Option Pricing and Portfolio Optimization

Modern Methods of Financial Mathematics

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Application of the product rule results in:

$$
\begin{aligned}
dX(t) &= d(Z(t)\cdot Y(t)) = Z(t)\cdot dY(t) + Y(t)\cdot dZ(t) + d\langle Y, Z\rangle_t \\
&= Z(t)\cdot\left(\frac{1}{Z(t)}\left(a(t) - \sum_{j=1}^{m} S_j(t)\cdot\sigma_j(t)\right)dt \right. \\
&\qquad \left. + \frac{1}{Z(t)}\sum_{j=1}^{m}\sigma_j(t)\, dW_j(t)\right) \\
&\quad + Y(t)\cdot Z(t)\left(A(t)\, dt + S(t)'\, dW(t)\right) + S(t)'\cdot Z(t)\cdot\frac{1}{Z(t)}\sigma(t)\, dt \\
&= (a(t) + A(t)\cdot X(t))\, dt + \left(X(t)\cdot S(t)' + \sigma(t)'\right) dW(t).
\end{aligned}
$$

Hence, $X(t)$ solves the stochastic differential equation (2.11).

iii) Let $\tilde{X}(t)$, $X(t)$ be two solutions of the stochastic differential equation. Then

$$\hat{X}(t) := \tilde{X}(t) - X(t)$$

solves the homogeneous equation

$$
\begin{aligned}
d\hat{X}(t) &= \hat{X}(t)\left(A(t)\, dt + \sum_{j=1}^{m} S_j(t)\, dW_j(t)\right), \\
\hat{X}(0) &= 0.
\end{aligned}
$$

But by i) we then have $\hat{X}(t) = 0$ for all $t \geq 0$ P-almost surely.

□

2.2. Trading Strategy and Wealth Process

Now, after the introduction of Itô calculus we are able to model stock prices with time dependent, random rates of return and volatilities. This will be the basis of all following continuous-time market models.

General assumptions:
Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. We assume that there is an m-dimensional Brownian motion $\{(W(t), \mathcal{F}_t)\}_{t\in[0,\infty)}$ defined on that space. The dynamics of the bond and stock prices will be modeled according to

$$P_0(t) = p_0 \cdot \exp\left(\int_0^t r(s)\, ds\right) \qquad \text{"bond"}$$

$$P_i(t) = p_i \cdot \exp\left(\int_0^t \left(b_i(s) - \tfrac{1}{2}\sum_{j=1}^m \sigma_{ij}^2(s)\right) ds + \sum_{j=1}^m \int_0^t \sigma_{ij}(s)\, dW_j(s)\right) \qquad \text{"stock"}$$

for $t \in [0,T], T > 0, i = 1, \ldots d$. The functions $r(t)$, $b(t) = (b_1(t), \ldots, b_d(t))'$, $\sigma(t) = (\sigma_{ij}(t))_{ij}$ should all be progressively measurable processes with respect to $\{F_t\}_t$ which are component-wise uniformly bounded in (t, ω). Further, we assume $\sigma(t)\sigma(t)'$ to be uniformly positive definite (i.e. there is a positive constant $K > 0$ with $x'\sigma(t)\sigma(t)'x \geq Kx'x$ for all $x \in \mathbb{R}^d$ and all $t \in [0.T]$ P-a.s.).

By Theorem 2.54, variation of constants, these prices are the unique solutions of the following stochastic differential equations (note that the above boundedness assumptions on $r(t)$, $b(t)$, $\sigma(t)$ ensure the applicability of Theorem 2.54):

$$\begin{aligned} dP_0(t) &= P_0(t) \cdot r(t)\, dt \\ P_0(0) &= p_0 \qquad \text{"bond"} \\ dP_i(t) &= P_i(t)\left(b_i(t)\, dt + \sum_{j=1}^m \sigma_{ij}(t)\, dW_j(t)\right), \quad i = 1, \ldots d \\ P_i(0) &= p_i \qquad \text{"stock"} \end{aligned}$$

In particular, these equations give us the representations of the prices as Itô processes.

Remark. Note that in the above general assumptions we have not required a deterministic rate of return $r(t)$ of the bond. This rate $r(t)$ is allowed to be a stochastic process. Thus, the bond is no longer riskless. However, by the boundedness assumption on $r(t)$, the risk involved in the bond price is

much smaller than that of the stock prices which are driven by Brownian motions.

Possible actions of the investors. After having modeled the prices we shall now model the actions of the market participants. Here, we assume that the investor has the following possibilities of actions:

(1) He can rebalance his holdings, i.e. he can sell some securities and invest the money in other securities. This action will be modeled by the *portfolio process* or by the *trading strategy*.

(2) He is also allowed to consume parts of his wealth which will be incorporated in our setting by the *consumption process*.

2.2.1. Requirements on a market model. We now collect some requirements that should be valid in a realistic market model:

(1) The investor should not be able to foresee events; especially, he is not allowed to have knowledge of future prices.

(2) The actions of a single investor should not have any impact on the stock prices; this is the so-called *small investor hypothesis*.

(3) The investor is endowed with a fixed initial capital at time $t = 0$. Of course, different investors can have different endowments.

(4) Money which is not invested into stocks has to be invested in bonds.

(5) The investors have to act in a self-financing way, i.e. each change in their wealth is due to gains or losses from investment and /or consumption. There is no secret source for new money, but also no secret sink where it can vanish.

We make some further assumptions which will prove to be very convenient for the development of the theory:

(6) The securities are perfectly divisible.

(7) Negative positions in securities are possible. A negative position in the bond will be interpreted as a credit. A negative stock position means that we have sold some stock short, i.e. we have sold shares that we do not possess and thus owe these shares to another investor. Of course, they have to be delivered later.

(8) Rebalancing of the holdings does not lead to transaction costs.

Negative bond positions and credit interest rates. If the interest rate $r(t)$ is constant, then the possibility to have a negative bond position means that it is possible to borrow money for the same rate as we would get for investing in bonds. For some institutions this situation is indeed close to reality. In our market model the interest may depend on the market

situation (i.e. on $(t,\omega) \in [0,T] \times \Omega$), but not on the fact that we have a positive or a negative bond position.

Mathematical realizations of some requirements. We recapitulate that our market consists of a bond and d stocks. The initial capital of the investor is $x > 0$ (requirement (3)). Now at time $t = 0$ he can buy a selection of securities $\varphi(0) = (\varphi_0(0), \varphi_1(0), \ldots, \varphi_d(0))'$ where $\varphi_i(0)$ denotes the number of shares of security i held at time $t = 0$. Similarly, the components of the vector $\varphi(t) = (\varphi_0(t), \varphi_1(t), \ldots, \varphi_d(t))'$ denote the numbers of shares of the different securities held at time t. This vector process $\varphi(t)$ is called a trading strategy. Requirement (1) forces the trading strategies to be progressively measurable with respect to $\{\mathcal{F}_t\}_t$. Decisions on buying and selling always have to be made on the basis of the information available at time t, and this is modeled by $\{\mathcal{F}_t\}_t$. Requirement (5) implies that only self-financing trading strategies should be used. This will be made more precise in the following definitions.

A discrete-time example of a self-financing trading strategy. In the continuous-time model we shall define a self-financing trading strategy by a requirement which at first sight does not seem to be very intuitive. Therefore this requirement will be motivated by the following discrete-time example.

Now consider a market consisting of one stock and a riskless bond. We look at a two-period model given by the time points $t = 0, 1, 2$. The vector $(\varphi_0(t), \varphi_1(t))'$ denotes the numbers of shares of bond and stock held at time t, respectively. As we allow negative positions and perfectly divisible assets, $\varphi_0(t)$ and $\varphi_1(t)$ can attain every arbitrary real number. Let $C(t)$ be the consumption of the investor at time t where we assume $C(0) = 0$. $X(t)$ denotes his wealth at time t, $X(0) = x$ his initial capital. $P_0(t)$, $P_1(t)$ are the bond and stock prices at time t, respectively.

At time $t = 0$ the investor uses his initial capital to buy shares of bond and stock, i.e. we have

$$X(0) = x = \varphi_0(0) \cdot P_0(0) + \varphi_1(0) \cdot P_1(0).$$

At time $t = 1$ the security prices have changed and the investor consumes parts of his wealth. Thus his current wealth is given by

$$X(1) = \varphi_0(0) \cdot P_0(1) + \varphi_1(0) \cdot P_1(1) - C(1).$$

In total we have

$$X(1) = x + \varphi_0(0) \cdot (P_0(1) - P_0(0)) + \varphi_1(0) \cdot (P_1(1) - P_1(0)) - C(1),$$

i.e. his wealth at time $t = 1$ equals his initial wealth plus gains or losses from the price changes of stock and bond minus consumption. Further, at that

time, the investor is allowed to rebalance his remaining holdings. However, we must have

$$X(1) = \varphi_0(1) \cdot P_0(1) + \varphi_1(1) \cdot P_1(1)$$

as the investor is required to invest his whole remaining capital at the market. Similarly, at time $t = 2$, we have

$$X(2) = \varphi_0(2) \cdot P_0(2) + \varphi_1(2) \cdot P_1(2), \tag{2.15}$$

and further

$$X(2) = x + \sum_{i=1}^{2} [\varphi_0(i-1) \cdot (P_0(i) - P_0(i-1)) + \varphi_1(i-1) \cdot (P_1(i) - P_1(i-1))] - \sum_{i=1}^{2} C(i). \tag{2.16}$$

In the discrete-time model the condition characterizing a self-financing trading strategy can be formulated as: "wealth before rebalancing minus consumption has to equal wealth after rebalancing", i.e.

$$\varphi_0(i) \cdot P_0(i) + \varphi_1(i) \cdot P_1(i) = \varphi_0(i-1) \cdot P_0(i) + \varphi_1(i-1) \cdot P_1(i) - C(i). \tag{2.17}$$

In a continuous-time setting, such a requirement makes no sense as we can trade securities at each time instant. In particular, it is impossible to distinguish between "before" and "after" at the same time instant. Instead of requirement (2.17) we take the approach to define the wealth process corresponding to the strategy $\varphi(t)$, namely $X(t)$, via equation (2.15) (i.e. "wealth = total wealth of the shares held"). Equation (2.16) will then be used to define a self-financing trading and consumption strategy where we shall of course replace all sums by the corresponding integrals. This suggests the requirement

$$X(t) = x + \int_0^t \varphi_0(s)\, dP_0(s) + \int_0^t \varphi_1(s)\, dP_1(s) - \int_0^t c(s)\, ds.$$

Note that the price processes are Itô processes. This leads to natural conditions on the $\varphi_i(s)$ to ensure existence of the integrals (see Definition 2.55).

Our considerations so far will now be made precise from a mathematical point of view.

Definition 2.55. (1) A *trading strategy* φ is an $\mathbb{R}^{d+1}$-valued progressively measurable process with respect to $\{\mathcal{F}_t\}_{t\in[0,T]}$

$$\varphi(t) := (\varphi_0(t), \varphi_1(t), ..., \varphi_d(t))'$$

satisfying

$$\int_0^T |\varphi_0(t)|\, dt < \infty \text{ a.s. } P. \tag{2.18}$$

$$\sum_{j=1}^{d} \int_0^T (\varphi_i(t) \cdot P_i(t))^2\, dt < \infty \text{ a.s. } P \text{ for } i = 1, ..., d. \tag{2.19}$$

The value $x := \sum_{i=0}^{d} \varphi_i(0) \cdot p_i$ is called *initial value* of φ.

(2) Let φ be a trading strategy with initial value $x > 0$. The process

$$X(t) := \sum_{i=0}^{d} \varphi_i(t) P_i(t)$$

is called *wealth process* corresponding to φ with *initial wealth* x.

(3) A non-negative progressively measurable process $c(t)$ with respect to $\{\mathcal{F}_t\}_{t\in[0,T]}$ with

$$\int_0^T c(t)\, dt < \infty \text{ a.s. } P \tag{2.20}$$

is called a *consumption rate process* (for short: *consumption process*).

Definition 2.56. A pair (φ, c) consisting of a trading strategy φ and a consumption rate process c is called *self-financing* if the corresponding wealth process $X(t)$. $t \in [0,T]$, satisfies:

$$X(t) = x + \sum_{i=0}^{d} \int_0^t \varphi_i(s)\, dP_i(s) - \int_0^t c(s)\, ds \text{ a.s. } P. \tag{2.21}$$

"current wealth" = "initial wealth" + "gains / losses" − "consumption"

Remark 2.57. We have

$$\int_0^t \varphi_0(s)\, dP_0(s) = \int_0^t \varphi_0(s) P_0(s) r(s)\, ds$$

$$\int_0^t \varphi_i(s)\, dP_i(s) = \int_0^t \varphi_i(s) P_i(s) b_i(s)\, ds + \sum_{j=1}^{m} \int_0^t \varphi_i(s) P_i(s) \sigma_{ij}(s)\, dW_j(s), \quad i = 1, ..., d.$$

The conditions (2.18), (2.19) and (2.20) in the definitions and the boundedness requirements on r, b and σ thus ensure existence of all the integrals occuring on the right-hand side of equation (2.21).

Definition 2.58. Let (φ, c) be a self-financing pair consisting of a trading strategy and a consumption process with corresponding wealth process $X(t) > 0$ P-a.s. for all $t \in [0,T]$. Then the $\mathbb{R}^d$-valued process

$$\pi(t) := (\pi_1(t), \ldots, \pi_d(t))', \; t \in [0,T] \text{ with } \pi_i(t) = \frac{\varphi_i(t) \cdot P_i(t)}{X(t)}$$

is called a *self-financing portfolio process corresponding to the pair* (φ, c).

Remark 2.59. (1) The portfolio process denotes the fractions of total wealth invested in the different stocks. Therefore, the fraction of wealth invested in the bond is given by

$$\left(1 - \pi(t)' \underline{1}\right) = \frac{\varphi_0(t) \cdot P_0(t)}{X(t)} \text{ where } \underline{1} := (1, \ldots, 1)' \in \mathbb{R}^d.$$

(2) Given the knowledge of the wealth $X(t)$ and the prices $P_i(t)$, it is possible for an investor to describe his activities via a self-financing pair (π, c). More precisely, in this case portfolio process and trading strategy are equivalent descriptions of the same action.

The wealth equation. The description of the trading activities by the portfolio process enables us to derive a simple stochastic differential equation for the wealth process. Let therefore (φ, c) be a self-financing pair consisting of a trading strategy and a consumption process. By additionally using Definition 2.58 we have

$$dX(t) = \sum_{i=0}^{d} \varphi_i(t)\, dP_i(t) - c(t)\, dt$$

$$= \varphi_0(t) P_0(t)\, r(t)\, dt + \sum_{i=1}^{d} \varphi_i(t) P_i(t) \left(b_i(t)\, dt + \sum_{j=1}^{m} \sigma_{ij}(t)\, dW_j(t) \right) - c(t)\, dt$$

$$= \left(1 - \pi(t)' \underline{1}\right) X(t) r(t)\, dt + \sum_{i=1}^{d} X(t) \pi_i(t) \left(b_i(t)\, dt + \sum_{j=1}^{m} \sigma_{ij}(t)\, dW_j(t) \right) - c(t)\, dt$$

$$= \left(1 - \pi(t)' \underline{1}\right) X(t) r(t)\, dt + X(t) \pi(t)' b(t)\, dt + X(t) \pi(t)' \sigma(t)\, dW(t) - c(t)\, dt.$$

We thus obtain the *wealth equation*

$$
\begin{aligned}
(2.22)\quad dX(t) &= [r(t)X(t) - c(t)]\,dt \\
&\quad + X(t)\pi(t)'\left((b(t) - r(t)\,\underline{1})\,dt + \sigma(t)\,dW'(t)\right) \\
X(0) &= x
\end{aligned}
$$

By Theorem 2.54, the variation of constants formula, this stochastic differential equation has a unique solution given some suitable integrability requirements for $\pi(t)$. As b, r, σ are uniformly bounded and as the consumption process c satisfies requirement (2.20), we only need the condition

$$
(2.23)\qquad \int_0^T \pi_i^2(t)\,dt < \infty \text{ a.s. } P
$$

for $i = 1, \ldots, d$ to ensure uniqueness and existence of the solution of the stochastic differential equation (2.22).

Now we are able to give an alternative definition of a portfolio process. We describe it as the process of the fractions of wealth invested in the different assets without referring to a trading strategy.

Definition 2.60. The progressively measurable $\mathbb{R}^d$-valued process $\pi(t)$ is called a *self-financing portfolio process* corresponding to the consumption process $c(t)$ if the corresponding wealth equation (2.22) possesses a unique solution $X(t) = X^{\pi,c}(t)$ with

$$
(2.24)\qquad \int_0^T (X(t) \cdot \pi_i(t))^2\,dt < \infty \text{ a.s. } P \text{ for } i = 1, \ldots, d.
$$

Remark 2.61. The condition (2.24) on the portfolio process and the wealth process is indeed exactly the integrability condition (2.19) on the trading strategy and the price processes. If the portfolio process satisfies condition (2.23) then continuity of the corresponding wealth process yields that condition (2.24) is satisfied, too. Thus, condition (2.24) is weaker than condition (2.23). Note further that if there is no consumption, then condition (2.23) implies strict positivity of the wealth process. This can be seen by looking at the explicit form of the solution of equation (2.22) under assumption (2.23). However, condition (2.24) allows for portfolio processes that can lead to the ruin of the investor (i.e. it allows $X(t) = 0$ for some $t \in [0, T]$ or even negative values of $X(t)$). This will be of relevance in connection with the replication approach to option pricing.

Definition 2.62. A self-financing pair (φ, c) or (π, c) consisting of a trading strategy φ or a portfolio process π and a consumption process c will be called *admissible for the initial wealth* $x > 0$, if the corresponding wealth process

satisfies

$$X(t) \geq 0 \text{ a.s. } P \text{ for all } t \in [0,T].$$

The set of admissible pairs (π, c) will be denoted by $\mathcal{A}(x)$.

An example. After having presented so many theoretical definitions we now look at a seemingly simple example. Consider a constant portfolio process and a consumption rate that is proportional to the current wealth

$$\pi(t) \equiv \pi \in \mathbb{R}^d \text{ constant}, \; c(t) = \gamma \cdot X(t),$$

with some $\gamma > 0$, $X(t)$ being the wealth process corresponding to (π, c). Hence, the investor rebalances his holdings in such a way that the fractions of wealth invested in the different stocks and in the bond remain constant over time. Further, the velocity of the increase in consumption ("the consumption rate") is always proportional to the current wealth of the investor. Please note that although $\pi(t)$ remains constant over time, the investor has to trade at each time instant as the stock prices change at each time instant and of course each in a different way. The wealth equation corresponding to this example has the form

$$\begin{aligned} dX(t) &= (r(t) - \gamma) X(t)\ dt + X(t) \pi' ((b(t) - r(t)\ \underline{1})\ dt + \sigma(t)\ dW(t)) \\ X(0) &= 0. \end{aligned}$$

By Theorem 2.54, "variation of constants", this results in a wealth process of

$$X(t) = x \cdot \exp\left(\int_0^t \left[r(s) - \gamma + \pi' (b(s) - r(s) \cdot \underline{1}) - \tfrac{1}{2} \left\| \pi' \sigma(s) \right\|^2 \right] ds \right.$$
$$\left. + \int_0^t \pi' \sigma(s)\ dW(s) \right).$$

In particular, $X(t)$ is strictly positive and we have $(\pi, c) \in \mathcal{A}(x)$. So if the investor wants to avoid bankruptcy such a strategy is a good one. It will also prove to be optimal from other viewpoints (see Chapter 5).

2.3. Properties of the Continuous-Time Market Model

In this section, we shall concentrate on the special case of $d = m$. Hence, the dimension of the underlying Brownian motion equals the number of stocks. We shall restrict ourselves to the case that past and present prices are the only sources of information for our investors. Mathematically, we

shall model this by choosing the Brownian filtration as $\{\mathcal{F}_t\}_{t\in[0,T]}$. We are particularly interested in the set of all final wealths $X(T)$ that we can attain when starting with an initial capital of x. The surprising result will be that every final wealth $X(T)$ can be generated given a sufficient initial capital of x.

General assumption for this section:

$$d = m$$

First we need some notation:

Notation:

$$\gamma(t) := \exp\left(-\int_0^t r(s)\, ds\right)$$

$$\theta(t) := \sigma^{-1}(t)\,(b(t) - r(t)\,\underline{1})$$

$$Z(t) := \exp\left(-\int_0^t \theta(s)'\, d\mathrm{W}(s) - \tfrac{1}{2}\int_0^t \|\theta(s)\|^2\, ds\right)$$

$$H(t) := \gamma(t) \cdot Z(t)$$

Please note that the uniform boundedness of b and r together with the uniform positive definiteness of $\sigma\sigma'$ imply the uniform boundedness of $\|\theta(t)\|^2$. $\theta(t)$ might be interpreted as a kind of relative risk premium for stock investment. The process $H(t)$ will play a crucial role in connection with option pricing. Note that $H(t)$ is positive, continuous, and progressively measurable with respect to $\{\mathcal{F}_t\}_{t\in[0,T]}$. Further, it is the unique solution of the stochastic differential equation

$$dH(t) = -H(t)\left(r(t)\, dt + \theta(t)'\, d\mathrm{W}(t)\right), \qquad H(0) = 1. \tag{2.25}$$

Note that (2.25) is the Itô process representation of $H(t)$.

The following theorem can be regarded as one of the main theorems presented in this book. It shows a remarkable property of the market model in the case $d = m$.

Theorem 2.63 (Completeness of the market). (1) *Let the self-financing pair (π, c) consisting of a portfolio process π and a consumption process c be admissible for an initial wealth of $x \geq 0$, i.e. we have $(\pi, c) \in \mathcal{A}(x)$. Then the corresponding wealth process $X(t)$*

satisfies

$$E\left(H(t)X(t)+\int_0^t H(s)c(s)\,ds\right)\leq x \text{ for all } t\in[0,T].$$

(2) *Let $B\geq 0$ be an $\mathcal{F}_T$-measurable random variable and $c(t)$, $t\in[0.T]$, a consumption process satisfying*

$$x:=E\left(H(T)B+\int_0^T H(s)c(s)\,ds\right)<\infty.$$

Then there exists a portfolio process $\pi(t)$, $t\in[0.T]$, with $(\pi,c)\in\mathcal{A}(x)$ and the corresponding wealth process $X(t)$ satisfies

$$X(T)=B \text{ a.s. } P.$$

Implications of Theorem 2.63. In view of part (1) the process $H(t)$ can be regarded as the appropriate discounting process that determines the initial wealth at time $t=0$

$$E\left(\int_0^T H(s)\cdot c(s)\,ds\right)+E(H(T)\cdot B)$$

which is necessary to be able to attain future aims (such as living according to a given consumption process or obtaining a wealth B at time $t=T$). Thus, part (1) puts bounds on the desires of an investor given his initial capital $x\geq 0$. Part (2) proves that future aims which are feasible in the sense of part (1) can indeed be realized. Part (2) thus says that each desired final wealth in $t=T$ can be attained exactly via trading according to an appropriate self-financing pair (π,c) if one possesses sufficient initial capital. This property of our model will be called *completeness* and we shall talk of a *complete market*.

Proof of Theorem 2.63.
(1) Let $(\pi,c)\in\mathcal{A}(x)$. Application of the product rule. Corollary 2.53, to $H(t)\cdot X(t)$ yields (note the explicit forms of the wealth equation (2.22) for $X(t)$ and the stochastic differential equation (2.25) for $H(t)$):

$$\begin{aligned}(2.26)\quad & H(t)\cdot X(t)+\int_0^t H(s)c(s)\,ds\\ &=x+\int_0^t H(s)\,dX_s+\int_0^t X(s)\,dH(s)+\langle X,H\rangle_t+\int_0^t H(s)c(s)\,ds\end{aligned}$$

$$= x + \int_0^t H(s)X(s)\left[r(s) + \pi(s)'\left(b(s) - r(s)\cdot\underline{1}\right) - r(s)\right.$$

$$\left. -\pi(s)'\sigma(s)\theta(s)\right]\, ds + \int_0^t H(s)X(s)\left[\pi(s)'\sigma(s) - \theta(s)'\right]\, dW(s)$$

$$= x + \int_0^t H(s)X(s)\left[\pi(s)'\sigma(s) - \theta(s)'\right]\, dW(s).$$

Due to $(\pi, c) \in \mathcal{A}(x)$ the left-hand side of equation (2.26) is non-negative. Hence, by Theorem 2.22 the local martingale which forms the right-hand side of the equation is a super-martingale. Thus:

$$E\left(H(t)X(t) + \int_0^t H(s)c(s)\, ds\right)$$

$$= E\left(x + \int_0^t H(s)X(s)\left(\pi(s)'\sigma(s) - \theta(s)'\right)\, dW(s)\right) \leq x.$$

(2) Define

$$X(t) := \frac{1}{H(t)} \cdot E\left(\int_t^T H(s)c(s)\, ds + H(T)\cdot B \mid \mathcal{F}_t\right).$$

Thus, $X(t)$ is $\mathcal{F}_t$-measurable and we have $X(T) = B$ P-a.s., $X(t) \geq 0$. As $\{\mathcal{F}_t\}_t$ is the Brownian filtration, the conditional expectation of a random variable given $\mathcal{F}_0$ is P-a.s. constant and so it coincides with its unconditional expectation. This yields $X(0) = x$ P-almost surely. Now define

$$M(t) := X(t)\cdot H(t) + \int_0^t H(s)c(s)\, ds$$

$$= E\left(\int_0^T H(s)c(s)\, ds + H(T)B \mid \mathcal{F}_t\right).$$

This process is an $\{\mathcal{F}_t\}_t$-martingale with $M(0) = x$ P-almost surely. By Corollary 2.70 (see the next section on the martingale representation theorem), M can be represented as an Itô integral

$$M(t) = x + \int_0^t \Psi(s)' \, dW(s) \text{ a.s. } P \text{ for all } t \in [0,T].$$

where $\Psi(t)$ is an $\{\mathcal{F}_t\}_t$-progressively measurable, $\mathbb{R}^d$-valued process satisfying

$$\int_0^T \|\Psi(t)\|^2 \, dt < \infty \text{ a.s. } P.$$

In total, we arrive at

$$X(t) H(t) + \int_0^t H(s) c(s) \, ds = x + \int_0^t \Psi(s)' \, dW(s) \text{ a.s. } P. \tag{2.27}$$

In particular $X(t)$ can be chosen such that it possesses continuous paths (note that $H(t)$, $\int H(s)c(s)ds$, and the Itô integral are continuous with respect to t and that $H(t)$ is strictly positive). By the following Lemmas 2.64 and 2.65, comparison of the representations (2.26) and (2.27) yields the fact that $X(t)$ defined as above is the wealth process corresponding to the pair $(\pi, c) \in \mathcal{A}(x)$ with:

$$\pi(t) = \begin{cases} (\sigma(t)^{-1})' \left(\dfrac{\Psi(t)}{H(t)X(t)} + \theta(t) \right) & \text{if } X(t) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

□

Lemma 2.64. *$X(t)$ and $\pi(t)$ as in the proof of part (2) of Theorem 2.63 satisfy*

$$\int_0^T (\pi_i(t) X(t))^2 \, dt < \infty \text{ a.s. } P, \; i = 1, \ldots, d.$$

Proof. We have

$$\begin{aligned} \|\pi(t) X(t)\| &\leq \left\| \left(\sigma(t)^{-1}\right)' \frac{\Psi(t)}{H(t)} + \left(\sigma(t)^{-1}\right)' \theta(t) X(t) \right\| \\ &\leq \underbrace{\left\| \left(\sigma(t)^{-1}\right)' \frac{\Psi(t)}{H(t)} \right\|}_{\alpha(t)} + \underbrace{\left\| \left(\sigma(t)^{-1}\right)' \theta(t) X(t) \right\|}_{\beta(t)}. \end{aligned}$$

i) As $\sigma(t)\sigma(t)'$ is uniformly positive definite, we have

$$\alpha^2(t) \leq \frac{1}{K} \cdot \frac{\|\Psi(t)\|^2}{|H(t)|^2}.$$

Continuity and strict positivity of $H(t)$ on $[0,T]$ implies

$$|H(t)| \geq \min_{s\in[0,T]} H(s) = H(s^*) > 0$$

for P-almost all $\omega \in \Omega$ with $s^* = s^*(\omega)$. As the integral $\int_0^T \|\Psi(t)\|^2 dt$ is finite for P-almost all $\omega \in \Omega$. we have

$$\int_0^T \alpha^2(t)\, dt \leq \frac{1}{K} \int_0^T \frac{\|\Psi(t)\|^2}{\left(\min_{s\in[0,T]} H(s)\right)^2} ds < \infty$$

for P-almost all $\omega \in \Omega$.

ii) $\beta(t)$ satisfies

$$\beta^2(t) = X^2(t) \cdot (b(t) - r(t)\cdot \underline{1})' \cdot \left(\sigma^{-1}(t)'\sigma^{-1}(t)\right)' \cdot \left(\sigma^{-1}(t)'\sigma^{-1}(t)\right) \cdot (b(t) - r(t)\underline{1}).$$

As $\sigma(t)\sigma(t)'$ is uniformly positive definite, we have

$$\beta^2(t) \leq X^2(t) \cdot \frac{1}{K^2} \|b(t) - r(t)\cdot \underline{1}\|^2.$$

As b and r are uniformly bounded and $X(t)$ is continuous on $[0,T]$, this yields

$$\int_0^T \beta^2(t)\, dt \leq \frac{1}{K^2} \int_0^T X^2(t) \|b(t) - r(t)\cdot \underline{1}\|^2 dt \leq \frac{1}{K^2} \left(\max_{t\in[0,T]} X(t)\right)^2 \cdot \int_0^T \|b(t) - r(t)\cdot \underline{1}\|^2 dt < \infty$$

for P-almost all $\omega \in \Omega$.

iii) Due to i), ii), and the relation $(a^2+b^2)/2 \geq 2ab$, we arrive at

$$\int_0^T \|\pi(t) X(t)\|^2 dt \leq 2\left(\int_0^T \alpha^2(t)\, dt + \int_0^T \beta^2(t)\, dt\right) < \infty$$

for P-almost all $\omega \in \Omega$. □

Lemma 2.65. *Let $\pi(t)$, $X(t)$, $c(t)$ be as in the proof of part (2) of Theorem 2.63. If $X(t)$ solves the stochastic differential equation*

$$d(H(t)X(t)) = H(t)X(t)\left(\pi(t)'\sigma(t) - \theta(t)'\right) dW(t) - H(t)c(t)\, dt$$
$$X(0) = x.$$

then $X(t)$ is the wealth process corresponding to (π, c) with $X(0) = x$.

Proof. Let $\tilde{X}(t) := X(t)H(t)$. First assume that we have $X(t) > 0$ for all $t \in [0,T]$ P-almost surely. Together with our assumptions we have

$$\begin{aligned} d\tilde{X}(t) &= H(t)\left[X(t)\left(\pi(t)'\sigma(t) - \theta(t)'\right) dW(t) - c(t)\,dt\right] \\ &= H(t)\left(\frac{\Psi(t)'}{H(t)}\,dW(t) - c(t)\,dt\right) \\ &= \Psi(t)'\,dW(t) - H(t)c(t)\,dt. \end{aligned}$$

Application of the Itô formula, Theorem 2.49, yields

$$d\left(\frac{1}{H(t)}\right) = \frac{1}{H(t)}\left[\left(r(t) + \|\theta(t)\|^2\right) dt + \theta(t)'\,dW(t)\right].$$

and by the product rule, Corollary 2.53, we have

$$\begin{aligned} dX(t) &= d\left(\frac{\tilde{X}(t)}{H(t)}\right) = \tilde{X}(t)\; d\left(\frac{1}{H(t)}\right) + \frac{1}{H(t)}\,d\tilde{X}(t) + d\left\langle \tilde{X}, \tfrac{1}{H}\right\rangle_t \\ &= \left(\frac{\tilde{X}(t)}{H(t)}\left(r(t) + \|\theta(t)\|^2\right) - c(t) + \frac{\theta(t)'\Psi(t)}{H(t)}\right) dt \\ &\qquad + \left(\frac{\tilde{X}(t)}{H(t)}\theta(t)' + \frac{\Psi(t)'}{H(t)}\right) dW(t) \\ &= \left(X(t)\left(r(t) + \theta(t)'\left(\theta(t) + \frac{\Psi(t)}{H(t)X(t)}\right)\right) - c(t)\right) dt \\ &\qquad + X(t)\left(\theta(t) + \frac{\Psi(t)}{H(t)X(t)}\right)' dW(t) \\ &= \left(X(t)\left[r(t) + \pi(t)'\left(b(t) - r(t)\cdot\underline{1}\right)\right] - c(t)\right) dt \\ &\qquad + X(t)\,\pi(t)'\sigma(t)\; dW(t). \end{aligned}$$

Hence, $X(t)$ solves the wealth equation corresponding to (π, c). Due to $X(t) \geq 0$ and Lemma 2.64, we also have $(\pi, c) \in \mathcal{A}(x)$.

If we now assume that $X(t)$ attains the value of zero for some $(t_0, \omega_0) \in [0,T] \times \Omega$, then due to $H(t) > 0$ for all $t \in [0,T]$, $c(t) \geq 0$ for all $t \in [0,T]$ and $B \geq 0$, we must have

$$\begin{aligned} c(t,\omega_0) &= 0 \text{ for all } t \geq t_0. \\ B(\omega_0) &= 0. \end{aligned}$$

Consequently $X(t,\omega_0)$ retains the value of zero on $[t_0,T]$. This implies

$$\Psi(t,\omega_0) = 0,\ \pi(t,\omega_0) = 0 \text{ for all } t \in [t_0,T].$$

In this case, we have $dX(t) = 0$ for all $t \geq t_0$. Due to $X(t) = 0$, $\pi(t) = 0$, $c(t) = 0$ this then also coincides with the right-hand side of the wealth equation. □

Remark 2.66. (1) Note that $1/H(t)$ is the wealth process corresponding to the pair

$$(\pi(t), c(t)) = \left(\sigma^{-1}(t)'\theta(t), 0\right)$$

with an initial wealth of $x := 1/H(0) = 1$ and a final wealth of $B := 1/H(T)$ (compare Exercise (10)).

(2) Further it can be shown that the portfolio process π constructed in the proof of part (2) of Theorem 2.63 is the unique (up to indistinguishability with respect to P) portfolio process with $(\pi, c) \in \mathcal{A}(x)$ and $X(T) = B$ P-a.s. (compare also with Remark 2.69).

Excursion 4: The Martingale Representation Theorem

General assumptions for this section:
Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. On this space we have an m-dimensional Brownian motion $\{(W_t, \mathcal{F}_t)\}_{t\in[0,\infty)}$ with $\{\mathcal{F}_t\}_t$ being the Brownian filtration.

Definition 2.67. A real-valued martingale $\{(M_t, \mathcal{F}_t)\}_{t\in[0,T]}$ with respect to the Brownian filtration $\{\mathcal{F}_t\}_t$ is called a *Brownian martingale.*

Theorem 2.68 (Martingale representation theorem). *Let $\{(M_t, \mathcal{F}_t)\}_{t\in[0,T]}$ be a Brownian martingale with*

$$EM_t^2 < \infty \text{ for all } t \in [0, T],$$

i.e. a square-integrable Brownian martingale. Then there exists a progressively measurable $\mathbb{R}^m$-valued process $\Psi(t)$, $t \in [0, T]$, with

$$E\left(\int_0^T \|\Psi(t)\|^2 dt\right) < \infty$$

and

$$M_t = M_0 + \int_0^t \Psi(s)' dW(s) \text{ a.s. } P.$$

Proof. As M_0 is $\mathcal{F}_0$-measurable it must be constant P-almost surely. Thus, we can w.l.o.g. assume $M_0 = 0$. As M_t is a martingale, this also yields

Optimal Portfolios

5.1. Introduction and Formulation of the Problem

So far, in the continuous-time market model we were only looking for trading strategies that generate a given payoff profile ("replication") or a lower bound for a payment ("hedging strategy"). The costs of this replication strategy then determined the price of the payoff profile (i.e. the price of the corresponding contingent claim). Now we look at the opposite situation: We are given a fixed initial capital and search for an admissible self-financing pair of portfolio and consumption processes which yields a payment stream as lucrative as possible. We had already considered a similar task in the one-period model at the very beginning of the book. There we looked for a portfolio yielding the biggest possible return for a given bound on its variance. This time we shall look at a more general situation by introducing the so-called portfolio problem.

General formulation of the portfolio problem. For a given initial capital of $x > 0$, the continuous-time *portfolio problem* in our given continuous-time financial market consists of the determination of an optimal consumption and investment strategy. More precisely: The investor has to determine *how many* shares of *which* security he has to hold at *which time instant* and *how much* of his wealth he is allowed to consume to *maximize his utility* from *consumption* during the period $[0, T]$ and / or from the *terminal wealth* at the time horizon $t = T$. Thus, the portfolio problem contains a choice problem ("which" security), a problem of volumes ("how many" shares, "how much"...), and a component which is dynamic with respect to time ("which

time"). We shall give the investor the possibility to decide on his actions at each time instant $t \in [0, T]$. The corresponding flow of information which forms the basis of his decisions should be the observation of past and present prices. He should not have any knowledge of future security prices and also he should not have any insider information.

General assumptions for this chapter:
We make the same assumptions as in Chapter 2, Section 2, p. 57.
Further, we always consider a self-financing pair (π, c) consisting of a portfolio process π and a consumption process c to be admissible with initial wealth $x > 0$, for short: $(\pi, c) \in \mathcal{A}(x)$ (see also Definition 2.62).

Solution approaches in the continuous-time market model. Mainly we have to distinguish between two approaches to solve the portfolio problem in the continuous-time market setting of Chapter 2. The (historically) first approach is the stochastic control method which goes back to Robert Merton: see [**MERT 69**], [**MERT 90**]. Merton's main idea consists of interpreting the portfolio problem as a stochastic control problem to which he then applied standard methods of stochastic control theory. We shall present this method in Section 4 (of course preceded by an excursion on optimal stochastic control). On the basis of the stochastic control method, new methods for treating portfolio problems under transaction costs were developed during the last years (see [**KORN 97**]).

The so-called martingale method represents the second main approach for solving the continuous-time portfolio problem. It was introduced by Cox & Huang (see [**CO/HU 89**]), Karatzas, Lehoczky & Shreve (see [**K/L/S 87**]), and Pliska (see [**PLIS 86**]) in different variants during the eighties. A common feature of these variants is the decomposition of the portfolio problem into the determination of the optimal payments (consumption and / or final wealth) by methods of convex analysis followed by the determination of the corresponding portfolio process. In the complete market model of Section 2.3 the existence of this portfolio process is ensured by Theorem 2.63. We shall present the martingale method in Section 2. In Section 3 we shall give an application to a portfolio problem where options are traded instead of stocks.

Formulation of the problem. To judge the advantages of a payment stream in an objective way, we introduce a functional J which measures the utility of such a payment stream. A "good" payment stream is then characterized by a large value of J. So for a given initial wealth $x > 0$ an

investor looks for a self-financing pair $(\pi, c) \in \mathcal{A}(x)$ which maximizes the expected utility from consumption and / or terminal wealth.

$$J(x;\pi,c) = E\left(\int_0^T U_1(t,c(t))\,dt + U_2(X(T))\right) \tag{5.1}$$

Here, $X(t)$ is the wealth process corresponding to x and (π, c). U_1 and U_2 are assumed to be utility functions in the sense of the following definition.

Definition 5.1. (1) Let $U : (0,\infty) \to \mathbb{R}$ be a strictly concave and continuously differentiable function satisfying

$$U'(0) := \lim_{x \downarrow 0} U'(x) = +\infty,\ U'(\infty) := \lim_{x\to\infty} U'(x) = 0.$$

Then U is called a *utility function.*

(2) A continuous function $U : [0,T] \times (0,\infty) \to \mathbb{R}$ such that for all $t \in [0,T]$ the function $U(t,.)$ is a utility function in the sense of part (1), is also called a *utility function.*

Examples of utility functions.

(1) $U(x) = \ln(x)$

(2) $U(x) = \sqrt{x}$

(3) $U(x) = x^\alpha$ for $0 < \alpha < 1$

(4) $U(t,x) = e^{-\rho t} \cdot U_1(x)$, $\rho > 0$ with U_1 a utility function as in (1) or (2).

Remark 5.2. (1) By the above definition, a utility function is strictly increasing. Thus, each additional unit of wealth leads to additional utility. Moreover, we require a utility function to be strictly concave. This in particular implies that $U'(x)$ is strictly decreasing. Thus we have a decreasing marginal utility, i.e. the gain of utility from one additional unit of money decreases with increasing x. The marginal utility in $x = 0$ is infinite, which can be paraphrased as "a tiny bit is much more than nothing" while the vanishing marginal utility in $x = \infty$ models a saturation effect.

(2) The methods of portfolio optimization presented in the following can also be applied to a wider class of utility functions as the one given in Definition 5.1. This wider class would also contain the popular but highly criticized quadratic utility function

$$U(x) = -\frac{1}{2}(x-a)^2.$$

However, this would need a more complicated notation. For the treatment of such utility functions we refer to [**KORN 97**].

Note that for an arbitrary $(\pi, c) \in \mathcal{A}(x)$ the expectation in $J(x;\pi,c)$ is not necessarily defined. We could now restrict the class of self-financing pairs (π, c) to all those for which the expectation in $(\pi, c) \in \mathcal{A}(x)$ is finite. However, an infinite positive expected utility is an investor's dream if it can be reached. We shall therefore allow for such a possibility by only requiring the following weak integrability condition for a feasible pair (π, c).

Definition 5.3. The problem

$$\max_{(\pi,c)\in\mathcal{A}'(x)} J(x;\pi,c) \tag{5.2}$$

with

$$\mathcal{A}'(x) = \left\{(\pi,c) \in \mathcal{A}(x) \,\middle|\, E\left(\int_0^T U_1(t,c(t))^- \, dt + U_2(X(T))^-\right) < \infty\right\}$$

is called the *continuous-time portfolio problem.*

Remark 5.4. (1) By restricting to the set $\mathcal{A}'(x)$, the integral in (5.2) is always defined. Also, the expectation exists but can be equal to infinity.

(2) In the case of positive utility functions, $U_1(t,.) > 0$ and $U_2(.) > 0$, the equality $\mathcal{A}(x) = \mathcal{A}'(x)$ is trivially satisfied.

In the following we shall present two methods for solving the portfolio problem. The first such method, the so-called martingale method, relies heavily on the completeness of the market. The second method, the so-called stochastic control method, is an application of standard methods of stochastic control theory to the portfolio problem (5.2).

5.2. The Martingale Method

General assumptions for this section:
Additionally to the general assumptions for this chapter we will use the notation of Section 2.3, p. 65. In particular, we have $d = m$. Thus, all assumptions of Theorem 2.63, "completeness of the market", are satisfied.

The main idea. The martingale method is mainly based on a separation of the dynamical (with respect to time) problem (5.2) into a static optimization problem ("determination of the optimal payoff profile") and a representation problem ("compute the portfolio process corresponding to the optimal payoff profile").

Motivation. We start by looking at a portfolio problem without consumption (i.e. we assume $c \equiv 0$, $U_1 \equiv 0$). Let the self-financing pair $(\pi, 0)$ be admissible for an initial wealth of $x > 0$. By Theorem 2.63, completeness of the market, we have for each corresponding wealth process $X^\pi(T)$

$$E\left(H\left(T\right)X^\pi\left(T\right)\right) \leq x \quad \text{for } T \geq 0.$$

Let the final payment $B \geq 0$ be $\mathcal{F}_T$-measurable with $E(H(T)B) = x$. Of course such random variables exist. A trivial example is

$$B := \frac{x}{E\left(H\left(T\right)\right)}.$$

By Theorem 2.63 there exists a portfolio process $(\pi, 0) \in \mathcal{A}(x)$ with $B = X^\pi(T)$ P-a.s. Now define

$$\mathcal{B}(x) := \left\{B \geq 0 \;\middle|\; B\ \mathcal{F}_T\text{-measurable. } E\left(H\left(T\right)B\right) \leq x,\ E\left(U_2\left(B\right)^-\right) < \infty\right\}.$$

Obviously, $\mathcal{B}(x)$ represents the set of all final wealths with $E(U_2(B)^-) < \infty$ that can be generated by trading in the securities starting with some initial wealth $y \in (0, x]$ and satisfying $E(U_2(B)^-) < \infty$. So, for determining the optimal final wealth $X^\pi(T)$ in our portfolio problem

$$\max_{(\pi,0)\in\mathcal{A}'(x)} E\left(U_2\left(X^\pi\left(T\right)\right)\right), \tag{5.3}$$

it is enough to maximize over all random variables $B \in \mathcal{B}(x)$, i.e. it is sufficient to solve the problem

$$\max_{B\in\mathcal{B}(x)} E\left(U_2\left(B\right)\right). \tag{5.4}$$

Note that in (5.4), we only have to optimize over a set of random variables. The time appearing in (5.3) (note that we optimized over a class of stochastic processes!) has disappeared. Therefore, we call (5.4) a static optimization problem. If now B^* is an optimal final wealth in (5.4), then to solve the portfolio problem completely we have to solve the representation problem

- Find a $(\pi, 0) \in \mathcal{A}'(x)$ with $X^\pi(T) = B^*$ a.s. P.

First we shall present the solution of the optimization problem (5.4). To motivate it we give a brief review of the Lagrangian method of (deterministic) non-linear constrained optimization.

Solving optimization problems with the help of the Lagrangian method. Let the function $f : \mathbb{R}^n \to \mathbb{R}$ be strictly concave, let $g : \mathbb{R}^n \to \mathbb{R}^k$ be convex, and let $f, g \in C^1$. Then we have

$$\hat{x} \text{ solves the optimization problem}$$

$$\max_{x \in \mathbb{R}^n} f(x)$$

$$\text{subject to} \quad g(x) = 0$$

$$\Leftrightarrow \text{ there exists a } \hat{\lambda} \in \mathbb{R}^k \text{ such that } \left(\hat{x}, \hat{\lambda}\right) \in \mathbb{R}^{n+k} \text{ satisfies}$$

$$\frac{\partial}{\partial x_i} f(x) - \sum_{j=1}^{k} \lambda_j \frac{\partial}{\partial x_i} g(x) = 0, \; i = 1, \ldots, n$$

$$g_i(x) = 0, \; i = 1, \ldots, k.$$

In words: $\left(\hat{x}, \hat{\lambda}\right) \in \mathbb{R}^{n+k}$ is a zero of the derivative of the Lagrangian function

$$L(x, \lambda) = f(x) - \lambda' g(x).$$

The Lagrangian method for the portfolio problem - motivation. We now set up an analogue to the above Lagrangian function in the setting of the optimization problem (5.4).

$$L(B, y) := E(U_2(B) - y \cdot (H(T) B - x))$$

with $y > 0$. Formally differentiating L with respect to B and y and interchanging the expectation with the differentiation process yields the equations

$$0 = L_B(B, y) = E\left(U_2'(B) - yH(T)\right),$$
$$0 = L_y(B, y) = x - E(H(T) B).$$

We will not justify this differentation, but use it only to derive the form of the optimal terminal wealth heuristically (of course we shall prove optimality later!). A random variable B satisfying

$$U_2'(B) - yH(T) = 0 \text{ a.s. } P$$

would solve the first equation. As by assumption the range of $U_2'(.)$ equals $\mathbb{R}^+$ and as $U_2'(.)$ is strictly decreasing, it can be inverted on $\mathbb{R}^+$ and we obtain

$$B = \left(U_2'\right)^{-1}(yH(T)). \tag{5.5}$$

Putting this into the second equation yields

$$0 = x - \underbrace{E\left(H(T) \cdot \left(U_2'\right)^{-1}(y \cdot H(T))\right)}_{=:\chi(y)}.$$

Now, if we can solve this equation uniquely for y then we have found a possible candidate for the optimal final wealth via equation (5.5). Therefore we define

$$Y(u) := \chi^{-1}(u), \qquad I_2 := \left(U_2'\right)^{-1}$$

and obtain as a possible candidate for the optimal final wealth

$$B^* = I_2\left(Y(x) \cdot H(T)\right) > 0.$$

Now it will be our aim to prove that B^* is indeed the optimal final wealth.

Notation. Going back to the general optimization problem we define in analogy to the above considerations:

$$\begin{aligned} I_2(y) &:= \left(U_2'\right)^{-1}(y) && \text{for } y \in (0, \infty), \\ I_1(t, y) &:= \left(U_1'\right)^{-1}(t, y) && \text{for } y \in (0, \infty),\ t \text{ fixed}, \\ \chi(y) &:= E\left(\int_0^T H(t)\, I_1\left(t, y \cdot H(t)\right) dt + H(T)\, I_2\left(y \cdot H(T)\right)\right) \end{aligned}$$

where $U_1'(\ldots)$ denotes the partial derivative with respect to x, the second component.

The properties of $\chi(y)$ will be summarized in the following lemma.

Lemma 5.5. *Assume $\chi(y) < \infty$ for all $y > 0$. Then χ is continuous on $(0, \infty)$, strictly decreasing and satisfies*

$$\chi(0) := \lim_{y \downarrow 0} \chi(y) = \infty, \quad \chi(\infty) := \lim_{y \to \infty} \chi(y) = 0.$$

Proof. (1) Continuity of χ follows from continuity of H, I_1, I_2 and the dominated convergence theorem.

(2) $I_1(t, .)$, $I_2(.)$ are both strictly decreasing on $(0, \infty)$. As we have $H(t) > 0$ for all $t \in [0, T]$ we thus also have that $\chi(y)$ is strictly decreasing in y.

(3) Due to

$$\begin{aligned} \lim_{y \downarrow 0} I_1(t, y) &= \lim_{y \downarrow 0} I_2(y) = +\infty, \\ \lim_{y \to \infty} I_1(t, y) &= \lim_{y \to \infty} I_2(y) = 0. \end{aligned}$$

and the monotonicity of I_1, I_2 the assertions on $\chi(0)$ and $\chi(\infty)$ are a consequence of the dominated convergence theorem.

□

Remark 5.6. Lemma 5.5 in particular implies the existence of

$$Y(x) := \chi^{-1}(x)$$

on $(0, \infty)$ with

$$Y(0) := \lim_{x \downarrow 0} Y(x) = +\infty \quad \text{and} \quad Y(\infty) := \lim_{x \to \infty} Y(x) = 0.$$

The following simple lemma will prove to be quite useful for the proof of the main theorem of the martingale method, Theorem 5.8.

Lemma 5.7. *Let U be a utility function with $I := (U')^{-1}$. Then we have*

$$U(I(y)) \geq U(x) + y(I(y) - x) \quad \text{for } 0 < y, x < \infty.$$

Proof. As U is concave we have

$$U(I(y)) \geq U(x) + U'(I(y))(I(y) - x) = U(x) + y(I(y) - x).$$

□

Now we are ready to give the solution of the portfolio problem.

Theorem 5.8 (Optimal consumption and optimal terminal wealth). *Consider the portfolio problem* (5.2). *Let $x > 0$ and $\chi(y) < \infty$ for all $y > 0$. Set $Y(x) := \chi^{-1}(x)$. Then, for*

$$B^* := I_2(Y(x) \cdot H(T)), \qquad \text{"optimal terminal wealth"}$$
$$c^*(t) := I_1(t, Y(x) \cdot H(t)), \qquad \text{"optimal consumption"}$$

there exists a self-financing portfolio process $\pi^*(t)$, $t \in [0,T]$, such that we have

$$(\pi^*, c^*) \in \mathcal{A}'(x), \quad X^{x,\pi^*,c^*}(T) = B^* \text{ a.s. } P.$$

and such that (π^*, c^*) solves the portfolio problem (5.2). *Here, $X^{x,\pi^*,c^*}(t)$ is the wealth process corresponding to the pair (π^*, c^*) and the initial wealth x.*

Proof. i) By definition of B^* and $c^*(t)$ we have

$$E\left(\int_0^T H(t)\, c^*(t)\, dt + H(T)\, B^*\right) = x.$$

Note that we have $Y(x) \cdot H(t) > 0$ and that due to $I_1, I_2 > 0$ also B^* and c^* are positive. The existence of a portfolio process π^* corresponding to the payoff profile $(c^*(t), B^*)$ satisfying $(\pi^*, c^*) \in \mathcal{A}(x)$ is implied by Theorem 2.63 on complete markets.

ii) Next, we show that the unique (up to indistinguishability) portfolio process π^* even satisfies $(\pi^*, c^*) \in \mathcal{A}'(x)$. By Lemma 5.7 we have

$$U_1(t, c^*(t)) \geq U_1(t, 1) + Y(x) \cdot H(t) \cdot (c^*(t) - 1).$$

$$U_2(B^*) \geq U_2(1) + Y(x) \cdot H(T) \cdot (B^* - 1).$$

From this, the obvious relation $a \geq b \Rightarrow a^- \leq b^- \leq |b|$, and the fact that $Y(x) \cdot H(t)$, B^* and c^* are positive, we conclude the following chain of inequalities

$$\begin{aligned}
&E\left(\int_0^T U_1(t, c^*(t))^- \, dt + U_2(B^*)^-\right) \\
&\qquad \leq E\left(\int_0^T \left(|U_1(t,1)| + Y(x) \cdot H(t)\,(c^*(t)+1)\right) dt \right. \\
&\qquad\qquad \left. + |U_2(1)| + Y(x) \cdot H(T)\,(B^*+1)\right) \\
&\qquad = |U_2(1)| + \int_0^T |U_1(t,1)| \, dt \\
&\qquad\qquad + Y(x)\left(x + E(H(T)) + \int_0^T E(H(t)) \, dt\right) < \infty.
\end{aligned}$$

To see the last inequality note that $U_1(t,x)$ is continuous and that the expected value $E(H(t))$ is bounded on $[0,T]$.

iii) Finally, we prove optimality of (π^*, c^*). Therefore, consider an arbitrary pair $(\pi, c) \in \mathcal{A}'(x)$ with wealth process $X^{x,\pi,c}$. From

$$U_1(t, c^*(t)) \geq U_1(t, c(t)) + Y(x) \cdot H(t)\,(c^*(t) - c(t))$$

$$U_2(B^*) \geq U_2(X^{x,\pi,c}(T)) + Y(x) \cdot H(T)\,(B^* - X^{x,\pi,c}(T))$$

we conclude

$$\begin{aligned}
&E\left(\int_0^T U_1(t, c^*(t)) \, dt + U_2(B^*)\right) \\
&\qquad \geq J(x; \pi, c) + Y(x) \cdot \left(E\left(\int_0^T H(t)\, c^*(t) \, dt + H(T)\, B^*\right)\right. \\
&\qquad\qquad \left. - E\left(\int_0^T H(t)\, c(t) \, dt + H(T)\, X^{x,\pi,c}(T)\right)\right)
\end{aligned}$$

$$= J(x;\pi,c) + Y(x) \cdot \underbrace{\left(x - E\left(\int_0^T H(t)\,c(t)\,dt + H(T)\,X^{x,\pi,c}(T)\right)\right)}_{\geq 0}$$

$$\geq J(x;\pi.c).$$

To see the last inequality use part (1) of Theorem 2.63 on complete markets. □

Example: "Logarithmic utility". We now consider the special choice of the logarithmic utility function. Note that values smaller than 1 have a negative utility! We then have

$$U_1(t,x) = U_2(x) = \ln(x)$$

$$\Rightarrow I_1(t,y) = I_2(y) = \frac{1}{y}$$

$$\Rightarrow \chi(y) = E\left(\int_0^T H(t) \cdot \frac{1}{y \cdot H(t)}\,dt + H(T) \cdot \frac{1}{y \cdot H(T)}\right) = \frac{1}{y}(T+1)$$

$$\Rightarrow Y(x) = \chi^{-1}(x) = \frac{1}{x}(T+1).$$

With this and Theorem 5.8 both the optimal consumption and the optimal final wealth are given by

$$c^*(t) = I_1(t, Y(x) \cdot H(t)) = \frac{x}{T+1} \cdot \frac{1}{H(t)},$$

$$B^* = I_2(Y(x) \cdot H(T)) = \frac{x}{T+1} \cdot \frac{1}{H(T)}.$$

In this special example, we can even calculate the portfolio process explicitly. To do so, note that by the proof of part (2) of Theorem 2.63 we have

$$(5.6)\quad H(t) \cdot X^{x,\pi^*,c^*}(t) = E\left(\int_t^T H(s)\,c^*(s)\,ds + H(T)\,B^* \mid \mathcal{F}_t\right)$$

$$= x \cdot \frac{T-t+1}{T+1}.$$

This implies

$$x = x \cdot \frac{T-t+1}{T+1} + x \cdot \frac{t}{T+1} = H(t) \cdot X^{x,\pi^*,c^*}(t) + \int_0^t H(s)\,c^*(s)\,ds.$$

Application of Itô's formula to the product $H(t) \cdot X^{x,\pi^*,c^*}(t)$ on the right-hand side of the above equation yields (compare with the proof of part (2) of Theorem 2.63)

$$x = x + \int_0^t \underbrace{H(s) \cdot X^{x,\pi^*,c^*}(s) \left(\pi^*(s)' \sigma(s) - \theta(s)'\right)}_{=:f(s)} dW(s).$$

Hence we must have

$$f(s) = 0 \text{ a.s. } P \text{ for all } s \in [0,T].$$

As $H(s) \cdot X^{x,\pi^*,c^*}(s)$ is positive we thus obtain

$$\pi^*(t) = \left(\sigma(t)'\right)^{-1} \theta(t) \text{ for all } t \in [0,T].$$

In the special case of $d = 1$ and constant coefficients r, b, σ we get

$$\pi^*(t) = \frac{b-r}{\sigma^2} \quad \text{"local risk premium for stock investment"}.$$

Here, it is important to point out that the simplicity of the form of $\pi^*(t)$ is a bit misleading. Following a constant portfolio process indeed means that we have to trade at each time instant as stock and bond prices move in different ways. The constant quotient $(1-\pi)/\pi$ of bond wealth divided by stock wealth would immediately be no longer valid if one would not do the suitable trading ("rebalancing") actions. Note that from (5.6) we obtain another representation of the consumption rate

$$c^*(t) = \frac{1}{T-t+1} X^{x,\pi^*,c^*}(t).$$

So, the consumption rate is proportional to the current wealth of the investor and inversely proportional to the remaining time $(T-t)$.

The method of determining π^* in the above example can be generalized. Indeed, we have:

Theorem 5.9 (Solution of the representation problem (5.2)). *Let the portfolio problem (5.2) be given. Let $x > 0$ and assume $\chi(y) < \infty$ for all $y > 0$. Further, c^* and B^* should be given as in Theorem 5.8. If there exists a function $f \in C^{1,2}([0,T] \times \mathbb{R}^d)$ with $f(0,0,\ldots,0) = x$ and*

$$\frac{1}{H(t)} \cdot E\left(\int_t^T H(s)\, c^*(s)\, ds + H(T)\, B^* \mid \mathcal{F}_t\right) = f(t, W_1(t), \ldots, W_d(t)),$$

then for all $t \in [0,T]$ we have

$$\pi^*(t) = \frac{1}{X^{x,\pi^*,c^*}(t)} \sigma^{-1}(t) \nabla_x f(t, W_1(t), \ldots, W_d(t)).$$

Here, $\nabla_x f$ denotes the gradient of $f(t, x_1, ..., x_d)$ with respect to the x-coordinates.

Proof. Application of the multi-dimensional Itô-formula, Theorem 2.52, yields

$$\frac{1}{H(t)} \cdot E\left(\int_t^T H(s) c^*(s)\, ds + H(T) B^* \,|\mathcal{F}_t \right) = f(t, W_1(t) ..... W_d(t))$$

$$= f(0. ...., 0) + \sum_{i=1}^{d} \int_0^t f_{x_i}(s, W_1(s) ..... W_d(s))\, dW_i(s)$$

$$+ \int_0^t \left(f_t(s, W_1(s) ...., W_d(s)) + \tfrac{1}{2} \sum_{i=1}^{d} f_{x_i x_i}(s, W_1(s) ...., W_d(s))\, ds \right).$$

On the other side, as in the proof of Theorem 2.63 on complete markets, we also have

$$\frac{1}{H(t)} \cdot E\left(\int_t^T H(s) c^*(s)\, ds + H(T) B^* \,|\mathcal{F}_t \right) = X^{x,\pi^*,c^*}(t)$$

$$= x + \int_0^t \left(\left(r(s) + \pi^*(s)' (b(s) - r(s) \cdot \underline{1}) \right) X^{x,\pi^*,c^*}(s) - c(s) \right) ds$$

$$+ \int_0^t X^{x,\pi^*,c^*}(s)\, \pi^*(s)' \sigma(s)\, dW(s).$$

To see the last equality, note that (π^*, c^*) is an admissible, self-financing pair, and thus the wealth process obeys the wealth equation (2.22). Comparison of the integrands of the stochastic integrals of both representations yields the assertion by uniqueness of the representation of an Itô process. □

With the help of the proof of Theorem 5.8 we can also give the solution of both the pure consumption and the pure terminal wealth maximization problem.

Corollary 5.10. (1) *The optimal terminal wealth B^* of the problem*

(5.7) $$\max_{(\pi,0) \in \mathcal{A}'(x)} E(U_2(X^{x,\pi}(T)))$$

is given by

$$B^* := I_2(Y(x) \cdot H(T))$$

where in the definition of $\setminus(y)$ we then have to set $I_1(t, y) \equiv 0$.

(2) *The optimal consumption process* $c^*(t)$ *of the problem*

$$\max_{(\pi,c)\in\mathcal{A}'(x)} E\left(\int_0^T U_1(t,c(t))\,dt\right) \tag{5.8}$$

is given by

$$c^*(t) := I_1(t, Y(x)\cdot H(t))$$

where in the definition of $\chi(y)$ *we then have to set* $I_2(y)\equiv 0$.

5.3. Optimal Option Portfolios

As an application of the martingale method for portfolio optimization in the complete market which has just been introduced in the preceding section. we shall now consider a portfolio problem where instead of stocks we are only allowed to trade in options on these stocks (see [**KO/TR 99**]).

General assumptions for this section:
We use the same notations as in Chapter 2. Section 3. p. 65. Further. we assume $d = m$. Thus, the assumptions of Theorem 2.63 on complete markets are satisfied. Further on. we restrict ourselves to the case of constant market coefficients r, b, σ.

Description of the market model. We consider a market where a bond, d stocks. and d options on these stocks are traded. However. we assume here that we are only allowed to hold a portfolio consisting of the bond and the options. We further assume that the options have a price process of the form

$$f^{(i)}(t, P_1(t), \ldots, P_d(t)), \quad i = 1, \ldots, d, \quad f \in C^{1,2}. \tag{5.9}$$

and where the option prices satisfy the requirements of Proposition 3.19 (this is particularly satisfied for European puts and calls in the Black-Scholes model). Let

$$\varphi(t) = (\varphi_0(t), \varphi_1(t), \ldots, \varphi_d(t))$$

be an admissible trading strategy in bond and options. I.e. the integrals

$$\int_0^t \varphi_0(s)\; dP_0(s)$$

$$\int_0^t \varphi_i(s)\; df^{(i)}(s, P_1(s), \ldots, P_d(s))$$

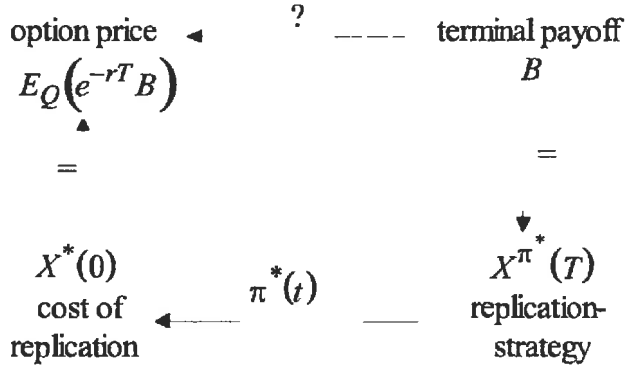


Figure 5.1. Scheme 1: Option pricing

are assumed to be defined and $\varphi(t)$ is assumed to be $\mathcal{F}_t$-progressively measurable. The corresponding wealth process $X(t)$ is given by

$$X(t) = \varphi_0(t) P_0(t) + \sum_{i=1}^{d} \varphi_i(t) f^{(i)}(t, P_1(t), ..., P_d(t)).$$

Let U be a utility function. We look at the problem

$$\max_{\varphi} E(U(X(T))) \tag{5.10}$$

Motivation. To motivate the solution of the problem (5.10), we first take a look at the following diagrams representing the solution of the option pricing problem via the replication approach and the solution of the portfolio problem via the martingale approach. This will then inspire the third diagram showing the solution of problem (5.10).

Figure 5.1: Option pricing
Starting out from the final payment B of an option, we are looking for the option price. To find it, we first replicate the final payment via following a suitable portfolio strategy leading to a final wealth that coincides with the final option payment. Then the costs of setting up the cheapest replication strategy yield the option price.

Figure 5.2: Portfolio optimization with stocks
We are given an initial wealth of x. This will be invested according to a portfolio process $\pi^*(t)$ with the aim to obtain a terminal wealth which promises the highest possible utility (we ignore the possibility of consumption here). To do so, we first determine an optimal final payoff B^* and then look for a replication strategy for B^*.

Figure 5.3: Portfolio optimization with options
As in the stock portfolio problem above, we look for an optimal final wealth

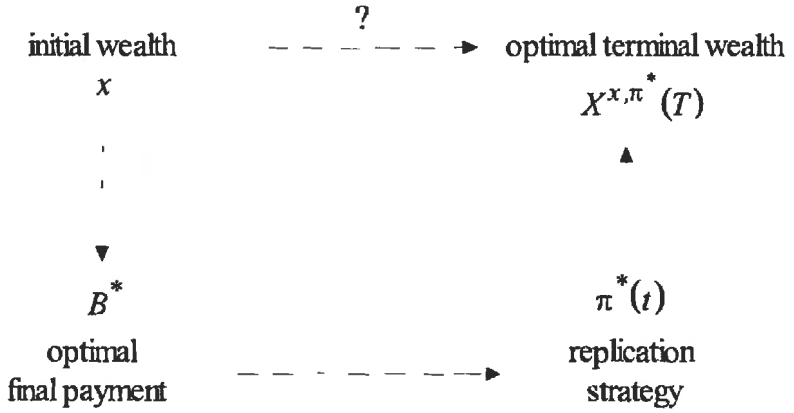


Figure 5.2. Scheme 2: Portfolio optimization with stocks

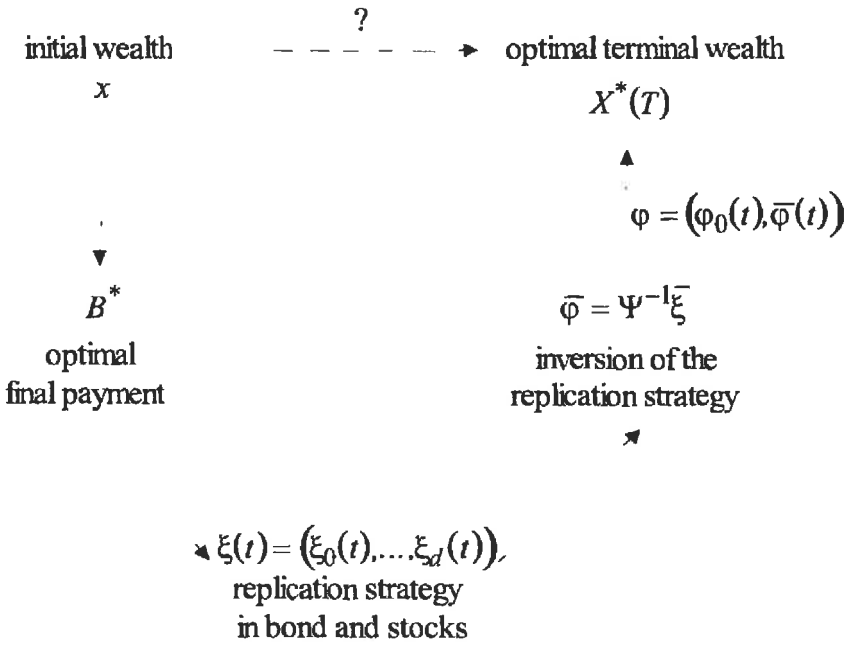


Figure 5.3. Scheme 3: Portfolio optimization with options

starting with an initial capital of x. To do so, we first determine an optimal payoff B^* and then a replication strategy $\xi(t) = (\xi_0(t), ..., \xi_d(t))$ in bond and stocks for the payoff B^*. As stocks should not appear in our portfolio, we have to replicate the stock position by bond and options. This so-obtained bond and option strategy yields the optimal terminal wealth $X^*(T)$. This is formalized in the following theorem.

Theorem 5.11. *Let the Delta matrix* $\Psi(t) = (\Psi_{ij}(t))_{ij}$, $i, j = 1, ..., d$, *with*

$$\Psi_{ij} := f_{p_j}^{(i)}(t, P_1(t), ..., P_d(t))$$

be regular for all $t \in [0,T)$. *Then the option portfolio problem* (5.10) *possesses the following explicit solution:*

(1) *The optimal terminal wealth* B^* *coincides with the optimal terminal wealth of the corresponding stock portfolio problem* (5.2).

(2) *Let* $\xi(t) = (\xi_0(t), \ldots, \xi_d(t))$ *be the optimal trading strategy in the corresponding stock portfolio problem* (5.2). *Then, the optimal trading strategy* $\varphi(t) = (\varphi_0(t), \varphi_1(t), \ldots, \varphi_d(t))$ *in the option portfolio problem* (5.10) *is given by*

$$\bar{\varphi}(t) = \left(\Psi(t)'\right)^{-1} \cdot \bar{\xi}(t),$$

$$\varphi_0(t) = \frac{\left(X(t) - \sum_{i=1}^{d} \varphi_i(t) f^{(i)}(t, P_1(t), \ldots, P_d(t))\right)}{P_0(t)}.$$

with $\bar{\varphi}(t) := (\varphi_1(t), \ldots, \varphi_d(t))$ *and* $\bar{\xi}(t) := (\xi_1(t), \ldots, \xi_d(t))$.

Proof. i) By Proposition 3.19 we have

$$f^{(i)}(t, P_1(t), \ldots, P_d(t)) = \sum_{j=0}^{d} \Psi_{ij}(t) \cdot P_j(t). \quad i = 1, \ldots, d,$$

where

$$\Psi_{i0} := \frac{f^{(i)}(t. P_1(t), \ldots, P_d(t)) - \sum_{j=1}^{d} \Psi_{ij}(t) P_j(t)}{P_0(t)}.$$

As the trading strategy $(\Psi_{i0}(t), \Psi_{i1}(t), \ldots, \Psi_{id}(t))$ is self-financing we also have

$$df^{(i)}(t, P_1(t), \ldots, P_d(t)) = \sum_{j=0}^{d} \Psi_{ij}(t)\ dP_j(t).$$

Let now φ be an admissible and self-financing trading strategy in bond and options. Then, the corresponding wealth process satisfies

$$dX(t) = \varphi_0(t) \cdot dP_0(t) + \sum_{i=1}^{d} \varphi_i(t)\, df^{(i)}(t, P_1(t), \ldots P_d(t)).$$

With the above representation of the option prices we obtain

$$dX(t) = \left(\varphi_0(t) + \sum_{i=1}^{d} \varphi_i(t)\,\Psi_{i0}(t)\right) dP_0(t) + \sum_{j=1}^{d} \left(\sum_{i=1}^{d} \varphi_i(t)\,\Psi_{ij}(t)\right) dP_j(t)$$
$$=: \zeta_0(t)\,dP_0(t) + \sum_{j=1}^{d} \zeta_j(t)\,dP_j(t).$$

Hence. the trading strategy ζ in bond and stocks is self-financing. As φ is admissible for the option portfolio problem (and as Ψ is a replication strategy) this implies that ζ is admissible for the stock portfolio problem.

ii) Let $\xi(t)$ be the optimal trading strategy of the corresponding stock portfolio problem with optimal wealth process $X(t)$. Then we get

$$X(T) = B^* \text{ a.s. } P,$$

$$dX(t) = \xi_0(t)\,dP_0(t) + \sum_{i=1}^{d} \xi_i(t)\,dP_i(t).$$

To obtain a trading strategy in bond and options admitting the same wealth process $X(t)$, we make the ansatz

$$dX(t) = \varphi_0(t)\,dP_0(t) + \sum_{i=1}^{d} \varphi_i(t)\,df^{(i)}(t, P_1(t), \ldots, P_d(t))$$

and obtain similarly to part (1) of the proof

$$dX(t) = \left(\varphi_0(t) + \sum_{j=1}^{d} \varphi_j(t)\,\Psi_{j0}(t)\right) dP_0(t) + \sum_{i=1}^{d} \left(\sum_{j=1}^{d} \varphi_j(t)\,\Psi_{ji}(t)\right) dP_i(t).$$

Comparison of the coefficients of the dP_i-terms of both representations of $X(t)$ yields the desired form of the last d components of $\varphi(t)$. Comparison of the dt-coefficients of both representations of $dX(t)$ yields

$$\xi_0(t) = \varphi_0(t) + \sum_{j=1}^{d} \varphi_j(t) \cdot \Psi_{j0}(t).$$

Due to

$$\xi_0(t) = \frac{X(t) - \sum_{i=1}^{d} \xi_i(t) P_i(t)}{P_0(t)} = \frac{X(t) - \sum_{i=1}^{d} \left(\sum_{j=1}^{d} \varphi_j(t) \Psi_{ji}(t) \right) P_i(t)}{P_0(t)} -$$

we also have

$$\begin{aligned}\varphi_0(t) &= \frac{X(t) - \sum_{i=0}^{d} \sum_{j=1}^{d} \varphi_j(t) \Psi_{ji}(t) P_i(t)}{P_0(t)} \\ &= \frac{X(t) - \sum_{j=1}^{d} \varphi_j(t) \sum_{i=0}^{d} \Psi_{ji}(t) P_i(t)}{P_0(t)} \\ &= \frac{X(t) - \sum_{j=1}^{d} \varphi_j(t) f(t, P_1(t) \ldots, P_d(t))}{P_0(t)}.\end{aligned}$$

This shows that $(\varphi_0(t), \varphi_1(t), \ldots, \varphi_d(t))$ is self-financing. To show that $\varphi(t)$ is admissible it is enough to show that the stochastic integrals

$$\sum_{i=1}^{d} \int_0^t \varphi_i(s)\, df^{(i)}(s, P_1(s), \ldots, P_d(s))$$

are defined. Using the representation of the df-terms, the explicit form of $\varphi(t)$, and the admissibility of the strategy $\xi(t)$ in the stock portfolio problem, we get the admissibility of $\varphi(t)$ for the option portfolio problem.

iii) As we have shown in ii), following the bond and option strategy $\varphi(t)$ leads to the same utility as by using $\xi(t)$ in the stock portfolio problem. On the other hand, we cannot obtain a higher expected utility in the option portfolio problem. To see this, note that the existence of a strategy with a higher utility in the option portfolio problem would induce the existence of a bond and stock strategy ζ (as in i)) yielding a higher expected utility as ξ in the stock portfolio problem. But this would contradict optimality of $\xi(t)$. □

Remark 5.12. (1) Under the given assumptions, the optimal final wealth only depends on the utility functions but not on the choice of the tradable securities.

(2) The optimal strategy depends heavily on the traded options via the delta matrix (more precisely: via the replication strategy for the options).

Example: "Logarithmic utility". Let us choose the utility function

$$U(x) = \ln(x).$$

We specifically consider the Black-Scholes model with $d = 1$. From the example "logarithmic utility" in the previous section we know that the stock position in the optimal trading strategy is given by

$$\xi_1(t) = \frac{\pi^*(t) \cdot X(t)}{P_1(t)} = \frac{b-r}{\sigma^2} \cdot \frac{X(t)}{P_1(t)}.$$

Thus, the optimal trading strategy in bond and options is given in dependence of the delta-matrix as

$$\varphi_1(t) = \frac{b-r}{\sigma^2} \cdot \frac{X(t)}{\Psi_1(t) \cdot P_1(t)}.$$

If we now introduce the option portfolio process $\pi_{opt}(t)$, then we obtain the remarkable consequence

$$(5.11) \quad \pi_{opt}(t) := \frac{\varphi_1(t) \cdot f^{(1)}(t, P_1(t))}{X(t)} = \frac{b-r}{\sigma^2} \cdot \frac{X(t) \cdot f^{(1)}(t, P_1(t))}{X(t) \cdot \Psi_1(t) \cdot P_1(t)}$$
$$= \frac{b-r}{\sigma^2} \cdot \frac{f^{(1)}(t, P_1(t))}{f_{p_1}^{(1)}(t. P_1(t)) \cdot P_1(t)} = \pi_{stock}(t) \cdot \frac{f^{(1)}(t, P_1(t))}{f_{p_1}^{(1)}(t, P_1(t)) \cdot P_1(t)}.$$

Here, $\pi_{stock}(t)$ denotes the optimal portfolio process in the corresponding stock portfolio problem (5.2). The optimal portfolio process in (5.10) thus only differs from the one in (5.2) by the factor

$$\frac{f^{(1)}(t, P_1(t))}{f_{p_1}^{(1)}(t, P_1(t)) \cdot P_1(t)}.$$

In particular we have

Proposition 5.13. *With the choice of $U(x) = \ln(x)$, in the Black-Scholes model with $d = 1$ we have*

(1) $\pi_{opt}(t) = \pi_{stock}(t)$ *for all* $t \in [0, T]$
$\Leftrightarrow f^{(1)}(t, P_1(t)) = k \cdot P_1(t)$ *for a constant* $k \in \mathbb{R} \setminus \{0\}$.

(2) *In the case of a European call option we have*
$\pi_{opt}(t) < \pi_{stock}(t)$ *for all* $t \in [0, T]$.

Proof. (1) Follows directly from the above relations (5.11) between π_{opt} and π_{stock}.

(2) We have

$$f^{(1)}(t, P_1(t)) = \Phi(d_1(t)) \cdot P_1(t) - \Phi(d_2(t)) \cdot e^{-r(T-t)} \cdot K$$
$$< \Phi(d_1(t)) \cdot P_1(t) = f_{p_1}^{(1)}(t, P_1(t)) \cdot P_1(t).$$

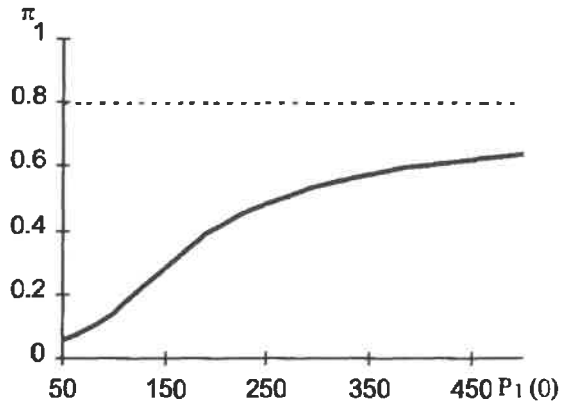


Figure 5.4. Portfolio with options

Together with (5.11) this yields the assertion as we also have

$$\frac{f^{(1)}(t, P_1(t))}{f_{p_1}^{(1)}(t, P_1(t)) \cdot P_1(t)} < 1.$$

□

Remark 5.14. Part (1) of Proposition 5.13 states that in the Black-Scholes model, $\pi_{opt}(t)$ is constant if and only if we have the degenerate case that the payoff of the contingent claims simply is a multiple of the underlying stock price. Part (2) says that with the choice of a European call option, in the option portfolio problem the optimal capital which is invested in the risky asset is always smaller than the corresponding risky position in the stock portfolio problem.

Example. We now look at a European call and the market coefficients

$$r = 0, \quad b = 0.05, \quad \sigma = 0.25, \quad T = 1, \quad K = 100. \quad P_0(0) = 1.$$

Figure 5.4 shows that part of the wealth which is optimally invested in the stock or in the option in the stock portfolio problem or in the option portfolio problem, respectively. More precisely, $\pi_{opt}(0)$ (bold line) and $\pi_{stock}(0)$ (dotted line) are given as functions of the underlying stock price $P_1(0)$. We observe:

- The deeper the option is in the money (i.e. $P_1(0) > K$), the closer $\pi_{opt}(0)$ gets to the optimal stock portfolio component $\pi_{stock}(0)$.
- The more the option is out of the money (i.e. $P_1(0) < K$), the smaller $\pi_{opt}(0)$ gets.