Content

Non Life Insurance Mathematics

I.) Basic Probability Theory

Randomness plays a central role in insurance math.:

Ex:
- occurence of a claim (e.g. death, ...)
- time of the occurence
- height of the claim
- number of claims in a certain time interval

$\Rightarrow$ Repetition of some basic Probability Theory !

I 1) Probability Space, Random Variables

Starting point: Probability triple $(\Omega, \mathcal{F}, \mathbb{P})$

- set of all elementary events : Ω, $\omega \in \Omega$
- $\mathcal{F}$ ... σ-algebra on Ω: set of all observable events (measurable sets)
- $\mathbb{P}$ ... measure on $\mathcal{F}$:

  $\mathbb{P}: \mathcal{F} \longrightarrow [0,1]$ with

  (i) $\mathbb{P}(\Omega) = 1$, $\mathbb{P}(\emptyset) = 0$

  (ii) $\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$ for $A_i \cap A_j = \emptyset$

  σ – additive

Remark:

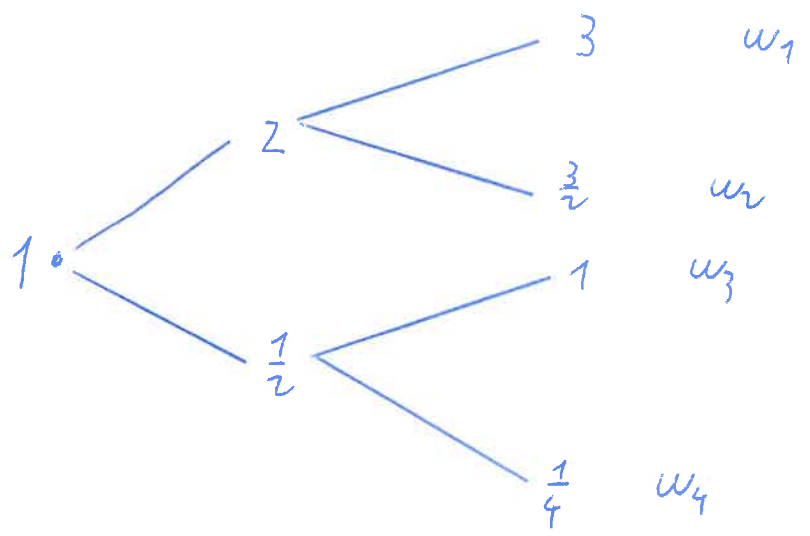
- $\mathcal{F}$ describes the information structure of an „experiment",
  i.e. by observation one can decide, whether an even occures or not.

- sometimes this structure changes in time
  (usually more information at later times) $\Rightarrow$

- Extension of the triple to $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t=0}^{T}, \mathbb{P})$
  where $\mathbb{F}$ is a filtration, i.e. a growing family of σ-algebras:

$$s < t \Rightarrow \mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$$

Ex: stock price



$\Omega = \{w_1, w_2, w_3, w_4\}$

$P(w_i) = \frac{1}{4}$

$\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \{\emptyset, \Omega, \{w_1, w_2\}, \{w_3, w_4\}\}$, $\mathcal{F}_3 = \mathcal{P}(\Omega)$

$\downarrow$

powerset

a random quantity will be described by a <u>random variable</u>,

i.e. a Borel-mb. map $\Omega \to E$, E ... state space (usually: $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{N}$, ...)

<u>Ex:</u>

- R ... damage of a <u>risk</u> (insurance policy) / period
- $B = (R_1, \dots R_n)$ ... <u>insurance portfolio</u>
- $S = R_1 + \dots R_n$ ... <u>total loss</u>
- N ... <u>number of claims</u> in a portfolio
- X ... <u>individual claim</u>

state space: $E = \mathbb{R}$ ... R, S, X

$\mathbb{N}_0$ ... N

$\mathbb{R}^n$ ... B

I.2) Distribution function and density

Def.: F is called <u>distribution function</u> of a real-valued r.v. (random variable) X, if

$F: \mathbb{R} \to [0,1]$ with $F(x) = \mathbb{P}(X \le x)$, s.t. we have

- $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$
- $F(\infty) = \lim_{x \to \infty} F(x) = 1$
- $F(x) \le F(y)$ for $x \le y$ (<u>monotonicity</u>)
- $\lim_{x \searrow x_0} F(x) = F(x_0)$ .... <u>right continuity</u>

- in some of the older literature: $F(x) := \mathbb{P}(X < x)$ ($\Rightarrow F$ is left continuous)

__Def.__ If there $\exists$ a x_p with $F_X(x) = p$: x_p ... p-quantil of F

$\exists$ several types of random variables (not uniquely defined in the literature):

1.) continuous random variable

if:

- $F' = f$ $\exists$ and f is continuous
- $F(x) = \int_{-\infty}^{x} f(s)\,ds$
- $f \geq 0$ (because $F \nearrow$)
- $1 = \int_{-\infty}^{\infty} f(s)\,ds$

Lemma: $X \triangleq f_X$, $Y = h(X)$, $h \ldots$ strictly monotone and $\in C^1$

$\Rightarrow$ density of Y: $g_Y(y) = f(h^{-1}(y)) \cdot |h^{-1}(y)'|$

Remark: in $\mathbb{R}^d$: h is bijective and $|\cdot| \longrightarrow$ functional determinant

2.) discrete random variable

state space consists of at most denumerable many values $\{x_i\}_{i \in I}$ without accumulation point $\Rightarrow$

distr. function F is a step function, with jumps at x_i (heights: $\mathbb{P}(\{x_i\})$)

We have:

- $p_i := \mathbb{P}(X = x_i) > 0$
- $F(x) = \sum_{i:\, x_i \leq x} p_i$
- $1 = \sum_{i \in I} p_i$

3.) mixed random variables

- derivative of F does not exist at at most denumerable points $\{x_i\}$
- $f(x) = F'(x)$ for $x \neq x_i$ ... modified density

We have:
- $f(x) \geq 0$, $p_i > 0$
- $F(x) = \int_{-\infty}^{x} f(s)\,ds + \sum_{i: x_i \leq x} p_i$
- $1 = \int_{-\infty}^{\infty} f(s)\,ds + \sum_{i \in I} p_i$

Remark: Unification with the help of the <u>Stieltjes integral</u>

g continuous: $\displaystyle\int_{-\infty}^{\infty} g(x)\,dF(x) := \int_{-\infty}^{\infty} g(x) f(x)\,dx + \sum_{i\in I} g(x_i)\,p_i$

Rem: can be defined for more general functions g (Baire functions)

Ex:

$$1 = \int_{-\infty}^{\infty} 1\,dF(x) = \begin{cases} \displaystyle\int_{-\infty}^{\infty} f(x)\,dx & X \text{ cont. r.v.} \\ \displaystyle\sum_{i\in I} p_i & X \text{ discrete r.v.} \\ \displaystyle\int_{-\infty}^{\infty} f(x)\,dx + \sum_{i\in I} p_i & X \text{ mixed r.v.} \end{cases}$$

I.3) Expectations

Def.: $$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \, dF(x) = \begin{cases} \int_{-\infty}^{\infty} x f(x) \, dx & X \text{ continuous} \\ \sum_{i \in I} x_i p_i & X \text{ discrete} \\ \int_{-\infty}^{\infty} x f(x) \, dx + \sum_{i \in I} x_i p_i & X \text{ mixed} \end{cases}$$

as long as the r.h.s. is well defined $\cancel{(\infty - \infty)}$

Ex: Cauchy r.v.: $f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$: $\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{1+x^2} \, dx$

(in the sense of a „principal value" one can set the integral above equal to 0

$$\lim_{N \to \infty} \int_{-N}^{N}$$

One has :

- $\mathbb{E}[aX + bY] = a \cdot \mathbb{E}[X] + b\,\mathbb{E}[Y]$, $a, b \in \mathbb{R}$: Linearity
- $\mathbb{E}[h(X)] = \int_{-\infty}^{\infty} h(x)\, dF(x)$ h ... measurable fct.

Some terms:

- n-th moment : $\mathbb{E}[X^n]$
- n-th central moment : $\mathbb{E}[(X - \mathbb{E}[X])^n]$

measures of variation

- $n = 2$: variance : $\mathbb{V}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \ldots = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$
- standard deviation : $\sqrt{\mathbb{V}(X)}$

relative meas. of variation : $VKO(X) := \dfrac{\sqrt{\mathbb{V}(X)}}{|\mathbb{E}[X]|}$ for $\mathbb{E}[X] \neq 0$

if $\mathbb{E}[X] = 0$: one could take, e.g., $\dfrac{\sqrt{\mathbb{V}(X)}}{\mathbb{E}[|X|]}$

- <u>standardized r.v.</u>: $X^* = \dfrac{X - \mathbb{E}[X]}{\mathbb{V}(X)} \Rightarrow \mathbb{V}(X^*) = 1, \quad \mathbb{E}[X^*] = 0$

- <u>skewness</u> $SCH(X) = \dfrac{\mathbb{E}[(X - \mathbb{E}[X])^3]}{\sqrt{\mathbb{V}(X)}} = \mathbb{E}[(X^*)^3]$

  - measure for the deviation from symmetry
  - <u>Rem:</u> Exponent: $\frac{3}{2}$ $\Rightarrow$ $SCH(X)$ is dimensionless
  - $SCH(X)$ is invariant w.r.t. affine transformations
  - $SCH(X) > 0$ <u>skewed to the right</u>

    $SCH(X) < 0$ <u>" " left</u>

Covariance measures dependence structures

Def:
- $\mathrm{cov}(X,Y) = \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$
- X, Y ... uncorrelated if $\mathrm{cov}(X,Y)=0$

One has:
- $\mathbb{V}(X+Y) = \mathbb{V}(X) + \mathbb{V}(Y) + 2\,\mathrm{cov}(X,Y)$
- $\mathbb{V}(X) = \mathrm{cov}(X,X)$
- Let $\vec{X} = (X_1, \dots, X_n) \Rightarrow \mathrm{cov}(X_i, X_j)$: Variance-Covariance matrix

Rem: Neglecting dependence structures can be „dangerous"

If $\mathrm{COV}(X,Y) > 0 \Rightarrow \mathbb{V}(X+Y) \overset{!}{>} \mathbb{V}(X) + \mathbb{V}(Y)$

Extreme Case: $X = Y \Rightarrow \mathbb{V}(X+Y) = 2 \cdot (\mathbb{V}(X) + \mathbb{V}(Y))$

(even worse, if we have more than two r.v.)

$\Rightarrow$ E.g.: Worse estimation in the Tschebychev Inequality

$Z = X+Y$

$$\mathbb{P}(|Z - \mathbb{E}[Z]| > \varepsilon) \leq \frac{\mathbb{V}(Z)}{\varepsilon^2} \quad !$$

I.4) Independence

Let the distribution function of the vector $\vec{X} = (X_1, \ldots X_n)$ be defined by

$$F_{\vec{X}}(x_1, \ldots, x_n) = \mathbb{P}(X_1 \leq x_1, \ldots, X_n \leq x_n)$$

Def: X_i are independent, if:

$$\vec{F}(x_1, \ldots x_n) = \prod_{i=1}^{n} F_i(x_i), \quad \text{where} \quad F_i(x_i) = \mathbb{P}(X_i \leq x_i)$$

- continuous r.v.: $f_{\vec{X}}(x_1, \ldots x_n) = \prod_{i=1}^{n} f_i(x_i)$, $\quad f_i$ density of X_i
- discrete r.v. $\quad \mathbb{P}(X_1 = x_1, \ldots X_n = x_n) = \prod_{i=1}^{n} \mathbb{P}(X_i = x_i)$

One has: X, Y independent $\Rightarrow \mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)] \cdot \mathbb{E}[h(Y)]$ (1)

g, h ... measurable fct.

$\Rightarrow$ Independence $\Rightarrow$ Uncorrelatedness

Proof.: $\operatorname{cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \overset{(1)}{=} \mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[X]\mathbb{E}[Y] = 0$

$\Leftarrow$ only for Gaussian r.v.

I.5) Conditional Probabilities and Expectations

Let $A, B \in \mathcal{F} \Rightarrow \mathbb{P}(A|B) := \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$ if $\mathbb{P}(B) \neq 0$

Consider now r.v. Y under the condition $X = x$: $Y/_{X=x}$

- distribution in the <u>discrete case</u>: $p(y|x) = \mathbb{P}(Y=y|X=x) = \dfrac{\mathbb{P}(X=x, Y=y)}{\mathbb{P}(X=x)}$

- <u>Continuous Case</u>: $g(y|x) = \dfrac{f(x,y)}{f_1(x)}$

with: f … joint density

f_1 … marginal density: $f_1(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

Consider now for each x the expectation and variance of the r.v. $Y/X=x$

i.e: $\mathbb{E}[Y/X=x]$, $\mathbb{V}(Y/X=x)$

these are called:
- conditional expectation
- " — variance

A move general view

Conditional expectation w.r.t. a σ-algebra

Let Y be an $\mathcal{F}$-mb. r.v. with $\mathbb{E}[|Y|] < \infty$

Let $\mathcal{G} \subset \mathcal{F}$ be a sub σ-algebra of $\mathcal{F}$

then the r.v. $Z = \mathbb{E}[Y|\mathcal{G}]$ is called conditional expectation w.r.t. $\mathcal{G}$

if: a) Z is $\mathcal{G}$ mb.

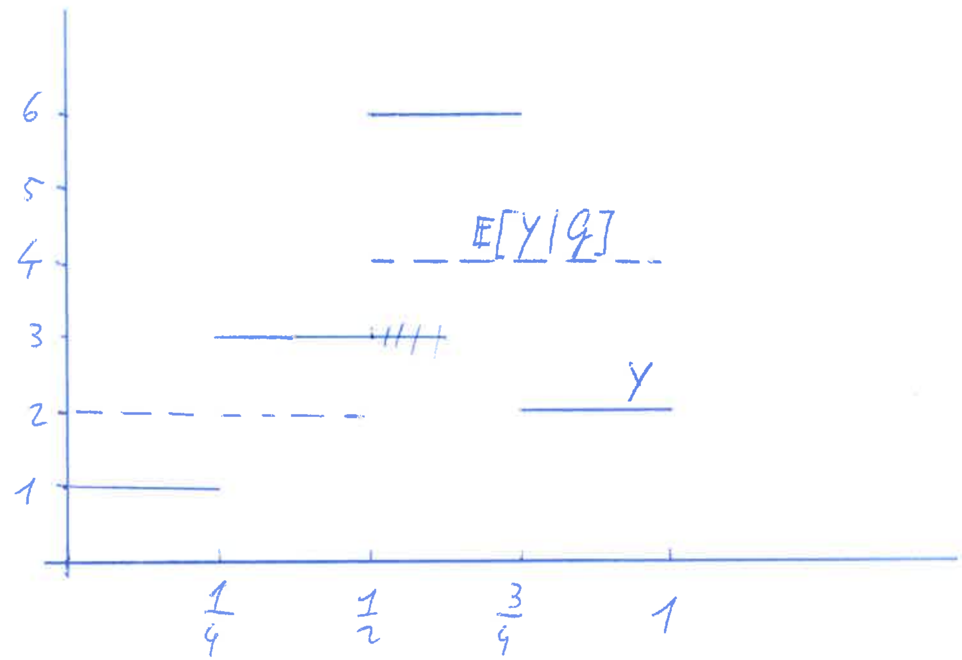
b) $\mathbb{E}[|Z|] < \infty$

c) $\forall\, G \in \mathcal{G}: \int_G Y \, d\mathbb{P} = \int_G Z \, d\mathbb{P}$

Remark:

- $\mathbb{E}[Y|\mathcal{G}]$ is the best approximation for Y among all $\mathcal{G}$-mb. r.v. in the space $L^2(\mathbb{P})$
- the r.v. $\mathbb{E}[Y|\mathcal{G}]$ is $\mathbb{P}$-a.s. unique

Ex:



$\mathcal{F} = \mathcal{P}(\{[0,\tfrac{1}{4}), [\tfrac{1}{4},\tfrac{1}{2}), [\tfrac{1}{2},\tfrac{3}{4}), [\tfrac{3}{4},1]\})$

$\mathcal{G} = \mathcal{P}(\{[0,\tfrac{1}{2}), [\tfrac{1}{2},1]\})$

Important rules

Let $\mathcal{H}$ be a sub σ-algebra of $\mathcal{G}$ $\Rightarrow$

- $\mathbb{E}[\mathbb{E}[Y|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[Y|\mathcal{H}]$ $\quad$ $\mathcal{H}$ … trivial σ-algebra $\Rightarrow$

- $\mathbb{E}[\mathbb{E}[Y|\mathcal{G}]] = \mathbb{E}[Y]$

- <u>furthermore:</u> Let $\mathcal{G}$ be independent of Y $\Rightarrow \mathbb{E}[Y|\mathcal{G}] = \mathbb{E}[Y]$

- <u>especially</u>: $\mathcal{G} = \sigma(X)$ … the σ-algebra generated by X

$\Rightarrow$ as above: $\mathbb{E}[Y|\mathcal{G}] =: \mathbb{E}[Y|X]$

Hence: $\mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}[Y]$

Lemma: $\mathbb{V}(Y) = \mathbb{V}(\mathbb{E}[Y|X]) + \mathbb{E}[\mathbb{V}[Y|X]]$

Proof: $\mathbb{V}[Y|X] = \mathbb{E}[Y^2|X] - (\mathbb{E}[Y|X])^2 \Rightarrow$

$$\mathbb{E}[\mathbb{V}[Y|X]] = \mathbb{E}[\mathbb{E}[Y^2|X]] - \mathbb{E}[(\mathbb{E}[Y|X])^2] =$$

$$\underline{\mathbb{E}[Y^2] - \mathbb{E}[(\mathbb{E}[Y|X])^2]} \qquad (1)$$

$$\mathbb{V}[\mathbb{E}[Y|X]] = \mathbb{E}[(\mathbb{E}[Y|X])^2] - (\mathbb{E}[\mathbb{E}[Y|X]])^2$$

$$= \underline{\mathbb{E}[(\mathbb{E}[Y|X])^2] - (\mathbb{E}[Y])^2} \qquad (2)$$

$(1) + (2) \Rightarrow \mathbb{V}[Y] = \mathbb{E}[\mathbb{V}[Y|X]] + \mathbb{V}[\mathbb{E}[Y|X]]$ □

I.6) Convolution

Let X, Y be indep. r.v. : ? distribution of $X+Y$?

1.) discrete case

assume: $X \mathrel{\hat{=}} p_X(x)$, $Y \mathrel{\hat{=}} p_Y(y)$ we want: p_{X+Y}

$$p_{X+Y}(z) = \mathbb{P}(X+Y=z) = \mathbb{P}\Big(\bigcup_{x+y=z} \{X=x, Y=y\}\Big) \overset{\sigma\text{-add.}}{=} \sum_{x+y=z} \mathbb{P}(\{X=x, Y=y\}) \overset{X,Y \text{ indep.}}{=}$$

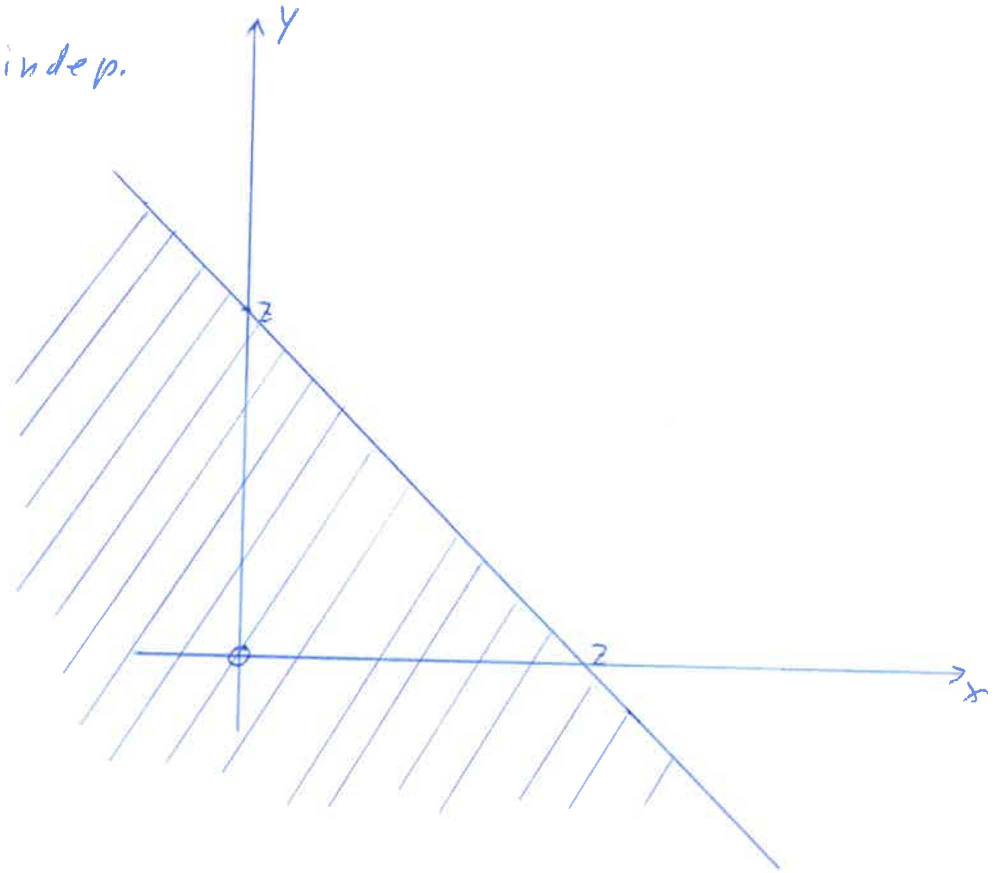
$$= \sum_{x+y=z} \mathbb{P}(X=x)\,\mathbb{P}(Y=y) = \sum_x \mathbb{P}(X=x)\,\mathbb{P}(Y=z-x) = \underline{\sum_x p_X(x) \cdot p_Y(z-x) =: (p_X * p_Y)(z)}$$

Convolution

Symmetry $\Rightarrow$ $(p_X * p_Y)(z) = \sum_y p_Y(y) \cdot p_X(z-y)$

2.1 Continuous Case

$X \triangleq F_X$ resp. f_X $\quad$ $Y \triangleq F_Y$ resp. f_Y $\quad$ indep.



$\Rightarrow \underline{\underline{F_{X+Y}(z)}} = \mathbb{P}(X+Y \leq z) = \left| \begin{array}{l} g(x,y)\ldots \\ \text{joint density} \end{array} \right|$

$= \iint_{x+y \leq z} g(x,y)\, d(x,y) \overset{\text{indep.}}{=} \int_{-\infty}^{\infty} dy \int_{-\infty}^{z-y} dx\, f_X(x) f_Y(y) =$

$\underline{\underline{\int_{-\infty}^{\infty} dy\, f_Y(y) \cdot F_X(z-y)}} \Rightarrow$

$\underline{f_{X+Y}(z)} = \int_{-\infty}^{\infty} f_Y(y) f_X(z-y)\, dy \overset{\text{Symmetry}}{=} \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x)\, dx =: \underline{(f_X * f_Y)(z)}$

I.7) Moment generating and cumulant generating functions

Def.: Let $X \geq 0$ r.v. $\Rightarrow$ $M(t) := \mathbb{E}[e^{tX}]$ is called moment generating function

$\forall t$, where the $\mathbb{E}$ $\exists$!

- We have: If $M(t)$ $\exists$ on an interval of pos. length $\Rightarrow$ distribution of X is uniquely determined
- $M(0) = 1$
- Name? formal calculation: $\mathbb{E}[e^{tX}] = \mathbb{E}\left[\sum_{k=0}^{\infty} \frac{(tX)^k}{k!}\right] =$

$$\sum_{k\geq 0}^{\infty} \frac{t^k}{k!} \mathbb{E}[X^k]$$

on the other hand: Taylor $\Rightarrow$ $M(t) = \sum_{k=0}^{\infty} \frac{M^{(k)}(0)}{k!} t^k$

$$\Rightarrow \boxed{M^{(k)}(0) = \mathbb{E}[X^k]}$$

Remarks:

- sufficient for the formal calculation: $\exists\, \varepsilon > 0: \mathbb{E}[e^{\varepsilon X}] < \infty$
- t can be chosen as a complex number, $t \in \mathbb{C}$:

Laplace Transform

one has: $M(t)$ ... analytic in: $\mathrm{Re}(t) < \varrho$ ... $\varrho \in \mathbb{R}_0^+ \cup \{+\infty\}$

Ex:
- Exp. Distribution: $\mathrm{Exp}(\lambda) \Rightarrow \varrho = \lambda$
- Pareto Distribution: $\mathrm{Pa}(\alpha, \beta) \Rightarrow \varrho = 0$
- Normal Distribution: $\mathcal{N}(\mu, \sigma) \Rightarrow \varrho = \infty$

- t ... purely imaginary: Fourier transform

no problems with $\exists$ of $\mathbb{E}$! $(|e^{isX}| = 1)$

- <u>Useful for Convolutions</u> !

Let $X \triangleq M_X(t)$, $Y \triangleq M_Y(t)$, X, Y indep.

$$\Rightarrow \underline{M_{X+Y}(t)} = \mathbb{E}\left[e^{t(X+Y)}\right] \underset{\text{indep.}}{=} \mathbb{E}\left[e^{tX}\right]\mathbb{E}\left[e^{tY}\right] = \underline{M_X(t)\, M_Y(t)}$$

Def: $L(t) := \ln M(t)$ ... <u>cumulant generating function</u>

One has (Exercise) $\mathbb{E}[X] = L'(0)$

$$\mathbb{V}(X) = L''(0)$$

$$\mathbb{E}\left[(X - \mathbb{E}[X])^n\right] = L^{(n)}(0)$$

I.8) Point estimator for parameters

Def: Sample: $\underline{X} := (X_1, \dots X_n)$ $\quad X_i$ indep. r.v.; $X_i \sim X$ (often iid)

Realisation $\underline{x} := (x_1, \dots x_n)$

Assumption: classical statistics! $\exists$ a true parameter θ!

for $X \sim F(x|\theta)$

$\theta \dots$ unknown

?: estimator $\hat{\Theta}(\underline{X})$?

Def: If we have: $\mathbb{E}[\hat{\Theta}(X_1, \dots X_n)] = \theta$: $\hat{\Theta}$ is called unbiased

„minimal requirement": otherwise systematic errors!

<u>Def.</u>: $\lim_{n\to\infty} \mathbb{E}[\hat{\Theta}(X_1, \ldots, X_n)] = \theta$ ... <u>asymptotically unbiased</u>

<u>Ex</u>:

- $\overline{X}_n := \dfrac{\sum_{i=1}^{n} X_i}{n}$ ... <u>sample mean</u> : unbiased for the expectation of X

- $S_n^2 := \dfrac{1}{(n-1)} \sum_{i=1}^{n} (X_i - \overline{X}_n)^2$ ... <u>sample variance</u> : unbiased for $\mathbb{V}(X)$

<u>Def</u>: an unbiased estimator is <u>efficient</u>, if it has minimal variance among all unbiased estimators

- under certain assumptions:

$$VCOV(\hat{\Theta}) \geq \left(-\mathbb{E}\left[\frac{\partial^2 \ln \ell(\theta|\underline{X})}{\partial \theta^2}\right]\right)^{-1} =: I_\theta^{-1}$$

Hessian matrix

I_θ ... Fisher information matrix

- estimators, where we have equality above:

<u>Frechet-Cramer-Rao efficient</u> (FRC-efficient)

- most common estimators: moment estimators, maximum likelihood estimator (ML)

– moment estimators: $? : \theta \in \mathbb{R}^m$

  - estimate the moments of X (e.g.: $\mathbb{E}[X] \longleftrightarrow \overline{X}_n$
    $\mathbb{V}(X) \longleftrightarrow S_n^2$
  - express parameters by the moments
  - replace moments by their estimators

Ex: $X \perp Exp(\lambda)$ ... $f(x|\lambda) = \lambda \cdot e^{-\lambda x}$ $\Rightarrow m = 1$

$$\lambda = \frac{1}{\mathbb{E}[X]} \Rightarrow \hat{\lambda} = \frac{1}{\overline{X}_n}$$

disadvantage:
- which moments should we take ??
- unbiased, efficient ?? not clear

– ML estimator:

Likelihood function: joint density of the sample, evaluated at the realisation, i.e.:

$$\ell(\theta|\underline{x}) = \begin{cases} \prod_{i=1}^{n} f(x_i|\theta) \; \dots & X \text{ continuous} \\ \prod_{i=1}^{n} p(x_i|\theta) \; \dots & X \text{ discrete} \end{cases} \qquad X_i \text{ iid}$$

ML-estimator : maximizer of $\ell(\theta|\underline{x})$

Remarks :
- sometimes one maximizes $\ln \ell$ instead of ℓ
- sometimes complicated to calculate ! BUT:

Theorem under certain assumptions we have for the ML-estimator $\hat{\Theta}_n$:

1.) $\hat{\Theta}_n$ is asymptotically unbiased : $\lim_{n\to\infty} \mathbb{E}[\hat{\Theta}_n] = \theta$

2.) $\hat{\Theta}_n$ is strongly consistent : $\lim_{n\to\infty} \hat{\Theta}_n = \theta$ a.s.

3.) $\hat{\Theta}_n$ is asymptotically normal : $\lim_{n\to\infty} \dfrac{\hat{\Theta}_n - \mathbb{E}[\hat{\Theta}_n]}{\sqrt{\mathbb{V}(\hat{\Theta}_n)}} \overset{(d)}{=} \mathcal{N}(0,1)$

4.) $\hat{\Theta}_n$ is asymptotically FRC-efficient : $\lim_{n\to\infty} \left(V\mathrm{cov}(\hat{\Theta}_n) - I_\theta^{-1} \right) = 0$

Remarks to 2.) a) • <u>local result</u> : Le Cam:

$\exists$ sequence of local maximizers $\longrightarrow \theta$

Conditions weaker than in b), but:

„What is the correct sequence"??, esp. if $m > 1$!

b) <u>global result</u> : Wald: strong assumptions

$\Rightarrow$ global maximum $\longrightarrow \theta$

guarantee uniqueness of global maximum!

Ex: $X \perp Exp(\lambda)$, but now: $\theta = \frac{1}{\lambda}$

$$\Rightarrow \ell(\theta|\underline{x}) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{x_i}{\theta}} = \frac{1}{\theta^n} e^{-\frac{n\bar{x}_n}{\theta}}$$

$$\Rightarrow \ln \ell(\theta|\underline{x}) = -n \ln\theta - \frac{n\bar{x}_n}{\theta}$$

$$(\ln \ell)' = -\frac{n}{\theta} + \frac{n\bar{x}_n}{\theta^2} \stackrel{!}{=} 0 \Rightarrow \hat{\theta} = \bar{x}_n$$

$$(\ln \ell)'' = \frac{n}{\theta^2} - \frac{2n\bar{x}_n}{\theta^3}\Big|_{\theta=\bar{x}_n} = \frac{n}{\bar{x}_n^2} - \frac{2n\bar{x}_n}{\bar{x}_n^3} = -\frac{n\cancel{\bar{x}_n}}{\bar{x}_n^2} < 0 \checkmark$$

Check the assertions of the theorem: 1.) $\overline{X}_n$ even unbiased:

$$\mathbb{E}[\overline{X}_n] = \mathbb{E}\left[\frac{X_1 + \dots X_n}{n}\right] = \mathbb{E}[X_i] = \theta \checkmark$$

2.) strongly consistent: strong law of large numbers ✓

3.) asymptotically normal: central limit theorem ✓

4.) FRC bound:

$$I_n(\theta) = -\mathbb{E}\left[\frac{\partial^2 \ln \ell}{\partial \theta^2}\right] = -\frac{n}{\theta^2} + \frac{2n}{\theta^2} = \frac{n}{\theta^2}$$

Theorem: $\rightarrow \mathbb{V}(\hat{\theta}_n) - \frac{\theta^2}{n} \longrightarrow 0$??

we have even:

$$\mathbb{V}(\hat{\theta}_n) - \frac{\theta^2}{n} = \frac{1}{n^2} n \underbrace{\mathbb{V}(X)}_{\theta^2} - \frac{\theta^2}{n} \equiv 0 \checkmark$$

Remark: If we estimate $Vcov(\hat{\Theta}_n)$ by $I_n(\theta)^{-1}$ one replaces θ by its estimator $\hat{\Theta}_n$, but usually $\theta \approx \hat{\Theta}_n \Rightarrow I_n(\theta) \approx I_n(\hat{\Theta}_n)$

I.g) χ^2 - Test

- test to check an assumption about a distribution
- Zero hypothesis: $\mathcal{H}_0 : X \hat{=} F(x|\theta)$
- Test with an <u>error probability of the first kind : small</u>, e.g $0.05 = \alpha$

  i.e.: $\mathbb{P}(\text{reject hypothesis} \mid \mathcal{H}_0 \text{ is correct}) = \alpha$

under certain assumptions: following procedure:

Let $\underline{x}$ be a realisation $(\in \mathbb{R}^n)$

- decomposition of the state space M_X in disjoint classes K_i:

  $M_X = \bigcup_{i=1}^{r} K_i$ with $r > m$ $(\theta \in \mathbb{R}^m!)$

- estimate the unknown parameter (ML, etc...) $\Rightarrow \hat{\Theta}$

- determine the absolute frequency o_i, $i = 1, \dots r$

  o_i ... number of data from $\underline{x}$ in K_i

- Determine the expected frequency e_i under $\mathcal{H}_0$:

$$e_i = n \cdot \mathbb{P}(X \in K_i \mid \mathcal{H}_0), \quad X \triangleq F(x \mid \hat{\Theta}) \qquad (e_i \geq 5 !!)$$

- determine the test statistics Z_n

$$Z_n := \sum_{i=1}^{r} \frac{(O_i - e_i)^2}{e_i}$$

If $\mathcal{H}_0$ is correct $\Rightarrow$ $Z_n \overset{(d)}{\Longrightarrow} \chi^2_{r-m-1}$

i.e. $\mathbb{P}(Z_n \le \chi_{r-m-1;p}) \xrightarrow[n\to\infty]{} p$ $\Rightarrow$

χ^2-distrib is continuous ! p-quantil

- Test with asympt. error probability α of the first kind : ($p = 1-\alpha$!)

$\rightarrow Z_n < \chi_{r-1-m;p} \Rightarrow$ accept $\mathcal{H}_0$

$\rightarrow Z_n \ge \chi_{r-1-m;p} \Rightarrow$ reject $\mathcal{H}$

I.10) Bayes Statistics – introduction

- classical statistics: $\exists$ a true parameter: $X \triangleq F(x|\theta)$, $\theta \in \mathbb{R}^m$
- Bayes statistics: • parameter is described by a r.v. Θ
  - distribution of Θ before the „experiment": a-priori distribution $\pi(\theta)$
    - e.g. claim data without considering the own claim data
  - Model: $X/_{\Theta=\theta} \triangleq F(x|\theta)$

    $\Theta \triangleq \pi(\theta)$

- „experiment": consider the own data $\Rightarrow$ $\underline{x}$ ... sample
- distribution after the experiment: <u>a-posteriori distr.</u> $\pi(\theta|\underline{x})$

hence: $\Theta|_{\underline{X}=\underline{x}} \sim \pi(\theta|\underline{x}) = ???$

<u>Bayes Theorem:</u> if $\pi(\theta)$ is continuous $\Rightarrow$

$$\pi(\theta|\underline{x}) = \frac{\ell(\theta|\underline{x}) \cdot \pi(\theta)}{\int_{\mathbb{R}^m} \ell(\theta|\underline{x}) \cdot \pi(\theta) \, d\theta}$$

<u>Proof:</u> numerator: joint dens. of $(\underline{X}, \Theta)$

denominator: marginal distribution of $\underline{X}$

<u>discrete case:</u> $\pi(\theta|\underline{x}) = \dfrac{\ell(\theta|\underline{x}) \cdot \pi(\theta)}{\sum\limits_{\theta} \ell(\theta|\underline{x}) \cdot \pi(\theta)}$

- in both cases: denominator is a constant w.r.t. θ!

$$\Rightarrow \pi(\theta|\underline{x}) \underset{\text{prop.}}{\propto} \ell(\theta|\underline{x}) \cdot \pi(\theta)$$

<u>Useful!</u> often it is clear that the numerator belongs to a certain known distribution

$\Rightarrow$

it is not necessary to calculate the denominator

Ex: <u>Poisson - Gamma model</u>

$X|_{\Lambda=\lambda} \sim Po(\lambda)$ $\qquad \Lambda \sim Ga(\alpha, \beta)$, i.e.:

$$\pi(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda\beta} \qquad \mathbb{P}(X=x \mid \Lambda=\lambda) = e^{-\lambda} \frac{\lambda^x}{x!}, \quad x \in \mathbb{N}_0$$

$$\Rightarrow \pi(\lambda \mid \underline{x}) \propto \pi(\lambda)\, \ell(\lambda \mid x_1, \dots, x_n) \propto \lambda^{\alpha-1} e^{-\lambda\beta} \prod_{i=1}^{n} \lambda^{x_i} e^{-\lambda}$$

$$= \lambda^{\alpha + \sum_{i=1}^{n} x_i - 1} \, e^{-\lambda(\beta+n)} \qquad \Rightarrow$$

$$\Lambda \mid \underline{X}=\underline{x} \sim Ga\left(\alpha + \sum_{i=1}^{n} x_i, \beta+n\right)$$

This is an <u>example for conjugated families</u> of distributions

Def: Let $\mathcal{F}$ be a class of a-priori distr.

If the a-posteriori distribution belong also to $\mathcal{F}$ $(\forall \underline{x})$

$\Rightarrow$ $\underline{\mathcal{F}}$ is <u>conjugated</u> to the <u>distribution of $X|\Theta$</u>

Parameter estimation:

- <u>a-posteriori Bayes estimator</u>: $\hat{\Theta} = \mathbb{E}[\Theta \mid \underline{X} = \underline{x}]$

  (a-posteriori expectation!)

  in the example above: $\hat{\lambda} = \dfrac{\sum_{i=1}^{n} x_i + \alpha}{n + \beta}$

- <u>Usage of loss functions</u>

  idea: penalize the difference between θ and $\hat{\Theta}$!, i.e.

  given: <u>loss function</u>: $L(\theta, \hat{\theta})$

  goal: $\int_{\mathbb{R}^m} L(\theta, \hat{\Theta}(\underline{x}))\, \pi(\theta \mid \underline{x})\, d\theta \longrightarrow \min$, i.e

  <u>Minimize the expected a-posteriori loss</u>!

Examples for loss functions

$$L(\theta, \hat{\theta}(\underline{x})) = \begin{cases} (\hat{\theta}(\underline{x}) - \theta)^2 & \ldots \quad \text{quadratic loss function} \\ |\theta(\underline{x}) - \theta| & \ldots \quad \text{absolute loss function} \\ |\theta(\underline{x}) - \theta|^p & p > 0 \\ \left.\begin{matrix} 0 \\ 1 \end{matrix}\right\} \text{ for } \begin{matrix} \theta = \hat{\theta} \\ \theta \neq \hat{\theta} \end{matrix} & \text{only meaningful in the discrete case!} \end{cases}$$

- Ex: (quadratic loss function): $\int_{\mathbb{R}^m} (\theta - \hat{\theta})^2 \pi(\theta|\underline{x})\, d\theta \longrightarrow \min$

$\frac{\partial}{\partial \hat{\theta}}$: $\Rightarrow -2 \int_{\mathbb{R}^m} (\theta - \hat{\theta})\, \pi(\theta | \underline{x})\, d\theta = 0 \Rightarrow \hat{\theta} = \int \theta\, \pi(\theta | \underline{x})\, d\theta$

... a posteriori Bayes estimator !

$\frac{\partial^2}{\partial \hat{\theta}^2}$: > 0 ✓

i.e.: $\boxed{\text{Usage of the } \underline{\text{quadratic loss function}} \Rightarrow \text{optimal estimator: } \underline{\text{a posteriori Bayes estimator}}}$

Rem: for the absolute loss function one gets the median of the a posteriori distr.

(Exercise)

Description of the (unconditional) distribution of after data, hence $X/_{\underline{X}=\underline{x}}$

- predictive distribution: $f(x|\underline{x})$
- averaging the conditional density of $X/_{\Theta=\theta}$ with the a posteriori distribution

$$\Rightarrow \quad f(x|\underline{x}) = \int_{\mathbb{R}^m} f(x|\theta)\,\pi(\theta|\underline{x})\,d\theta$$

- compare: uncond. distribution before data: averaging with a-priori distr.

hence: $\int_{\mathbb{R}^m} f(x|\theta)\cdot\pi(\theta)\,d\theta$

I.11 Remarks to modelling

1.) Formulate an exact question

2.) real problem → math. model

compromise:
- all relevant ingredients of reality should be incorporated in the model
- simple enough for solvability

3.) statistics
- estimation of parameters
- tests

4.) solution of the math. problem

5.) check whether the solution is plausible (realistic?, check dependence on parameter variation!)

Example: Insurance company wants to invest in the stock market

1. What is the aim of the company?

   e.g.: a) minimize the probability of ruin

   b) maximize dividend payments

2. real world → math. modell

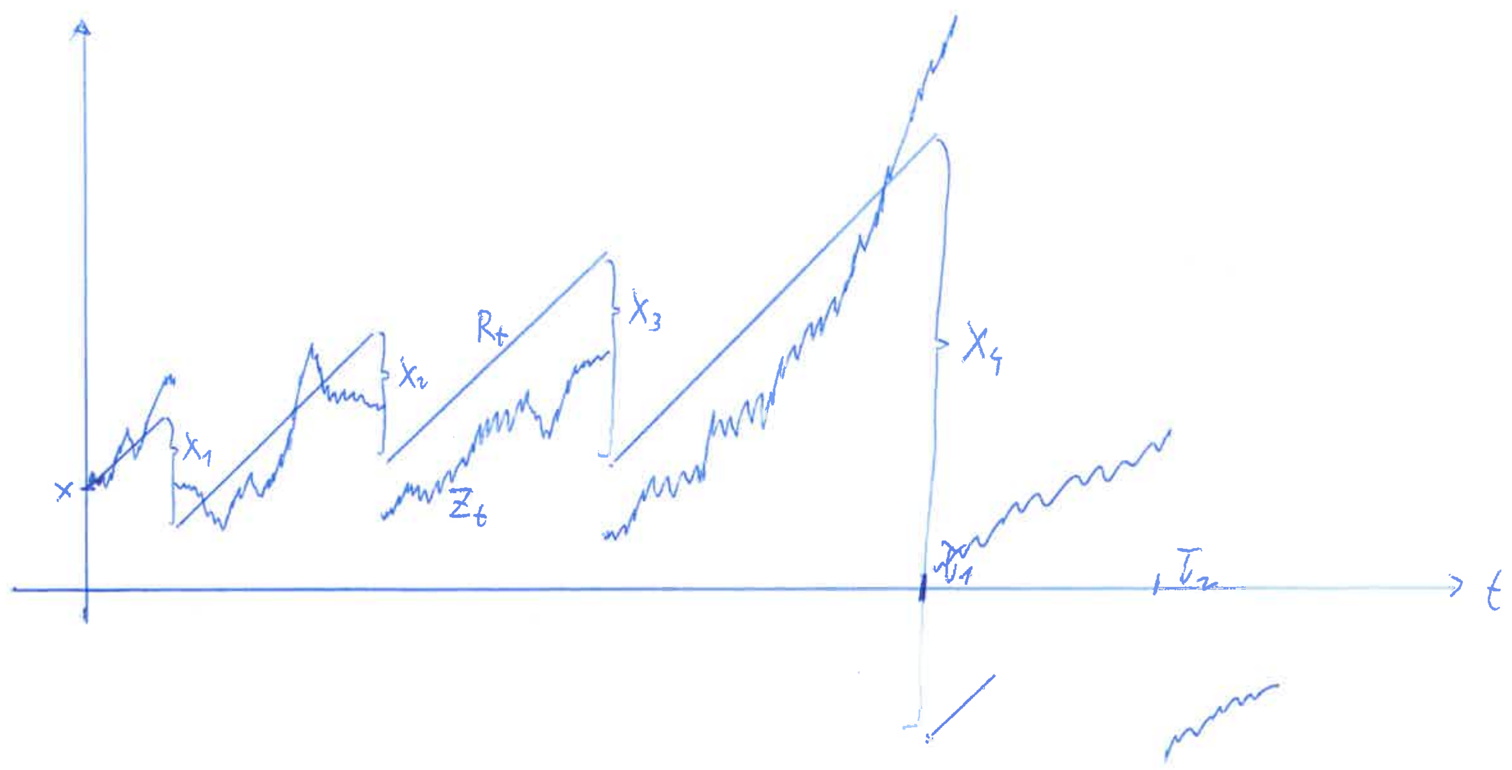
endowment of the company: $R_t = x - \sum_{i=1}^{N_t} X_i + ct$

x … initial endowment

N_t … number of claims until time t

c … premium density

X … individual claims



stock price S_t ... Black-Scholes model

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t \quad ; \quad \mu, \sigma \in \mathbb{R}^+$$

W_t ... Brownian motion

θ_t ... number of stocks in the portfolio

interest rate: $r=0$ (for convenience)

$\Rightarrow$ Endowment after the investment:

$$Z_t = x - \sum_{i=1}^{N_t} X_i + ct + \underbrace{\int_0^t \theta_s \cdot dS_t}_{\text{Gain of the stock trading}}$$

Goal: a) from 1.) $\mathbb{P}(\tilde{\tau}_2 < \infty) \to \min$ (1)

$$\tau_2 = \inf\{t>0 \mid Z_t < 0\} \quad \text{... time of ruin}$$

3 <u>statistics</u> : estimate the parameters: c, μ, σ and the distribution of N_t, X_i from time series.

4 <u>solution of (1)</u>

„stochastic control problem" (see lecture in the winter term)

$\Rightarrow$ for a certain class of claim distribution:

it is optimal to invest a certain fraction (dep. on the model parameters) in the stock

5 <u>Is the solution plausible</u> :

e.g.: is it true that: $\sigma \nearrow \Rightarrow$ fraction above $\searrow$

II.) Distribution of the aggregate loss

Policy portfolio: $B = (R_1, R_2, \dots, R_n)$, R_i risks (policies): loss / period and policy

often summarized as policy matrix, if many identical risks are in the portfolio:

Example: R_i characterized by $\underbrace{\text{insured sum}}_{IS}$ and $\underbrace{\text{probability of occurence}}_{p}$

$$\mathbb{P}(R = IS) = p$$
$$P(R = 0) = 1 - p$$

$\downarrow p$ \ $IS \rightarrow$	10000	50000
0.01	37	43
0.05	98	52

Aggregate loss: $S = R_1 + \dots R_n$ (usually time period = 1 year)

Models for S:

- individual model
- collective model
- approximations

II.1) Individual model

$$S = \sum_{i=1}^{n} R_i$$

often: R_i are independent, but not neccesarily iid

Principal of Collective Balancing : larger policy portfolios lead to a distribution of S with a smaller variability, i.e. VKO $\searrow$

Ex: R_i iid $$VKO(S) = \frac{\sqrt{V(S)}}{|\mathbb{E}[S]|} \underset{\text{iid}}{=} \frac{\sqrt{n \cdot V(R_i)}}{|n \cdot \mathbb{E}[R_i]|} = \frac{1}{\sqrt{n}} \cdot \frac{\sqrt{V(R_i)}}{|\mathbb{E}[R_i]|} = \frac{1}{\sqrt{n}} VKO(R_i)$$

i.e.: relative measure of variability $\longrightarrow 0$ for $n \to \infty$

Illustration with the help of Chebychev's inequality

Lemma (Cheb. ineq.) X r.v. with $\mathbb{E}[X^2] < \infty \Rightarrow \mathbb{P}(|X - \mathbb{E}[X]| \geq \varepsilon) \leq \dfrac{V(X)}{\varepsilon^2}$

Application: $\underline{\underline{\mathbb{P}\left(\dfrac{|S - \mathbb{E}[S]|}{\mathbb{E}[S]} \geq \varepsilon\right)}} = \mathbb{P}(|S - \mathbb{E}[S]| \geq \varepsilon \mathbb{E}[S]) \underset{\text{Cheb. in.}}{\leq} \dfrac{V(S)}{(\varepsilon \mathbb{E}[S])^2} = \underline{\underline{\dfrac{VKO(S)}{\varepsilon^2}}}$

ε fixed! $n \to \infty$!! $\Rightarrow$ $\xrightarrow[n \to \infty]{} 0$

„for large portfolios the relative deviation from the expectation $\to 0$"

- <u>strong law of large numbers</u>

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{\sum R_i}{n} = \mathbb{E}[R]\right) = 1$$

– more of theoretical interest, since no quantitative estimate about the probability of deviation (as in Cheb. inequality)

- Expectation of the risk is <u>not</u> sufficient as premium!

$$\frac{S-\mathbb{E}[S]}{\sqrt{\mathbb{V}(S)}} \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0,1) \Rightarrow$$

$$\mathbb{P}(S > \mathbb{E}[S]) \approx 0.5 \Rightarrow \underline{\text{safety loading}}\ c$$

Estimates for the „ruin probability" for R_i, resp. S:

- Risk: $\mathbb{P}(R > \mathbb{E}[R]+c) \le \mathbb{P}(|R-\mathbb{E}[R]| \ge c) \overset{\text{Cheb.}}{\le} \frac{\mathbb{V}(R)}{c^2}$

  $\geq$! for large n

- aggregate loss: $\mathbb{P}(S > \mathbb{E}[S]+n\cdot c) \le \mathbb{P}(|S-\mathbb{E}[S]| \ge nc) \le \frac{\mathbb{V}(S)}{n^2c^2} = \frac{\mathbb{V}(R)}{n\,c^2}$

II.2) Collective Model

Consider not the risks (policies) but the individual claims

X_i ... ~~height~~ size of an individual claim

N ... claim number : number of ind. claims / period

$\Rightarrow$ aggregate loss is a random sum

$$S = \sum_{i=1}^{N} X_i$$

Assumptions : $X_i \stackrel{d}{=} X$; X_i iid ; X_i indep. of N

$N = 0$ ... no claim ! $\Rightarrow$ " $\sum_{i=1}^{0} X_i := 0$ " convention

Distribution models for X

typically: (exception: large reinsurance companies, e.g. SWISSRE, ...)

- many small claims
- a few large claims
- right-skewed distributions

math. classification: large claim $\forall \varepsilon > 0: \mathbb{E}[e^{\varepsilon X_i}] = \infty$

small claim $\exists \varepsilon_0 > 0: \mathbb{E}[e^{\varepsilon_0 X_i}] < \infty$

i.e.: small claims have exponential moments

(moment generating function $\exists$)

- *Examples*: <u>Large claims</u>: Pareto, generalized Pareto, log normal, Burr, heavy Weibull distribution

  <u>Small claims</u>: exponential, mixture of exponential (i.e.: $F(x) = \sum_{i=1}^{n} \alpha_i F_i(x)$, $F_i \sim Exp(\gamma_i)$, $\sum \alpha_i = 1$)

  Γ- distribution

Distribution models for N

most important:

- Binomial distr.: $B(n,p)$: $\mathbb{P}(N=k) = \binom{n}{k} p^k (1-p)^{n-k}$, $p \in (0,1)$
- Poisson distr.: $Po(\lambda)$: $\mathbb{P}(N=k) = e^{-\lambda} \frac{\lambda^k}{k!}$
- negative Binomial distr.: $NB(\alpha,p)$: $\mathbb{P}(N=k) = \binom{\alpha+k-1}{k} p^\alpha (1-p)^k$, $\alpha > 0$, $p \in (0,1)$

Remark: • NB: $\lim\limits_{k\to\infty} \frac{\mathbb{P}(N=k)}{\mathbb{P}(N=k-1)} = \lim\limits_{k\to\infty} \frac{\frac{\Gamma(\alpha+k)}{k!\,\cancel{\Gamma(\alpha)}} \cancel{p^\alpha} (1-p)^{\cancel{k}}}{\frac{\Gamma(\alpha+k-1)}{(k-1)!\,\cancel{\Gamma(\alpha)}} \cancel{p^\alpha} \cancel{(1-p)^{k-1}}} = \lim\limits_{k\to\infty} \frac{(\alpha+k-1)(1-p)}{k} = (1-p) < 1$!

$$\Rightarrow \sum_{k=0}^{\infty} \mathbb{P}(N=k) < \infty$$, approx. decrease like a geom. distr.

- vague idea which distribution one observes:

$$\left.\begin{array}{lll} \text{if} & B(n,p): & \mathbb{E}[N] > \mathbb{V}(N) \\ & Po(\lambda): & \mathbb{E}[N] = \mathbb{V}(N) \\ & NB(\alpha,p): & \mathbb{E}[N] < \mathbb{V}(N) \end{array}\right\} \Rightarrow$$

Calculate the estimators: $\overline{X}_n, S_n^2$: if

$$\begin{array}{lll} \overline{X}_n > S_n^2 & \text{„}\Rightarrow\text{"} & Bn(n,p) \\ \overline{X}_n \approx S_n^2 & \text{„}\Rightarrow\text{"} & Po(\lambda) \\ \overline{X}_n < S_n^2 & \text{„}\Rightarrow\text{"} & NB(\alpha,p) \end{array}$$

Mixture of distributions :

Assumption : Policy number j : claim distribution $X_j \triangleq f(x|\lambda_j)$

Claim distribution depends on the policy !

$\Rightarrow$ Model this as : λ_j realisation of a random variable Λ !

The distribution of Λ is called <u>structure distribution</u>

if : $\Lambda \triangleq h(\lambda)$ ... density

$$\Rightarrow X \triangleq f(x) = \int_{-\infty}^{\infty} f(x|\lambda)\, h(\lambda)\, d\lambda$$

- similar for claim numbers:

$$N|_{\Lambda=\lambda} \sim p(n|\lambda) \qquad \Lambda \sim h(\lambda) \qquad \Rightarrow \qquad N \sim p(n) = \int_{-\infty}^{\infty} p(n|\lambda)\, h(\lambda)\, d\lambda$$

- Calculate $\mathbb{E}$ and $\mathbb{V}$ with the help of:

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|\Lambda]]$$

$$\mathbb{V}(X) = \mathbb{V}(\mathbb{E}[X|\Lambda]) + \mathbb{E}[\mathbb{V}(X|\Lambda)]$$

<u>Ex:</u> $X_{/\Lambda=\lambda} \sim Exp(\lambda) \qquad \Lambda \sim \Gamma(\alpha,\beta)$

$$\Rightarrow \underline{f(x)} = \int_0^\infty \lambda e^{-\lambda x} \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} d\lambda =$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \lambda^\alpha e^{-\lambda(\beta+x)} d\lambda = \left| \begin{array}{l} (\beta+x)\lambda = z \\ d\lambda = \frac{dz}{\beta+x} \end{array} \right|$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \frac{z^\alpha}{(\beta+x)^\alpha} e^{-z} \frac{dz}{\beta+x} = \frac{\beta^\alpha}{(\beta+x)^{\alpha+1}\Gamma(\alpha)} \underbrace{\int_0^\infty z^\alpha e^{-z} dz}_{\Gamma(\alpha+1)} = \underline{\frac{\alpha\beta^\alpha}{(\beta+x)^{\alpha+1}}}$$

$x \geq 0$!

$$\underline{F(x)} = \int_0^x f(z)\,dz = \alpha\beta^\alpha \int_0^x \frac{1}{(\beta+z)^{\alpha+1}}\,dz = \alpha\beta^\alpha(-\alpha)^{-1}(\beta+z)^{-\alpha}\Big|_{z=0}^{x} =$$

$$\underline{1-\frac{\beta^\alpha}{(\beta+x)^\alpha}} \quad \Rightarrow \quad X \triangleq Pa(\alpha,\beta)$$

Pareto distribution

Rem: Mixture of small claims yields a large claim!

(Pareto has no exp. moments!)

$$\underline{\mathbb{E}[X]} = \mathbb{E}[\mathbb{E}[X|\Lambda]] \overset{\text{exp. distr.}}{=} \mathbb{E}\left[\frac{1}{\Lambda}\right] = \int_0^\infty \frac{1}{\lambda}\frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}\,d\lambda = \left| \begin{array}{l} \beta\lambda = z \\ d\lambda = \frac{dz}{\beta} \end{array} \right|$$

$$= \frac{\beta^{\alpha-1}}{\Gamma(\alpha)} \underbrace{\int_0^\infty \left(\frac{z}{\beta}\right)^{\alpha-2} e^{-z}\,dz}_{\Gamma(\alpha-1)/\beta^{\alpha-2}} = \underline{\frac{\beta}{\alpha-1}} \quad , \; \exists \text{ only for } \alpha > 1 \,!$$

$$V(X) = \mathbb{E}[V(X|\Lambda)] + V(\mathbb{E}[X|\Lambda]) \overset{\text{exp. distr.}}{=}$$

$$\mathbb{E}\left[\frac{1}{\Lambda^2}\right] + V\left(\frac{1}{\Lambda}\right) = \mathbb{E}\left[\frac{1}{\Lambda^2}\right] + \mathbb{E}\left[\frac{1}{\Lambda^2}\right] - \left(\mathbb{E}\left[\frac{1}{\Lambda}\right]\right)^2$$

$$= 2\,\mathbb{E}\left[\frac{1}{\Lambda^2}\right] - \frac{\beta^2}{(\alpha-1)^2} \qquad (1)$$

$$\underline{\mathbb{E}\left[\frac{1}{\Lambda^2}\right]} = \int_0^\infty \frac{1}{\lambda^2}\frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}\,d\lambda = \left|\begin{array}{l}\beta\lambda = z\\ d\lambda = \frac{dz}{\beta}\end{array}\right| = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty \left(\frac{z}{\beta}\right)^{\alpha-3} e^{-z}\frac{dz}{\beta} =$$

$$\frac{\beta^2}{\Gamma(\alpha)}\Gamma(\alpha-2) = \underline{\frac{\beta^2}{(\alpha-1)(\alpha-2)}}$$

$$(1) \Rightarrow \underline{V(X)} = \frac{2\beta^2}{(\alpha-1)(\alpha-2)} - \frac{\beta^2}{(\alpha-1)^2} = \frac{\beta^2}{(\alpha-1)}\left[\frac{2}{(\alpha-2)} - \frac{1}{(\alpha-1)}\right] = \frac{\beta^2}{(\alpha-1)}\,\frac{\cancel{2\alpha}\;2\alpha - \cancel{2} - \alpha + \cancel{2}}{(\alpha-1)(\alpha-2)} = \underline{\frac{\alpha\beta^2}{(\alpha-1)^2(\alpha-2)}}$$

for $\alpha > 2$

Basics of Reinsurance

Reinsurance: anoter ins. comp. takes part of the risk

advantages
- reduction of the payments for claims (but payments to reinsurance comp.!)
- reduction of the variance of the portfolio
- increase of the safety (e.g. ruin prob. $\searrow$)

Basic forms:

Quota share reinsurance: each single claim is decomposed:

$$X = X_E + X_R$$

X_E ... paid by ins. comp.

X_R ... paid by reinsurer

with: $X_E = \alpha X$ $\quad X_R = (1-\alpha) X$ $\quad \alpha \in (0,1)$

α ... deductible

$(1-\alpha)$ ... quota

¿ distributions, $\mathbb{E}$, $\mathbb{V}$?

$$F_{X_E}(x) = \mathbb{P}(X_E \leq x) = \mathbb{P}(\alpha X \leq x) = \mathbb{P}\left(X \leq \frac{x}{\alpha}\right) = F_X\left(\frac{x}{\alpha}\right)$$

$$F_{X_R}(x) = \ldots = F_X\left(\frac{x}{1-\alpha}\right)$$

$$\mathbb{E}[X_E] = \alpha\, \mathbb{E}[X] \qquad \mathbb{E}[X_R] = (1-\alpha)\,\mathbb{E}[X]$$

$$\mathbb{V}(X_E) = \alpha^2\, \mathbb{V}(X) \qquad \mathbb{V}(X_R) = (1-\alpha)^2\, \mathbb{V}(X)$$

Excess of loss Reinsurance (XL - reinsurance)

starting point: individual claims

$$\left.\begin{array}{l}\text{Insur. comp.: takes everything until the priority (retention level) } M \\ \text{Reinsurer : takes the rest}\end{array}\right\} \Rightarrow$$

$$X_E = \begin{cases} X, & X \leq M \\ M, & X > M \end{cases}$$

$$X_R = \begin{cases} 0, & X \leq M \\ X - M, & X > M \end{cases}, \text{ i.e.: } X_R = (X-M)^+$$

Remark: even if X has a continuous distr. $\Rightarrow$

X_E, X_R have mixed distributions (~~no~~ probability mass at $X_E = M$, resp. $X_R = 0$!)

i.e.: $\mathbb{P}(X_E = M) = \mathbb{P}(X > M) > 0$ in general

$\mathbb{P}(X_R = 0) = \mathbb{P}(X \leq M) > 0$ in general

Calculation of $\mathbb{E}[X_E]$ (assuming X has density f)

$$\mathbb{E}[X_E] = \int_0^\infty x \, dF_{X_E}(x) = \int_0^M x f(x)dx + M \cdot \mathbb{P}(X_E = M) =$$

$$= \int_0^M x f(x)dx + M \cdot (1 - F_X(M)) =$$

$$= \int_0^\infty x f(x)dx - \int_M^\infty x f(x)dx + M \cdot \int_M^\infty f(x)dx =$$

$$= \mathbb{E}[X] - \int_M^\infty (x - M) f(x)\,dx = \mathbb{E}[X] - \int_0^\infty y f(y+M)\,dy = \mathbb{E}[X_E]$$

$X = X_E + X_R$

$$\Rightarrow \quad \mathbb{E}[X_R] = \int_0^\infty y f(y+M)\,dy$$

$V(X_E)$, $V(X_R)$ … see Exercises

influence of inflation ???

Let i ... inflation rate in %, let $k = 1 + \frac{i}{100}$

Assumption: M will not be adjusted $\Rightarrow$

$$X_E = \begin{cases} kX & kX \leq M \\ M & kX > M \end{cases} \Rightarrow$$

$$\mathbb{E}[X_E] = \int_0^{M/k} kx\,f(x)\,dx + M \int_{M/k}^{\infty} f(x)\,dx =$$

$$\int_0^{\infty} kx\,f(x)\,dx - k\left(\int_{M/k}^{\infty} \left(x f(x) - \frac{M}{k} f(x)\right) dx \right) =$$

$$k \cdot \left(\mathbb{E}[X] - \int_0^{\infty} y\,f\left(y + \frac{M}{k}\right) dy \right),$$ (for $k = 1$ we get the formula above)

XL from the point of view of the reinsurer

Ex.: $M = 100$

$x_i = 23,\ 68,\ 187,\ 54,\ 106,\ 143,\ 88,\ 122,\ 45 \Rightarrow$

$(x_i)_R = 0,\ 0,\ 87,\ 0,\ 6,\ 43,\ 0,\ 22,\ 0$

i.e. • reinsurer sees only 4 claims

• from his point of view: better description by

$$Z_R := X_R\big|_{X>M} = (X-M)\big|_{X>M}$$

distribution of Z_R ??

$$F_{Z_R}(z) = \mathbb{P}(Z_R \le z) = \mathbb{P}(X \le M+z \mid X > M) = \frac{\mathbb{P}(M < X \le M+z)}{\mathbb{P}(X > M)} =$$

$$= \frac{F_X(M+z) - F_X(M)}{1 - F_X(M)} \quad \overset{\frac{d}{dz}}{\Longrightarrow}$$

$$f_{Z_R}(z) = \frac{f_X(M+z)}{1 - F_X(M)}$$

The claim number changes too!

in the Example: $10 \to 4$

Distribution of N_R ?? (Claim number of the reinsurer)

Def: $\mathbb{1}_{\{X>M\}} := \begin{cases} 1 & X > M \\ 0 & X \leq M \end{cases}$ a r.v. with alternative distribution with parameter $p = \mathbb{P}(X>M)$

$$\Rightarrow \sum_{i=1}^{n} \mathbb{1}_{\{X_i > M\}} \stackrel{\Delta}{=} Bn(n,p)$$

One has: $N_R = \sum_{i=1}^{N} \mathbb{1}_{\{X_i > M\}}$ N is r.v.! $\Rightarrow$

$$\underline{\mathbb{P}(N_R = k)} = \sum_{n=k}^{\infty} \mathbb{P}(N_R = k \mid N = n)\, \mathbb{P}(N=n) = \underline{\sum_{n=k}^{\infty} \mathbb{P}(N=n) \cdot \binom{n}{k} p^k (1-p)^{n-k}}$$

For the standard claim numbers:

$$\left.\begin{array}{l} N \triangleq Bn(m, \bar{p}) \Rightarrow N_R \triangleq Bn(m, p\bar{p}) \\ N \triangleq Po(\lambda) \Rightarrow N_R \triangleq Po(\lambda p) \end{array}\right\} \text{Proof, see Exerc.}$$

$$N \triangleq NB(\bar{\alpha}, \bar{p}) \Rightarrow N_R \triangleq NB\left(\bar{\alpha}, \frac{\bar{p}}{\bar{p} + p(1-\bar{p})}\right) \qquad (*)$$

Proof of (*): $\mathbb{E}[e^{tN_R}] = \mathbb{E}[\mathbb{E}[e^{tN_R} | N]] \underset{\substack{\text{cond. distr. is Binomial} \\ (q := 1-p)}}{=} \mathbb{E}[(e^t p + q)^N] =$

$$\mathbb{E}[e^{\ln(e^t p + q) N}] \underset{\substack{\uparrow \\ N \triangleq NB(\bar{\alpha}, \bar{p})}}{=} \left(\frac{\bar{p}}{1-(1-\bar{p})(e^t p + q)} \right)^{\bar{\alpha}} \Rightarrow$$

to show:

$$\left(\frac{\bar{p}}{1-(1-\bar{p})(e^t p + q)} \right)^{\cancel{\bar{\alpha}}} \overset{?}{=} \left(\frac{\frac{\bar{p}}{\bar{p}+p(1-\bar{p})}}{1-\left(1-\frac{\bar{p}}{\bar{p}+p(1-\bar{p})}\right)e^t} \right)^{\bar{\alpha}} = \left(\frac{\frac{\bar{p}}{\bar{p}+p(1-\bar{p})}}{1-\frac{p(1-\bar{p})}{\bar{p}+p(1-\bar{p})}e^t} \right)^{\cancel{\bar{\alpha}}}$$

$$\Leftrightarrow$$

$$\frac{\cancel{\bar{p}}}{1-(1-\bar{p})(e^t p + q)} \overset{?}{=} \frac{\cancel{\bar{p}}}{\bar{p}+p(1-\bar{p}) - p(1-\bar{p})e^t}$$

Comparison of the denominator:

Coefficient of e^t: $-(1-\bar{p})p \overset{?}{=} -p(1-\bar{p})$ ✓

" " 1: $1-(1-\bar{p})q \overset{?}{=} \bar{p} + p(1-\bar{p})$

$$\Leftrightarrow$$

$$1-(1-\bar{p})(1-p) \overset{?}{=} \bar{p} + p - p\bar{p}$$

$$\Leftrightarrow$$

$$\bar{p} + p - p\bar{p} \overset{?}{=} \bar{p} + p - p\bar{p}$$ ✓

$\square$

Parameter estimation in the XL-reinsurance

situation : insurance company has saved only the own payments !

in the example above: $x_i = 23, 68, 100, 54, 100, 100, 88, 100, 45$

estimation of the parameter ???

problem with the method of moments : no information about the values above 100

$\Rightarrow$ choose adapted ML method

sample: $\underline{x} = (x_1, \ldots x_n, \underbrace{\geq M, \geq M, \ldots \geq M}_{m \text{ pieces!}})$

censored sample

(in the ex.: $m = 4$
$n = 5$)

Assumption : continuous distribution

Likelihood: $\ell(\theta|\underline{x}) = \prod_{i=1}^{n} f(x_i|\theta) \cdot \prod_{i=1}^{m} \mathbb{P}_\theta(X_i \geq M)$

$$= \prod_{i=1}^{n} f(x_i|\theta) \cdot \prod_{i=1}^{m} (1 - F_X(M|\theta))$$

Find θ, s.t. $\ell(\theta|\underline{x}) \to \max$!

Stop loss reinsurance (SL)

- starting point: aggregate loss S

- $S = S_E + S_R$

- analogously to XL: $(X \to S)$

$$S_E = \begin{cases} S & S \le d \\ d & S > d \end{cases} \qquad S_R = \begin{cases} 0 & S \le d \\ (S-d) & S > d \end{cases} = (S-d)^+$$

d ... stop loss point

Assumption: $\mathbb{N}_0$ valued distribution (resp. discrete distrib. with equidistant distances)

- Recursive Calculation of the net SL premium: $\mathbb{E}[S_R(d)]$ (as a function of d!)

$$\mathbb{E}[S_R(d)] = \sum_{s=d+1}^{\infty} (s-d)\,\mathbb{P}(S=s) = \sum_{s=0}^{\infty} (s-d)\,\mathbb{P}(S=s) - \sum_{s=0}^{d} (s-d)\,\mathbb{P}(S=s)$$

$$= \mathbb{E}[S] - d + \sum_{s=0}^{d-1} (d-s)\,\mathbb{P}(S=s) \qquad (1)$$

$$\Rightarrow \mathbb{E}[S_R(d+1)] = \mathbb{E}[S] - (d+1) + \sum_{s=0}^{d} (d+1-s)\,\mathbb{P}(S=s) \qquad (2)$$

$(2)-(1) \Rightarrow$

$$\mathbb{E}[S_R(d+1)] - \mathbb{E}[S_R(d)] = -1 + \mathbb{P}(S=d) + \sum_{s=0}^{d-1} \mathbb{P}(S=s)$$

$$= -1 + \sum_{s=0}^{d} \mathbb{P}(S=s) = F_S(d) - 1$$

$$\left.\begin{array}{l} \Rightarrow \mathbb{E}[S_R(d+1)] = \mathbb{E}[S_R(d)] - (1 - F_S(d)) =: \mathbb{E}[S_R(d)] - \bar{F}_S(d) \\ \quad \mathbb{E}[S_R(0)] = \mathbb{E}[S] \ \ldots \ \text{initial value} \end{array}\right\}$$

analogous formula $\exists$ for $\mathbb{V}(S_R(d))$!

(uses : $\mathbb{E}[S_R^2(d+1)] = \mathbb{E}[S_R^2(d)] - 2\mathbb{E}[S_R(d)] + \bar{F}_S(d)$, see Ex.

deductible of a policy holder

ins. company pays everthing above the deductible L, i.e.

$$X_W = \begin{cases} 0 & X \leq L \\ X-L & X > L \end{cases}$$

analogously to XL reinsurance:

ins.comp. $\longrightarrow$ policy holder

reinsurer $\rightarrow$ ins. company

Distribution of random sums

E.g.: aggregate loss: $S = \sum_{i=1}^{N} X_i$

$$S_E = \sum_{i=1}^{N} (X_E)_i$$

$$S_R = \sum_{i=1}^{N_R} (Z_R)_i = \sum_{i=1}^{N} (X_R)_i$$

We have:

X discrete $\rightarrow$ S discrete

X continuous $\rightarrow$ S mixed with $\mathbb{P}(S=0) = \mathbb{P}(N=0) > 0$

X mixed $\rightarrow$ S mixed

Calculation of the distribution of S

Assumption: X_i iid $X_i \perp X$; N, X_i indep.

idea: decompose $\{S \le s\} = \bigcup_{n=0}^{\infty} \{S \le s, N=n\} \xRightarrow{\text{indep., } \sigma\text{-add.}}$

$$\underline{\mathbb{P}(S \le s)} = \sum_{n=0}^{\infty} \mathbb{P}(S \le s, N=n) = \sum_{n=0}^{\infty} \mathbb{P}(S \le s \mid N=n) \cdot \mathbb{P}(N=n) =$$

$$\underline{\sum_{n=0}^{\infty} F^{n*}(s)\, \mathbb{P}(N=n)} \quad , \text{ where}$$

$$F^{n*}(s) = \mathbb{P}(X_1 + \dots + X_n \le s) \, , \qquad n \ge 1$$

$$F^{0*}(s) = \left\{ \begin{array}{ll} 0 & s < 0 \\ 1 & s \ge 0 \end{array} \right\} \Rightarrow \mathbb{P}(S=0) = \mathbb{P}(N=0)$$

- X has a discrete distribution on $\mathbb{N}$

$$\underline{\mathbb{P}(S=k)} = F_S(k) - F_S(k-1) = \sum_{n=0}^{\infty} \left(F^{n*}(k) - F^{n*}(k-1) \right) \mathbb{P}(N=n)$$

$$= \underline{\sum_{n=0}^{\infty} p^{n*}(k) \cdot \mathbb{P}(N=n)}, \quad \text{where}$$

$$p^{n*}(k) = \mathbb{P}(X_1 + \dots + X_n = k) \qquad n \geq 1$$

$$p^{0*}(k) = \begin{cases} 1 & k=0 \\ 0 & \text{else} \end{cases}$$

- X has continuous distr.

$$\Rightarrow \begin{cases} \mathbb{P}(S=0) = \mathbb{P}(N=0) \\ + \text{ modified density: } \sum_{n=1}^{\infty} f^{n*}(s) \mathbb{P}(N=n) \end{cases}$$

$f^{n*}(s)$ by induction

$f^{2*}(s)$ see above (convolution integral)

Formulas for $\mathbb{E}[S]$, $V(S)$, $M_S(t)$

$$\underline{\mathbb{E}[S]} = \mathbb{E}[\mathbb{E}[S|N]] = \mathbb{E}[N\,\mathbb{E}[X]] = \underline{\mathbb{E}[N]\,\mathbb{E}[X]}$$

$$\underline{V(S)} = \mathbb{E}[V(S|N)] + V(\mathbb{E}[S|N]) = \mathbb{E}[N \cdot V(X)] + V(N\,\mathbb{E}[X]) =$$

$$\underline{\mathbb{E}[N] \cdot V(X) + (\mathbb{E}[X])^2 \cdot V(N)}$$

$$\underline{M_S(t)} = \mathbb{E}[e^{tS}] = \mathbb{E}[\mathbb{E}[e^{tS}|N]] = \mathbb{E}[e^{t(X_1+\dots+X_N)}] \overset{X_i \text{ ind.}}{=} \mathbb{E}[M_X(t)^N] =$$

$$\mathbb{E}[e^{\ln M_X(t) \cdot N}] = \underline{M_N(\ln M_X(t))}$$

Compound Poisson Distribution (CP-distr.)

- $N \sim Po(\lambda) \Rightarrow S = \sum_{i=1}^{N} X_i$ has a CP-distr. (X_i iid, indep. of N !)

i.e.: $S \sim CP(\lambda, F_X)$, F_X ... distr. function of X_i

We have: $\mathbb{E}[N] = \mathbb{V}(N) = \lambda$, $M_N(t) = e^{\lambda(e^t - 1)}$ (see exerc.)

$$\Rightarrow \underline{\mathbb{E}[S]} = \mathbb{E}[N] \cdot \mathbb{E}[X] = \underline{\lambda \cdot \mathbb{E}[X]}$$

$$\underline{\mathbb{V}(S)} = \mathbb{E}[N]\,\mathbb{V}(X) + (\mathbb{E}[X])^2\,\mathbb{V}(N) = \lambda\,(\mathbb{V}(X) + (\mathbb{E}[X])^2) = \underline{\lambda \cdot \mathbb{E}[X^2]}$$

$$\underline{M_S(t)} = M_N(\ln M_X(t)) = \underline{e^{\lambda \cdot (M_X(t) - 1)}}$$

One has: $\mathbb{E}[X] > 0$, $X \geq 0 \Rightarrow \text{Skewness}(S) > 0$

- Further properties of the CP-distribution

Lemma: $S = S_1 + \dots + S_n$, S_i indep, $S_i \sim CP(\lambda_i, F_i)$

$\Rightarrow S \sim CP(\lambda, F)$, where $\lambda = \sum_{i=1}^{n} \lambda_i$ $\quad F(x) = \frac{1}{\lambda} \sum_{i=1}^{n} \lambda_i F_i(x)$

Proof: $F(x)$ is indeed distribution function, we have:

- $\lim_{x \to \infty} F(x) = \frac{1}{\lambda} \sum_{i=1}^{n} \lambda_i \overbrace{\lim_{x \to \infty} F_i(x)}^{1} = \frac{1}{\lambda} \sum_{i=1}^{n} \lambda_i = 1$ ✓
- $\lim_{x \to -\infty} F(x) = \frac{1}{\lambda} \sum_{i=1}^{n} \lambda_i \lim_{x \to -\infty} F_i(x) = \frac{1}{\lambda} \sum \lambda_i \cdot 0 = 0$ ✓
- monotone nondecreasing: F_i are mon. nond. + $\lambda_i > 0$ ✓
- right continuous, since the F_i are right cont. ✓

Calculate the moment generating function of F!

$$\underline{M(t)} = \int_{-\infty}^{\infty} e^{tx} \cdot dF(x) = \int_{-\infty}^{\infty} e^{tx}\, d\left(\frac{1}{\lambda}\sum_{i=1}^{n} \lambda_i F_i(x)\right) = \qquad \Big|\ d, \int \text{ are linear operators } \Big|$$

$$= \frac{1}{\lambda}\sum_{i=1}^{n} \int_{-\infty}^{\infty} e^{tx} \lambda_i \, dF_i(x) = \underline{\frac{1}{\lambda}\sum_{i=1}^{n} \lambda_i M_i(t)} \qquad M_i \Leftrightarrow F_i$$

on the other hand: $M_S(t) \overset{\text{indep.}}{=} \prod_{i=1}^{n} M_{S_i}(t) = \prod_{i=1}^{n} e^{\lambda_i (M_i(t)-1)} =$

$$e^{\sum_{i=1}^{n} [\lambda_i (M_i(t)-1)]} = e^{\lambda \cdot \left(\sum_{i=1}^{n} \frac{\lambda_i}{\lambda} M_i(t) - 1\right)} = e^{\lambda (M(t)-1)}$$

~~this~~ this is the moment gen. function of $CP(\lambda, F(\cancel{X}))$ ∎

- **Decomposition of a CP distribution, where X_i takes only finitely many values**

Lemma: Let N_i, $i = 1, \dots, n$ indep.

$N_i \triangleq Po(\lambda_i)$ $\quad k_i \neq k_j \quad i \neq j$

$$\Rightarrow \sum_{i=1}^{n} k_i N_i \triangleq CP(\lambda, p(\,)), \text{ where}$$

$$\lambda = \sum_{i=1}^{n} \lambda_i \qquad p(k) = \begin{cases} \frac{\lambda_i}{\lambda} & k = k_i \\ 0 & \text{else} \end{cases}$$

Proof: fix i $\Rightarrow k_i N_i = \sum_{j=1}^{N_i} k_i \triangleq CP(\lambda_i; p_i(k))$

$$\text{where } p_i(k) = \begin{cases} 1 & k = k_i \\ 0 & \text{else} \end{cases}$$

Lemma above
$\Rightarrow$ $$\sum_{i=1}^{n} k_i N_i \triangleq CP\left(\lambda = \sum_{i=1}^{n} \lambda_i \, ; \; P(k) = \sum_{i=1}^{n} \frac{\lambda_i}{\lambda} p_i(k)\right) \qquad \square$$

Remark: If X_i takes only finitely many values

$$\Rightarrow \left| \mathbb{P}(S=k) = \sum_{n=0}^{\infty} p^{n*}(k) \cdot \mathbb{P}(N=n) \right|$$

For fixed k ~~[illegible]~~ it is enough to calculate finitely many convolutions!

Compound Binomial and compound NB distribution

- Let $N \sim Bn(n,p)$ $\quad S = \sum_{i=1}^{N} X_i \quad X_i$ iid, indep. of N

$\Rightarrow \mathbb{E}[N] = np \quad V(N) = npq, \quad (q = 1-p), \quad M_N(t) = (p \cdot e^t + q)^n$

$\Rightarrow \underline{\mathbb{E}[S]} = \mathbb{E}[N] \cdot \mathbb{E}[X] = \underline{np \cdot \mathbb{E}[X]}$

$\underline{V(S)} = \mathbb{E}[N] \cdot V(X) + (\mathbb{E}[X])^2 \cdot V(N) = \underline{np\, V(X) + npq\,(\mathbb{E}[X])^2}$

$\underline{M_S(t)} = M_N(\ln M_X(t)) = \underline{(p \cdot M_X(t) + q)^n}$

Rem: S can be leftskewed (in contrast to CP!)

e.g.: $X \equiv 1 \Rightarrow S = N \Rightarrow SCH(S) = SCH(N) = \underset{\substack{\uparrow \\ \text{exerc.}}}{\ldots} = \dfrac{1-2p}{\sqrt{npq}}$ $\begin{array}{ll} > 0 & p < \frac{1}{2} \\ = 0 & p = \frac{1}{2} \\ < 0 & p > \frac{1}{2} \end{array}$

- Let $N \sim NB(\alpha, p) \overset{ex.}{\Rightarrow}$

$\mathbb{E}[N] = \frac{\alpha(1-p)}{p}$ $\quad V(N) = \frac{\alpha(1-p)}{p^2}$ $\quad M_N(t) = \left(\frac{p}{1-(1-p)e^t}\right)^\alpha$, for $t < \ln\frac{1}{1-p}$

$\Rightarrow$ for $CNB(\alpha, p, F_X)$: $M_S(t) = \left(\frac{p}{1-(1-p)M_X(t)}\right)^\alpha$, for $M_X(t) < \ln\frac{1}{1-p}$

$\frac{d}{dt}, \frac{d^2}{dt^2}, t=0 \Rightarrow$

$\mathbb{E}[S] = \dots$

$V(S) = \dots$

analogously to CP we have $SCH(S) > 0$!

Recursion formula of Panjer (1981)

- Recursion for the distribution of the aggregate loss S, if we have discrete, equidistant single claims; it holds for the most important claim numbers!

- Notation: $p_k = \mathbb{P}(X = k)$, $k \in \mathbb{N}$

  $g_n = \mathbb{P}(N = n)$, $n \in \mathbb{N}_0$

- Assumption: g_n satisfies $g_n = \left(a + \frac{b}{n}\right) g_{n-1}$, $a, b \ldots$ const. $n = 1, 2, \ldots$

„the (a,b)-class"

ad „(a,b) - class“ :

distributions in the (a,b) class

distribution	a	b	g_0
Poisson	0	λ	$e^{-\lambda}$
Binomial	$-\frac{p}{q}$	$(n+1)\frac{p}{q}$	q^n
NB	q	$(\alpha-1)q$	p^α
geometric (spec. case of NB, if $\alpha=1$)	q	0	p

Remark: one can show, that these are the only distrib. in this class (+ the one-point distr.)

One has : $p^{n*}(k) = \mathbb{P}(X_1 + \dots + X_n = k) = 0 \qquad k < n, \quad \text{since } X_i \geq 1$

$$\Rightarrow \mathbb{P}(S=k) = \sum_{n=0}^{k} p^{n*}(k)\, g_n$$

but we still have to calculate k convolutions (troublesome!, see ex.)

$\Rightarrow$

Theorem (Panjer) : Under the assumptions above :

$$\mathbb{P}(S=0) = \mathbb{P}(N=0) = g_0$$

$$P(S=k) = \sum_{j=1}^{k} \left(a + \frac{bj}{k}\right) p_j\, P(S=k-j) \qquad k \in \mathbb{N}$$

Proof: • We show the following 3 relations:

a) $\mathbb{E}\left[X_1 \,\middle|\, \sum_{i=1}^{n} X_i = k\right] = \frac{k}{n}$ for $p_k^{n*} = p^{n*}(k) \neq 0$

b) $\mathbb{E}\left[X_1 \,\middle|\, \sum_{i=1}^{n} X_i = k\right] = \sum_{j=1}^{k} j\, p_j\, p_{k-j}^{(n-1)*} \Big/ p_k^{n*}$ for $p_k^{n*} \neq 0$

c) $g_n\, p_k^{n*} = \sum_{j=1}^{k} \left(a + \frac{bj}{k}\right) p_j\, g_{n-1}\, p_{k-j}^{(n-1)*}$

ad a): $\mathbb{E}\left[X_1 \,\middle|\, \sum_{i=1}^{n} X_i = k\right] \underset{\text{symmetry}}{=} \frac{1}{n}\, \mathbb{E}\left[X_1 + \dots + X_n \,\middle|\, \sum_{i=1}^{n} X_i = k\right] = \frac{k}{n}$ ✓

ad b) $$p_j \, p^{(n-1)*}_{k-j} / p^{n*}_k = \frac{\mathbb{P}(X_1 = j)\, \mathbb{P}(X_2 + \dots + X_n = k-j)}{\mathbb{P}(X_1 + \dots + X_n = k)} =$$

$$\mathbb{P}(X_1 = j \mid X_1 + \dots + X_n = k)$$

if $X_1 + \dots X_n = k \Rightarrow j \in \{1, 2, \dots, k\} \Rightarrow$ b) ✓

ad c) Case 1: $p^{n*}_k = 0$

$\Rightarrow$ l.h.s. (c) $= 0$

r.h.s. (c): $p_j \, p^{(n-1)*}_{k-j} = \mathbb{P}(X_1 = j,\ X_2 + \dots + X_n = k-j) \leq$

$\mathbb{P}(X_1 + \dots + X_n = k) = p^{n*}_k = 0 \Rightarrow$ r.h.s (c) $= 0$ ✓

Case 2: $p_k^{n*} > 0$

$$g_n p_k^{n*} \overset{\text{Ass.}}{=} g_{n-1}\left(a + \frac{b}{n}\right) p_k^{n*} \overset{a)}{=} g_{n-1}\left(a + \frac{b}{k}\mathbb{E}\left[X_1 \,\middle|\, \sum_{i=1}^{n} X_i = k\right]\right) p_k^{n*} =$$

$$\overset{b)}{=} g_{n-1}\left(a + \frac{b}{k}\sum_{j=1}^{k} j\, p_j\, p_{k-j}^{(n-1)*} \big/ p_k^{n*}\right) \cdot p_k^{n*} \overset{\text{Convolution}}{=}$$

$$= g_{n-1}\left(a \cdot \sum_{j=1}^{k} p_j\, p_{k-j}^{(n-1)*} + \frac{b}{k}\sum_{j=1}^{k} j\, p_j\, p_{k-j}^{(n-1)*}\right) =$$

$$= \sum_{j=1}^{k} p_j\, g_{n-1}\, p_{k-j}^{(n-1)*}\left(a + \frac{bj}{k}\right) \Rightarrow \quad c) \checkmark$$

- Finally: ($k>0$, hence summation starts from $n=1$!)

$$\mathbb{P}(S=k) = \sum_{n=1}^{\infty} g_n p_k^{n*} \overset{c)}{=} \sum_{n=1}^{\infty} \sum_{j=1}^{k} \left(a + \frac{bj}{k}\right) p_j \, g_{n-1} \, p_{k-j}^{(n-1)*} =$$

$$\sum_{j=1}^{k} \left(a + \frac{bj}{k}\right) p_j \sum_{n=1}^{\infty} g_{n-1} p_{k-j}^{(n-1)*} = \sum_{j=1}^{k} \left(a + \frac{bj}{k}\right) p_j \, \mathbb{P}(S=k-j) \qquad \square$$

Special case: CP: $a=0$, $b=\lambda$

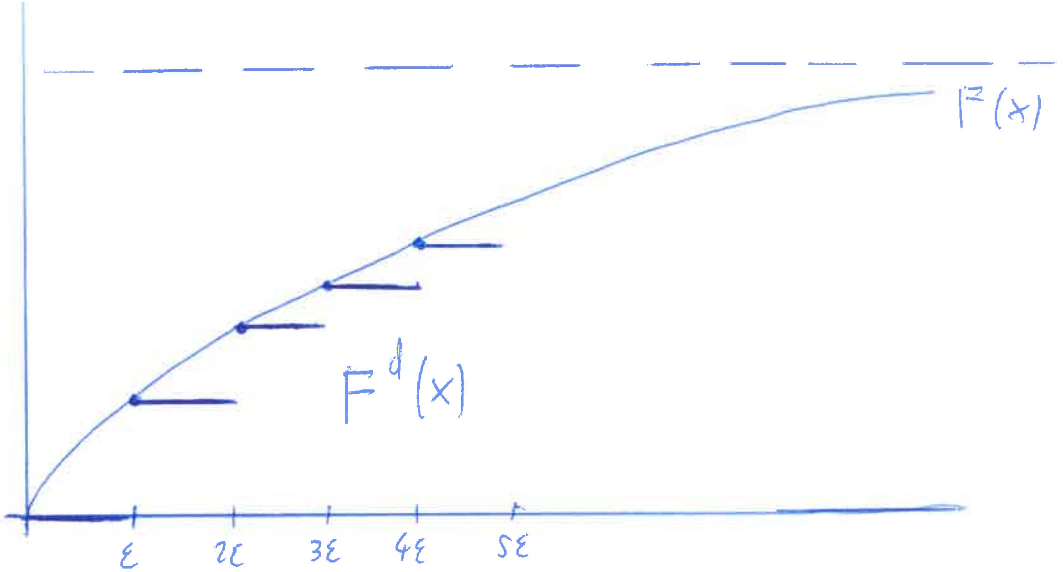
$$\Rightarrow \mathbb{P}(S=0) = \mathbb{P}(N=0) = e^{-\lambda}$$

$$\mathbb{P}(S=k) = \frac{\lambda}{k} \sum_{j=1}^{k} j \, p_j \, \mathbb{P}(S=k-j)$$

„Continuous Claims"

discretize X!

e.g.: X is r.v. in $\mathbb{R}^+$ with distr. function F



Def.: $\mathbb{P}(X^d = k\varepsilon) := \mathbb{P}((k-1)\varepsilon < X \le k\varepsilon)$, ε small!

- X^d has distribution function F^d with jumps at $k\varepsilon$
- X^d has state space $\varepsilon \cdot \mathbb{N}$ (not yet $\mathbb{N}$!)

$\Rightarrow$ <u>Scaling</u>: $X^{dd} := \dfrac{X^d}{\varepsilon}$

- apply <u>Panjer recursion on X^{dd}</u>
- retransformation

Further models for claim number processes

- up to now: $t=1$, now $t>0$, hence N_t ... number of claims until time t

A) inhomogeneous Poisson process

- Poisson process with fluctuating intensity

(e.g.: car dammges $\Leftrightarrow$ seasons

<u>Def.:</u> N_t is an <u>inhomogeneous Poisson process</u>, if we have:

(1) $N_0 = 0$ a.s.

(2) the process has <u>independent increments</u>,

i.e.: let $0 = t_0 < t_1 < \dots < t_n \Rightarrow$

$N(t_{i-1}, t_i] := N_{t_i} - N_{t_{i-1}}$ are independent (for $i \neq j$)

(3) $\exists$ a nondecreasing, rightcontinuous function

$\mu : [0,\infty) \to [0,\infty)$, $\mu(0) = 0$, s.t.

$N(s,t] \cancel{\Rightarrow} \triangleq Po(\mu(s,t])$

μ... <u>mean value function</u>

(4) N_t has a.s. rightcontinuous paths

Remark: • special case (standard) homogeneous Poisson process: $\mu(t) = t$

- $\mu(t)$ can be interpreted as an operational time

- one gets immediately the finite dimensional distributions: $k_i \in \mathbb{N}_0$

$$\mathbb{P}(N_{t_1} = k_1, N_{t_2} = k_1 + k_2, N_{t_3} = k_1 + k_2 + k_3, \dots N_{t_n} = k_1 + \dots + k_n) =$$

$$\mathbb{P}(N_{t_1} = k_1, N(t_1, t_2] = k_2, N(t_2, t_3] = k_3, \dots, N(t_{n-1}, t_n] = k_n) \overset{\text{indep.!}}{=}$$

$$\prod_{i=1}^{n} e^{-\mu(t_{i-1}, t_i]} \cdot \frac{(\mu(t_{i-1}, t_i])^{k_i}}{k_i!} \qquad (\text{convention } 0! = 1)$$

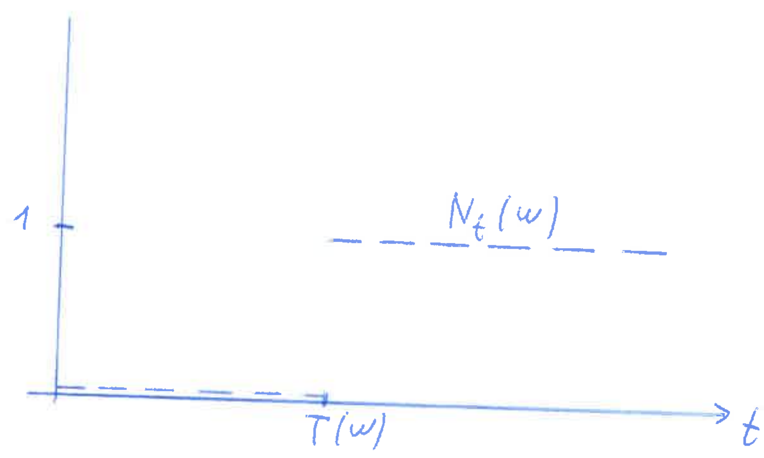
- If $\mu(t)$ is absolutely continuous, i.e. $\mu(s,t] = \int_s^t \lambda(z)\,dz \quad , \quad s < t$

  for some nonnegative λ

  $\Rightarrow$ λ ... intensity

- Let T be „the first arrival time" of N_t

  i.e.: $T = \inf\{t > 0 \mid N_t = 1\}$



$\Rightarrow \mathbb{P}(T > t) = \mathbb{P}(N_t = 0) = e^{-\mu(0,t]} \overset{\text{if } \mu \text{ is abs. continuous}}{=} e^{-\int_0^t \lambda(z)\,dz}$

Remark: (without proof) if $\mu(t)$ is continuous, strictly mon. increasing,

and $\lim_{t \to \infty} \mu(t) = \infty$

$\Rightarrow$ $\underline{N_{\mu^{-1}(t)}}$ is a homogeneous (standard) Poisson process,
(i.e: $\mu(t) = t$ or $\lambda \equiv 1$)

– conditions on $\lambda(s)$: $\lambda(s) > 0$, $\int_0^\infty \lambda(s)\,ds = \infty$

B) Renewal process

Def: a process N_t of the form

$$N_t := \underset{\substack{\uparrow \\ \text{number}}}{\#} \{ i \geq 1 , T_i \leq t \} , \qquad t \geq 0$$

where $T_n = W_1 + \dots + W_n$, $n \geq 1$

with W_i iid nonnegative random var. is called

renewal process

- T_n … arrival times

  W_i … interarrival times

Special Case: $W_i \sim Exp(\lambda)$

- Calculation of the one-dimensional distribution of N_t: $\mathbb{P}(N_t = n) = ?$

One has: $\{N_t = n\} = \{T_n \leq t < T_{n+1}\}$

$$T_n = W_1 + \dots + W_n \sim \Gamma(n, \lambda)$$

$$\Rightarrow \mathbb{P}(T_n \leq t) = 1 - e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} =: F(t) \qquad (1)$$

Check, whether (1) gives the correct density:

$$F'(t) = \lambda \cdot e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} - e^{-\lambda t} \underbrace{\sum_{k=1}^{n-1} \frac{\lambda^k t^{k-1}}{(k-1)!}}_{\lambda \sum_{k=0}^{n-2} \frac{(\lambda t)^k}{k!}} = \lambda \cdot e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} \quad \checkmark$$

is the density of the Γ distr. ☺!

Remark: $\underset{\xi}{\Gamma(n, \lambda)}$, $n \in \mathbb{N}$ ... Erlang Case

$\Rightarrow \mathbb{P}(N_t = n) = \mathbb{P}(T_n \leq t) - \mathbb{P}(T_{n+1} \leq t) \overset{(1)}{=} e^{-\lambda t} \frac{(\lambda t)^n}{n!}$

$\Rightarrow$ the one-dim. distribution is the distribution of $Po(\lambda t)$

(homogeneous Poisson process with intensity λ !)

without proof: N_t is !

Mean value and Variance in the Renewal model

Renewal model: $S_t := \sum_{i=1}^{N_t} X_i$ $\quad (N_t, X_i \text{ indep.})$

i.e.: a collective model (with variable t), where the claim number process is a renewal process

$$\mathbb{E}[N_t] = \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{N_t} X_i \,\middle|\, N_t\right]\right] = \mathbb{E}[N_t \cdot \mathbb{E}[X_i]] = \mathbb{E}[N_t] \cdot \mathbb{E}[X_i]$$

Special Case $\quad$ CP-model: $\mathbb{E}[S_t] = (\lambda t) \cdot \mathbb{E}[X_i]$

Remark: One has the elementary renewal theorem:

i.e.: $\frac{\mathbb{E}[N_t]}{t} \xrightarrow{t\to\infty} \lambda := \frac{1}{\mathbb{E}[W_i]}$

Landau symbol

$\Rightarrow$ Approximation: $\mathbb{E}[N_t] = (\lambda t) + o(t)$

$$\Rightarrow \mathbb{E}[S_t] = [(\lambda t) + o(t)] \cdot \mathbb{E}[X_i] =$$

$$(\lambda t) \cdot \mathbb{E}[X_i] \, (1 + o(1))$$

$$\mathbb{V}(S_t) = \mathbb{E}[N_t] \cdot \mathbb{V}(X_i) + \mathbb{V}(N_t) \cdot (\mathbb{E}[X_t])^2$$

Special cases: CP-process: $\mathbb{E}[N_t] = \lambda t$, $\mathbb{V}(N_t) = \lambda t$

$$\Rightarrow \mathbb{V}(S_t) = (\lambda t)\left(\mathbb{V}(X_i) + (\mathbb{E}[X_i])^2\right) = \lambda t \cdot \mathbb{E}[X_i^2]$$

- (without proof): One has, if N_t is a renewal process:

$$\lim_{t \to \infty} \frac{\mathbb{V}(N_t)}{t} = \frac{\mathbb{V}(W_i)}{(\mathbb{E}[W_i])^3} \Rightarrow$$

Approximation in the renewal model:

$$\lim_{t\to\infty} \frac{\mathbb{V}(S_t)}{t} = \mathbb{V}(X_i)\cdot \lim_{t\to\infty} \frac{\mathbb{E}[N_t]}{t} + (\mathbb{E}[X_i])^2 \cdot \lim_{t\to\infty} \frac{\mathbb{V}(N_t)}{t} =$$

$$\mathbb{V}(X_i)\cdot\lambda + (\mathbb{E}[X_i])^2\cdot \mathbb{V}(W_i)\cdot\lambda^3 =$$

$$\lambda\cdot\{\mathbb{V}(X_i) + \lambda^2\,\mathbb{V}(W_i)\,(\mathbb{E}[X_i])^2$$

hence: $\underline{\mathbb{V}(S_t) = \lambda t\cdot\{\mathbb{V}(X_i) + \lambda^2\,\mathbb{V}(W_i)\,(\mathbb{E}[X_i])^2\}\cdot(1+o(1))}$

Approximations of S (now we set again $t=1$)

- if one has few data

  $\rightarrow$ • estimate moments $\mathbb{E}[S]$, $V(S)$, ...

  • then, one of the following approximations:

– Approximation by the normal distribution:

individual model : $\dfrac{S_n - \mathbb{E}[S_n]}{\sqrt{V(S_n)}} \xrightarrow[\text{centr. limit th.}]{(d)} \mathcal{N}(0,1)$

CP-model (without proof): $S \sim CP(\lambda; X) \Rightarrow \dfrac{S - \mathbb{E}[S]}{\sqrt{\mathbb{V}(S)}} \xrightarrow[\lambda \to \infty]{(d)} \mathcal{N}(0,1)$

"version for processes": $S_t \sim CP(\lambda t; X) \Rightarrow \dfrac{S_t - \mathbb{E}[S_t]}{\sqrt{\mathbb{V}(S_t)}} \xrightarrow[\lambda \to \infty]{(d)} \mathcal{N}(0,1)$

estimators for these!

$$\Rightarrow S \approx \mathcal{N}(\mathbb{E}[S], \mathbb{V}(S)) =: \tilde{S}$$

disadvantages

- $\mathbb{P}(\tilde{S} < 0) > 0$
- Underestimation of heavy tails
- $SCH(\tilde{S}) = 0$!

Edgeworth expansion

Improvement of the „normal approx." !

Let Y_i iid copies of Y $\mathbb{E}[Y] = \mu$, $V(Y) = \sigma^2$

$$S_n = Y_1 + \dots + Y_n \,, \qquad S_n^* = \frac{S_n - \mu n}{\sqrt{n}\ \sigma}$$

- Taylor expansion of cumulant generating function of S_n^*
- inverting the terminated expansion yields

$$f(S_n^*, x) = \phi(x) \left\{ 1 + \frac{g_3 H_3(x)}{6\sqrt{n}} + \frac{g_4 H_4(x)}{24\, n} + \dots \right\}$$

where ϕ ... density of $N(0,1)$

H_i ... Hermite polynomials (e.g. $H_3(x) = 8x^3 - 12x$)

g_i ... constants depending on the moments of Y

Remark: Sometimes one uses the lognormal distribution ($S>0$, $SCH(S)>0$)

Approximation by the shifted Γ-distribution: $Ga^T(y_0, \alpha, \beta)$:

- uses $\mathbb{E}[S]$, $\mathbb{V}(S)$, $SCH(S)$!
- approximation by the Γ-distributio, shifted by y_0 !

  i.e: $S - y_0 \stackrel{d}{=} Ga(\alpha, \beta)$

- set the empirical moments equal to the theoretical ones:

hence:

$$\left.\begin{aligned} \widehat{\mathbb{E}[S]} &= \frac{\alpha}{\beta} + y_0 \\ \widehat{\mathbb{V}(S)} &= \frac{\alpha}{\beta^2} \\ \widehat{\mathbb{E}[(S-\mathbb{E}[S])^3]} &= \frac{2\alpha}{\beta^3} \end{aligned}\right\} \text{solve for } \alpha, \beta, y_0$$

disadvantage sometimes $y_0 < 0$!

Approximation of the Binomial distr. by the Poisson distr.

We use:

<u>Lemma:</u> $S_n \mathrel{\hat{=}} Bn(n, p_n)$, s.t. $\lim\limits_{n\to\infty} n p_n = \lambda$

$$\Rightarrow \mathbb{P}(S_n = k) \xrightarrow{n\to\infty} \mathbb{P}(S = k) \quad \text{where } S \mathrel{\hat{=}} Po(\lambda)$$

Ex: 110 risks $p = 0.005$ (with alternative distrib) $\Big\}$ indep.!
60 risks $p = 0.004$ (— " —)

$$S_n^{(1)} = R_1^{(1)} + \dots + R_{110}^{(1)} \mathrel{\hat{=}} \overset{\text{Lemma}}{\approx} Po(110 \cdot 0.005) = Po(0.55)$$

$$S_m^{(2)} = R_1^{(2)} + \dots + R_{60}^{(2)} \mathrel{\hat{=}} \approx Po(60 \cdot 0.004) = Po(0.24)$$

$$\Rightarrow S = S_n^{(1)} + S_m^{(2)} \underset{\text{indep!}}{\mathrel{\hat{=}}} Po(0.79)$$

Premium calculation

Problem: $S \to$ premium which mapping ??

Premium calculation principles

Insurance risk can be splitted into:

- „random risk": S is a random variable
- „estimation risk": distribution or moments have to be estimated from data
- „Prediction risk": change of the stochastic structure with time
  (e.g.: change of laws, techn. innovations, ...)

<u>Remark:</u> In non-life insurance: usually short periods (often 1 year): usually the „interest risk" is neglected.

<u>formal:</u> premium calculation principle $\mathcal{H}$:

functional from the space, where S „lives" (e.g. L^∞, L^1, L^2, etc.)

premium: $P = \mathcal{H}(S)$

<u>Ex:</u>

- <u>netto premium principle</u> (equivalence principle)

  $\mathcal{H}(S) = \mathbb{E}[S]$

  motivated by the „collective balance"

<u>BUT:</u> (see lecture „risk and ruin theory"): $\Rightarrow$ a.s. ruin of the insurance company!

Ex: • discrete random walk

$Z_n := U_1 + \dots + U_n$, $\quad U_i$ iid with $\mathbb{E}[U_i] = 0$

(net result / year, if premium $= \mathbb{E}$!)

one has: $\liminf_{n \to \infty} Z_n = -\infty$ a.s.

• analogously one has in the Cramer Lundberg model:

$Z_t = ct - \sum_{i=1}^{N_t} X_i$: if $ct = \mathbb{E}[N_t] \cdot \mathbb{E}[X_i]$

$$\Downarrow$$

$$\liminf_{t \to \infty} Z_t = -\infty \quad ! \quad \Rightarrow$$

$\Rightarrow$ one needs a „**security loading**"

- **expectation principle**

$$\mathcal{K}(S) = (1+\theta)\cdot \mathbb{E}[S] = \mathbb{E}[S] + \theta\,\mathbb{E}[S]$$

θ … relative security loading

- advantage: easy calculation, one needs only $\mathbb{E}[S]$
- disadvantage: no information on the variability of S enters!

- <u>variance principle</u>

$$\mathcal{H}(S) = \mathbb{E}[S] + \alpha \mathbb{V}(S), \qquad \alpha > 0 \quad (\text{unit: } [\mathbb{E}^{-1}])$$

takes the variab. into account!

- <u>standard deviation principle</u>

$$\mathcal{H}(S) = \mathbb{E}[S] + \beta \sqrt{\mathbb{V}(S)}, \qquad \beta > 0 \text{ dimensionless constant!}$$

- accounting principle of first order

replace the risk S by $\tilde{S}$ with (pessimistic assumption!)

$$F_{\tilde{S}}(s) = \mathbb{P}(\tilde{S} \le s) \le \mathbb{P}(S \le s) = F_S(s) \quad \Rightarrow \quad \bar{F}_{\tilde{S}}(s) \ge \bar{F}_S(s)$$

$\Updownarrow$

premium: expectation of $\tilde{S}$ (use $\mathbb{E}[S] = \int_0^\infty (1 - F_S(s))\,ds$!)

$$\underline{P = \mathcal{H}(S)} = \mathbb{E}[\tilde{S}] = \int_0^\infty (1 - F_{\tilde{S}}(s))\,ds \cdot \frac{\mathbb{E}[S]}{\mathbb{E}[S]} = \int_0^\infty (1 - F_S(s) + F_S(s) - F_{\tilde{S}}(s))\,ds \cdot \frac{\mathbb{E}[S]}{\int_0^\infty (1 - F_S(s))\,ds} =$$

$$= \mathbb{E}[S] \cdot \left(1 + \frac{\int_0^\infty (F_S(s) - F_{\tilde{S}}(s))\,ds}{\int_0^\infty (1 - F_S(s))\,ds} \right)$$

"kind of expectation principle

with $\theta = \theta(F_S, F_{\tilde{S}})$!

- Quantile principle

given: upper bound for $\mathbb{P}(P < S) = \varepsilon$

$$p := \inf\{P \mid \mathbb{P}(S > P) \leq \varepsilon)$$

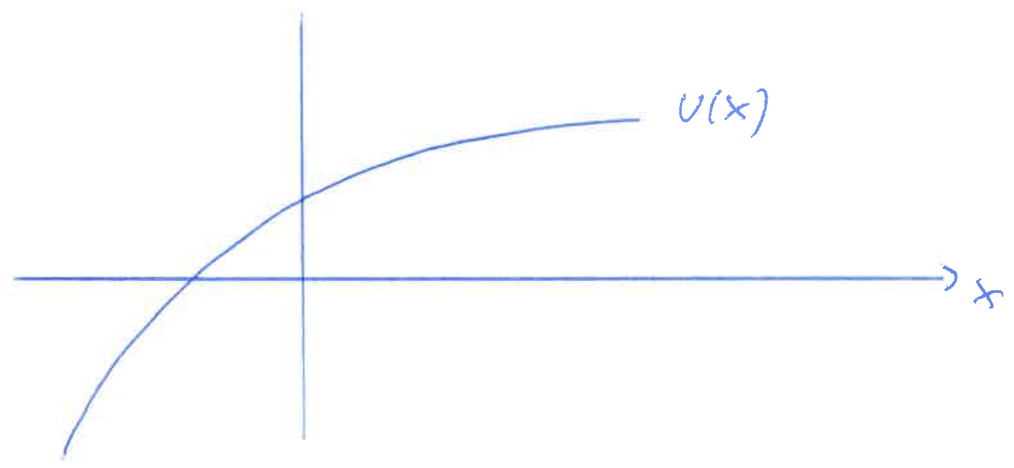
if $(1-\varepsilon)$ quantil $\exists$, then it is equal to the premium

- <u>principle of zero utility</u>

<u>given:</u> utility function $u: \mathbb{R} \to \mathbb{R}$ (money $\to$ utility)

<u>properties:</u> $u' > 0$: more money $\to$ more utility

$u'' \underset{(-)}{\leq} 0$: additional utility decreases with increasing money



Def: (absolute) risk aversion : $r(x) = -\frac{u''(x)}{u'(x)}$

Example for constant $r(x)$: $u(x) = \frac{1 - e^{-ax}}{a}$

$\Rightarrow u'(x) = e^{-ax} \Rightarrow u''(x) = -a e^{-ax} \Rightarrow r(x) \equiv a$

(relative) risk aversion : $\hat{r}(x) = -\frac{x\, u''(x)}{u'(x)}$

Ex: $u(x) = x^p$, $0 < p < 1$

$\Downarrow$

$u'(x) = p \cdot x^{p-1}$

$\Downarrow$

$u''(x) = p(p-1)x^{p-2} \Rightarrow \hat{r}(x) = (1-p)$

Consider one period model ; $t = 0, 1$

$t=0$: x ... initial endowment utility: $u(x)$

$t=1$ $x + P - S$ ... wealth after claim and premium: utility: $u(x+P-S)$

principle of zero utility : $\boxed{\text{utility at } t=0 = \text{expected utility at } t=1}$

i.e.: $u(x) = \mathbb{E}[u(x+P-S)]$

Solve this equation for P !

- special case: „exponential principle"

$$u(x) = \frac{1-e^{-ax}}{a}$$

$$u(x) = \mathbb{E}[u(x+p-S)]$$

$$\Leftrightarrow$$

$$\frac{\cancel{1}-e^{-ax}}{\cancel{a}} = \mathbb{E}\left[\frac{\cancel{1}-e^{-a(x+p-S)}}{\cancel{a}}\right]$$

$$\Leftrightarrow$$

$$-e^{-ax} = \mathbb{E}[-e^{-a(x+p-S)}]$$

$$\Leftrightarrow$$

$$\cancel{e^{-ax}} = e^{\cancel{-ax}-ap}\,\mathbb{E}[e^{aS}] \Leftrightarrow e^{ap} = \mathbb{E}[e^{aS}] \Leftrightarrow p = \frac{1}{a}\ln M_S(a)$$

<u>Remarks:</u> usable only for small claims (i.e: $\exists\, \varepsilon_0 > 0$, s.t. $\mathbb{E}[e^{\varepsilon_0 S}] < \infty$)

• <u>high risk aversion (limiting case $a \to \infty$)</u>

One has: $\lim_{a \to \infty} P(a) = \operatorname{ess\,sup} S =: \bar{S}$

<u>Proof:</u> (only for $\bar{S} < \infty$) (for $\bar{S} = \infty$ very similar)

Let $\delta > 0$ be arbitrary $\Rightarrow$

$\exists\, A \subset \Omega$ mb., s.t. $\mathbb{P}(A) =: \eta > 0$ and $S \geq \bar{S} - \delta$ on A

$$\Rightarrow P(a) = \frac{1}{a} \ln \mathbb{E}[e^{aS}] \geq \frac{1}{a} \ln \mathbb{E}[e^{aS} \mathbb{1}_A] \geq \frac{1}{a} \ln \mathbb{E}[e^{a(\bar{S}-\delta)} \mathbb{1}_A]$$

$$= \frac{1}{a} \left\{ \ln \left(e^{a(\bar{s}-\delta)} \cdot \mathbb{E}[\mathbb{1}_A] \right) \right\} = \frac{1}{a} \left\{ a(\bar{s}-\delta) + \ln \mathbb{P}(A) \right\} \geq$$

$$\bar{s} - \delta + \frac{\ln \eta}{a} \xrightarrow{a \to \infty} \bar{s} - \delta$$

$$\Rightarrow \liminf_{a \to \infty} P(a) \geq \bar{s} - \delta \overset{\delta \text{ arbitr.}}{\Rightarrow} \liminf_{a \to \infty} P(a) \geq \bar{s} \qquad (1)$$

moreover: $P(a) = \frac{1}{a} \ln \mathbb{E}[e^{as}] \leq \frac{1}{a} \ln \cancel{\mathbb{E}}[e^{a\bar{s}}] = \frac{1}{a} \cdot a \cdot \bar{s} = \bar{s} \qquad (2)$

$$(1) + (2) \Rightarrow \lim_{a \to \infty} P(a) = \bar{s} \quad \blacksquare$$

- **loss function principle**

  - minimization of the expected „loss"

  - loss function $L(S,P)$: describes the loss of the comp., if claim S and premium are different

  - risk function $R(p) = \mathbb{E}[L(S,p)]$

  - premium $\underline{P = \mathcal{H}(S) = \arg\min R(p)}$, i.e. the p which minimizes $R(p)$!

Example: quadratic loss function

$$L(S,p) = (S-p)^2$$

Rem: if $S > p$: „loss", because claim bigger than the premium

$S < p$: if the premium is to large $\longrightarrow$ competitive disadvantage

$$R(p) = \mathbb{E}[(S-p)^2] = \mathbb{E}[S^2 - 2pS + p^2] = \mathbb{E}[S^2] - 2p\,\mathbb{E}[S] + p^2$$

$$\frac{d}{dp} R(p) = -2\mathbb{E}[S] + 2p \overset{!}{=} 0 \Rightarrow p = \mathcal{H}(S) = \mathbb{E}[S]$$

$$\frac{d^2}{dp^2} R(p) = 2 > 0 \checkmark$$

i.e.: the premium is given by the net premium, if we use a quadratic loss function different loss functions: $L(S,p) = (S-p)^r$, $r>0$

$L(S,p) = \mathbb{1}_{\{S \neq p\}}$, in the discrete case!

quality criteria for $\mathcal{H}(S)$

- $\nexists$ an optimal principle
- selection of the principle depends on the situation
- BUT: the are desirable properties

a) $P \leq \operatorname{ess\,sup} S$

- nobody pays more than the maximal damage as premium

b) $P \geq \mathbb{E}[S]$

- otherwise the insurance comp. would be ruined a.s. (even if we have "=", see above)

c) $S \equiv$ const. $\Rightarrow P \equiv$ const.

- constant risk has no variability, hence no security loading

d) translation invariance: $\mathcal{H}(S+c) = \mathcal{H}(S) + c$, c... const.

- if one has additional to the claim some constant service (e.g. administration): this will be added 1:1

e) positive homogeneity

$$\mathcal{H}(c\cdot S) = c\cdot\mathcal{H}(S) \qquad \forall c>0$$

- often one impose this only for $0<c<1$

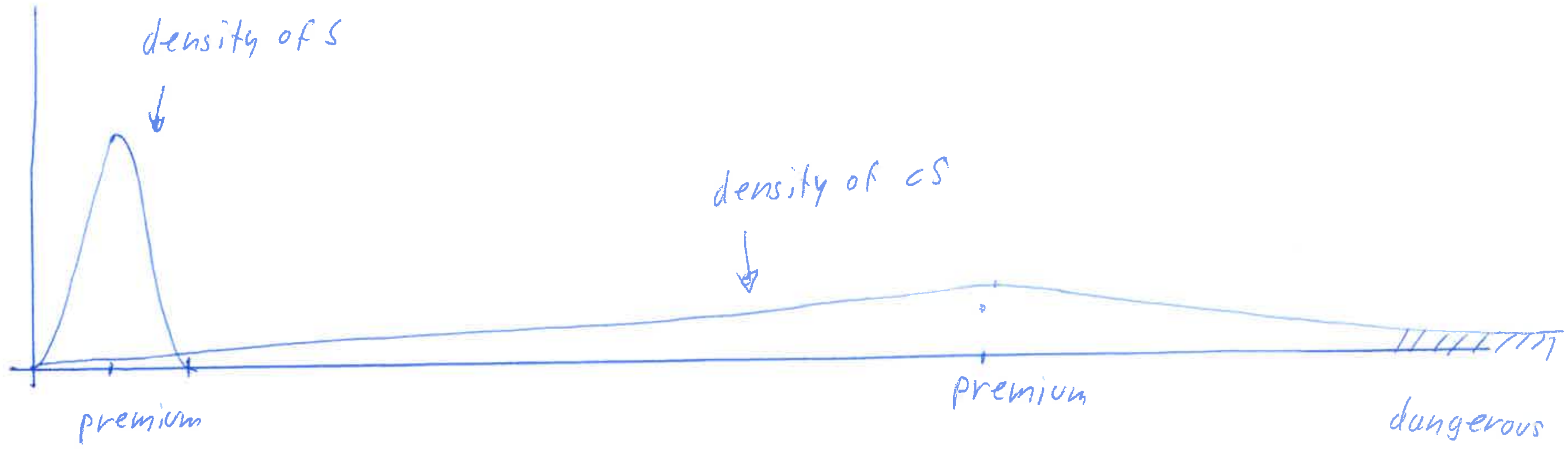
Remarks: 1.) if $\forall c>0$: $\mathcal{H}(c\cdot S) \le c\,\mathcal{H}(S) \Rightarrow \mathcal{H}(cS) = c\,\mathcal{H}(S)$

Proof.: $\mathcal{H}(S) = \mathcal{H}(\frac{1}{c}\cdot c\cdot S) \le \frac{1}{c}\mathcal{H}(c\cdot S)$

$\Rightarrow \mathcal{H}(cS) \ge c\,\mathcal{H}(S) \overset{ass.}{\Rightarrow}$

$\mathcal{H}(cS) = \mathcal{H}(S)$

2.) $c \gg 1$: not desirable : could cause ruin



f) <u>additivity</u> : $\mathcal{H}(S_1+S_2) = \mathcal{H}(S_1) + \mathcal{H}(S_2)$, S_i indep.!

for ~~the~~ independent risks : total premium = sum of the individual premiums

g) <u>subadditivity</u>

$$\mathcal{H}(S_1+S_2) \leq \mathcal{H}(S_1)+\mathcal{H}(S_2)$$

motivated by risk pooling

h) <u>iterativity</u>

$$\mathcal{H}(S) = \mathcal{H}(\mathcal{H}(S|X)) \quad \forall S, X \text{ r.v.}$$

i.e: r.h.s: apply $\mathcal{H}$ to the conditional r.v. $S|_{X=x}$

i.e: $\mathcal{H}(S|X=x) = g(x)$

- dann apply $\mathcal{H}$ on $g(X)$

- motivated by the iterativity of $\mathbb{E}$!

Ex: car insurance

$X = 1, 2, 3, 4 \leftrightarrow$ spring, summer, ...

premium has to be paid yearly:

interpretation of averaged „season premiums"

Ex (numerical)

$S \backslash X$	1	2	3	4
0	$\mathbb{P} = 0.9$	0.95	0.8	0.7
1000	$\mathbb{P} = 0.1$	0.05	0.2	0.3

$$\mathcal{H}(S|X) = \begin{cases} 100 & X = 1 \\ 50 & X = 2 \\ 200 & X = 3 \\ 300 & X = 4 \end{cases}$$

$$\mathcal{U}(\mathcal{U}(S|X)) = \frac{1}{4}(100 + 50 + 200 + 300) = \frac{650}{4}$$

<u>L.h.s:</u> $S = \begin{cases} 0 & \mathbb{P} = \frac{1}{4}(0.95 + 0.95 + 0.8 + 0.7) \\ 1000 & \mathbb{P} = \frac{1}{4}(0.1 + 0.05 + 0.2 + 0.3) = \frac{0.65}{4} \end{cases}$

$$\Rightarrow \mathcal{U}(S) = \frac{650}{4} \checkmark$$