin{aligned} \mathcal{L}^L H := \mathbf{1}_{II}H_x + \mathbf{1}_I H_y + \frac{1}{2}\Delta^{(\epsilon)}H + f(x,y) &= 0, \\ H(x,0) = H(0,y) &= 0, \\ \lim_{x\to\infty,y\to\infty} H(x,y) &= 0, \end{aligned} \tag{36}$$

with $f(x,y) = -R(,x,y)$, and R defined in Lemma 3.1. By Theorem 4.1, this solution is unique and an element of $W^{2,2}_{loc}(G)$. Moreover, we have

$$|||H(x,y)||_{L^\infty(G)} \leq C(-\epsilon \ln \epsilon), \tag{37}$$

for some positive constant C, not depending on ϵ. Consider now $\hat{V}(x,y) := \tilde{V}(x,y) - H(x,y)$. Since $\tilde{V}$ solves by construction $\mathcal{L}^L\tilde{V} + f = 0$, $\hat{V}$ is the unique solution of

$$\begin{aligned} \mathcal{L}^L\hat{V} &= 0, \\ \hat{V}(x,0) = \hat{V}(0,y) &= 0, \\ \lim_{x\to\infty,y\to\infty} \hat{V}(x,y) &= 1. \end{aligned} \tag{38}$$

The next lemma shows that $\hat{V}(x,y)$ can be interpreted as the survival probability of X_t and Y_t, if we use the admissible strategy $\hat{u}$.

Lemma 4.1 *One has*

$$\hat{V}(x,y) = \mathbf{P}_{x,y}\left(Z_t \in G, \forall t \in \mathbf{R}_0^+\right) = \mathbf{P}_{x,y}(\tau = \infty).$$

Proof. In this proof we denote by Z_t the state process, if we use the strategy $\hat{u}$. Let

$$\tau_n := \inf\{t > 0 \,|\, Z_t \notin (1/n, n) \times (1/n, n)\},$$

n large enough, s.t. we have $(x, y) \in (1/n, n) \times (1/n, n)$. Ito-Krylov's formula, see e.g. [9], gives

$$\hat{V}\left(Z_{t\wedge\tau_n}\right) = \hat{V}(x, y) + \int_0^{t\wedge\tau_n} \nabla\hat{V}(Z_s)\mathrm{diag}(1, \epsilon^2)\, d\overline{B}_s + \int_0^{t\wedge\tau_n} \mathcal{L}^L\hat{V}(Z_s)\, ds, \tag{39}$$

where $\overline{B}_t$ denotes $(B_t^{(1)}, B_t^{(2)})$.

We have, a.s., $\tau_n \to \tau$, for $n \to \infty$, hence $\tau_n \wedge t \to \tau \wedge t$. Now, since $\hat{V}, Z_t$, as well as the stochastic integral are continuous, we get - employing that the last integral in (39) vanishes by assumption -

$$\hat{V}\left(Z_{t\wedge\tau}\right) = \hat{V}(x, y) + \int_0^{t\wedge\tau} \nabla\hat{V}(Z_s)\mathrm{diag}(1, \epsilon^2)\, d\overline{B}_s. \tag{40}$$

Hence, $\hat{V}\left(Z_{t\wedge\tau}\right)$ is a bounded local martingale, hence a true martingale, even uniformly integrable. Therefore

$$\mathbf{E}_{x,y}\left[\hat{V}\left(Z_{t\wedge\tau}\right)\right] = \hat{V}(x, y) \tag{41}$$

holds. Now, one easily checks by Ito's Lemma (see also Proposition 3.1 [8]) that $-e^{-2X_{t\wedge\tau}}$ is a local supermartingale, bounded above and below. Therefore, this process is a true supermartingale, hence $\lim_{t\to\infty} -e^{-2X_{t\wedge\tau}}$ exists a.s., and $\lim_{t\to\infty} X_{t\wedge\tau}$ exists a.s. as well. Clearly, on the set $\{\tau = \infty\}$ this limit can not be finite, and we get

$$\begin{aligned} \lim_{t\to\infty} X_t &= \infty, \\ \lim_{t\to\infty} Y_t &= \infty, \end{aligned} \tag{42}$$

on $\{\tau = \infty\}$, since the same considerations hold for the process Y_t as well. All together, $\hat{V}_{/\partial G} = 0$, the third equation of (38) and (42) gives, after $t \to \infty$ in (41),

$$\mathbf{E}_{x,y}\left[\hat{V}(Z_\tau)\right] = \mathbf{P}_{x,y}(\tau = \infty) = \hat{V}(x, y),$$

which finishes the proof. □

Our final theorem asserts that the strategy $\hat{u}$ leads indeed to a uniform approximation of the value function of the problem.

Theorem 4.2 *The target functional, which we get by using the strategy* $\hat{u} = \binom{\mathbf{1}_{II}}{\mathbf{1}_I}$*, with the sets* I *and* II *defined in (35), i.e.* $\hat{V}(x, y) = \mathbf{P}_{x,y}\left(\tau^{\hat{u}} = \infty\right)$ *is a uniform approximation of the value function* $V(x, y)$ *for the problem (1)-(3):*

$$\left|\left|V(x, y) - \hat{V}(x, y)\right|\right|_{L^\infty(G)} \leq C(-\epsilon \ln \epsilon),$$

for some positive, ϵ*-independent constant* C*.*

Proof. An application of Proposition 4.1 and (37) yields

$$\left|\left|V(x, y) - \hat{V}(x, y)\right|\right|_{L^\infty(G)} \leq \left|\left|V(x, y) - \tilde{V}(x, y)\right|\right|_{L^\infty(G)} + \left|\left|\tilde{V}(x, y) - \hat{V}(x, y)\right|\right|_{L^\infty(G)} \leq C(-\epsilon \ln \epsilon),$$

concluding our proof. □

Acknowledgements: Support by the "Austrian Science Foundation" (Fonds zur Förderung der wissenschaftlichen Forschung), Project nr. P26487, is gratefully acknowledged.

References

[1] H. Albrecher, P. Azcue and N. Muler, *Optimal dividend strategies for two collaborating insurance companies*, Adv. in Appl. Probab. 49, no. 2 (2017), pp. 515-548.

[2] F. Avram, Z. Palmowski and M. Pistorius, *A two-dimensional ruin problem on the positive quadrant*, Insurance Math. Econom. 42, no. 1 (2008), pp. 227-234.

[3] W. Eckhaus, *Asymptotic analysis of singular perturbations*, Studies in Mathematics and its Applications 9, North-Holland Publishing Co., Amsterdam-New York, (1979).

[4] W.H. Fleming and H.M. Soner, *Controlled Markov processes and viscosity solutions*, Second edition, Stochastic Modelling and Applied Probability 25, Springer, New York, (2006).

[5] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, (1998).

[6] P. Grandits, *A two-dimensional dividend problem for collaborating companies and an optimal stopping problem*, Scand. Actuar. J. 2019, no. 1, pp. 80-96.

[7] P. Grandits, *An Alexandrov-Bakelman-Pucci estimate for an anisotropic Laplacian with positive drift in unbounded domains*, Preprint TUWIEN (2020), submitted, available at https://fam.tuwien.ac.at/∼pgrand/

[8] P. Grandits, *A ruin problem for a two-dimensional Brownian motion with controllable drift in the positive quadrant*, Teor. Veroyatn. Primen. 64, no. 4 (2019), pp. 811-823; reprinted in Theory Probab. Appl. 64, no.4 (2019), pp. 646-655.

[9] N. V. Krylov, *Controlled diffusion processes*, Translated from the 1977 Russian original, Stochastic Modelling and Applied Probability 14, Springer-Verlag, Berlin, (2009).

[10] H.P. McKean and L.A. Shepp, *The advantage of capitalism vs. socialism depends on the criterion*, Journal of Mathematical Sciences, 139, no. 3 (2006), pp. 6589-6594.