The market: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $(W_t, \mathcal{F}_t)_{t\in[0,\infty)}$ an m-dimensional brownian motion. The prices of the bond and the d tradable stocks at time $t \in [0,T]$ are modelled by

$$P_0(t) = p_0 \exp\left(\int_0^t r(s)ds\right),$$

$$P_i(t) = p_i \exp\left(\int_0^t \left(b_i(s) - \frac{1}{2}\sum_{j=1}^m \sigma_{ij}^2(s)\right) ds + \sum_{j=1}^m \int_0^t \sigma_{ij}(s)dW_j(s)\right),\ i = 1,\dots,d,$$

where r, $b = (b_1,\dots,b_d)'$ and $\sigma = (\sigma_{ij})_{\substack{i=1,\dots,d\\ j=1,\dots,m}}$ are progressively measurable with respect to $(\mathcal{F}_t)$ and uniformly bounded. Further $\sigma\sigma'$ is assumed to be uniformly positive definite. These price processes are the unique solutions to the SDEs

$$dP_0(t) = P_0(t)r(t)dt, \quad P_0(0) = p_0$$

$$dP_i(t) = P_i(t)\left(b_i(t)dt + \sum_{j=1}^m \sigma_{ij}(t)dW_j(t)\right), \quad P_i(0) = p_i, \quad i = 1,\dots,d.$$

In this market the investor can take two actions: They can rebalance their holdings and consume money.

Requirements/Assumptions: We operate under the following assumptions. Note that assumptions 6-8 are out of convenience and can be unrealistic.

1. The investor has no knowledge of future prices (progressive measurability of processes).
2. A single investor's actions have no effect on the market (*small investor hypothesis*).
3. At time $t = 0$ the investor has a fixed initial capital $x > 0$.
4. Money not invested in stocks has to be invested in bonds.
5. The investor acts in a self-financing way (see below).
6. The securities are perfectly divisible.
7. Negative positions in securities are possible.
8. There are no transaction costs.

Basic definitions: A *trading strategy* φ is an $\mathbb{R}^{d+1}$-valued progressively measurable process satisfying $\mathbb{P}$-a.s.

$$\int_0^T |\varphi_0(t)|dt < \infty \text{ and } \int_0^T (\varphi_i(t)\cdot P_i(t))^2 dt < \infty,\ i = 1,\dots,d.$$

The value $x := \sum_{i=0}^d \varphi_i(0)\cdot p_i$ is called the initial value of φ.
For $x > 0$ we then define $X(t) := \sum_{i=0}^d \varphi_i(t)P_i(t)$, the *wealth process* corresponding to φ with initial wealth x.
A non-negative progressively measurable process c with $\int_0^T c(t)dt < \infty$ $\mathbb{P}$-a.s. is called a *consumption process.*
The pair (φ, c) is called *self-financing*, if for all $t \in [0,T]$ we have

$$X(t) = x + \sum_{i=0}^d \int_0^t \varphi_i(s)dP_i(s) - \int_0^t c(s)ds,$$

i.e. the current wealth is given by the initial wealth plus the gains and losses in investments minus the consumption. Note that the integrability conditions imposed on the trading strategy and the consumption process in conjunction with the boundedness of r, b and σ ensure the existence of the integrals on the right-hand side.

Definition (self-financing portfolio process): Let (φ, c) be a self-financing pair consisting of a trading strategy and a consumption process with corresponding wealth process $X(t) > 0$ $\mathbb{P}$-a.s. for all $t \in [0,T]$. Then the $\mathbb{R}^d$-valued process

$$\pi(t) := (\pi_1(t), ..., \pi_d(t))', t \in [0,T] \quad \text{with} \quad \pi_i(t) = \frac{\varphi_i(t) \cdot P_i(t)}{X(t)}$$

is called a self-financing portfolio process corresponding to the pair (φ, c).

The wealth equation: Let (φ, c) be a self-financing pair consisting of a trading strategy and a consumption process. Deriving the wealth process from the definition above w.r.t. t, yields

$$dX(t) = \sum_{i=0}^{d} \varphi_i(t) dP_i(t) - c(t)dt.$$

The portfolio description and the definition above lead to a stochastic differential equation for the wealth process, the wealth equation:

$$dX(t) = [r(t)X(t) - c(t)]dt + X(t)\pi(t)'((b(t) - r(t)\underline{1})dt + \sigma(t)dW(t)), \qquad X(0) = x.$$

We know that b, r, σ are uniformly bounded and that the consumption process c is almost surely integrable on $[0,T]$. Therefore, we only need the following condition on the self-financing portfolio process π to ensure the existence of unique a solution to the wealth equation:

$$\int_0^T \pi_i^2(t)dt < \infty \ \mathbb{P}\text{-a.s.}.$$

This condition implies that if there is no consumption ($c(t) = 0$ for all $t \in [0,T]$), then the wealth process is strictly positive ($X(t) > 0$ for all $t \in [0,T]$). This means that there is no risk of ruin for an investor.

Definition (alternative definition of a self-financing portfolio process): The progressively measurable $\mathbb{R}^d$-valued process $\pi(t)$ is called a self-financing portfolio process corresponding to the consumption process $c(t)$ if the corresponding wealth equation possesses a unique solution $X(t) = X^{\pi,c}(t)$ with

$$\int_0^T (X(t) \cdot \pi_i(t))^2 dt < \infty \ \mathbb{P}\text{-a.s.} \quad \text{for } i = 1, ..., d.$$

This condition on the portfolio process and the wealth process is equivalent to the condition on the trading strategy and the prices in the definition of the trading strategy.
The last condition allows ruin for an investor. It allows the wealth process to be zero or even negative for some $t \in [0,T]$. This will be relevant in connection with the replication approach to option pricing.

Definition (admissible): A self-financing pair (φ, c) or (π, c) consisting of a trading strategy φ or a portfolio process π and a consumption process c will be called admissible for the initial wealth $x > 0$, if the corresponding wealth process satisfies

$$X(t) \geq 0 \ \mathbb{P}\text{-a.s.} \quad \text{for all } t \in [0,T].$$

The set of admissible pairs (π, c) will be denoted by $\mathcal{A}(x)$.

We now look at special case $d = m$ where the dimension of the underlying Brownian motion equals the number of stocks. Prices are the only source of information available to investors, which is modeled by the Brownian filtration $(\mathcal{F}_t)_{t\in[0,T]}$.
We are interested in the set of all final wealths $X(T)$ attainable from an initial capital x. The key result is that every such final wealth can be generated given a sufficiently large initial capital.

We now define a process $H(t) = Z(t)\,\gamma(t)$ which is positive, continuous and progressively measurable with respect to the Brownian filtration.
The process $H(t)$ is interpreted as a discounting process and satisfies:

$$dH(t) = -H(t)\big(r(t)\,dt + \theta(t)'dW(t)\big), \quad H(0) = 1,$$

Theorem 2.63 (Completeness of the Market)

(1) Let (π, c) be an self-financing pair admissible for an initial wealth $x \geq 0$. Then the corresponding wealth process $X(t)$ satisfies

$$\mathbb{E}\left(H(t)X(t) + \int_0^t H(s)c(s)\,ds\right) \leq x, \quad for\ all\ t \in [0,T].$$

(2) Let $B \geq 0$ be an $\mathcal{F}_T$-measurable random variable and $c(t)$ $t \in [0,T]$ a consumption process satisfying

$$x := \mathbb{E}\left(H(T)B + \int_0^T H(s)c(s)\,ds\right) < \infty.$$

Then there exists a portfolio process $\pi(t)$ $t \in [0,T]$ such that $(\pi, c) \in \mathcal{A}(x)$ and the corresponding wealth process $X(t)$ satisfies $X(T) = B$ a.s. P.

Implications of the Theorem

The process $H(t)$ acts as a stochastic discount factor determining the initial wealth at time t=0 which is necessary to be able to attain future aims.
Part (1) sets limits on attainable goals given an initial capital $x \geq 0$. Part (2) says that each desired final wealth at time t = T can be attained exactly by trading according to an appropriate self financing pair and sufficient initial capital.

Proof idea of Theorem 2.63

(1) Let $(\pi, c) \in \mathcal{A}(x)$
Using the differential equations for X(t) (wealth equation) and H(t) when applying the product rule gives:

$$H(t)X(t) + \int_0^t H(s)c(s)\,ds = x + \int_0^t H(s)X(s)\big(\pi(s)'\sigma(s) - \theta(s)'\big)dW(s).$$

The left hand side is not negative and the right hand side is a local-martingale. Therefore it is a super-martingale and the inequality follows.

(2) Define

$$X(t) = H(t)^{-1}\mathbb{E}\left(\int_t^T H(s)c(s)\,ds + H(T)B \,\middle|\, \mathcal{F}_t\right).$$

Then $X(T) = B$ and $X(0) = x$ P-a.s. Set

$$M(t) := H(t)X(t) + \int_0^t H(s)c(s)\,ds = x + \int_0^t \psi(s)'dW(s). \quad a.s.\ P \quad for\ all\ t \in [0,T].$$

The last equality follows from the Martingale Representation Theorem for some $\psi(t)$ which is progressively measurable and $\mathbb{R}^d$-valued with $\int_0^T ||\psi(t)||^2 dt < \infty$. The following two lemmas yield that, by comparing the two different representations from part (1) and (2), X(t) defined as above is the wealth process corresponding to

$$\pi(t) = \begin{cases} \sigma^{-1}(t)\left(\dfrac{\psi(t)}{H(t)X(t)} + \theta(t)\right), & X(t) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Thus $(\pi, c) \in \mathcal{A}(x)$ and $X(T) = B$ P-a.s. □

Lemma 2.64

$X(t)$ and $\pi(t)$ as in the proof of part (2) of the Theorem 2.63 satisfy

$$\int_0^T (\pi_i(t)X(t))^2\, dt < \infty \text{ P-almost surely, } i = 1, \ldots, d$$

Proof idea

We have

$$\begin{aligned} ||\pi(t)X(t)|| &\leq \left\| \left(\sigma(t)^{-1}\right)' \frac{\Psi(t)}{H(t)} + \left(\sigma(t)^{-1}\right)' \theta(t)X(T) \right\| \\ &\leq \underbrace{\left\| \left(\sigma(t)^{-1}\right)' \frac{\psi(t)}{H(t)} \right\|}_{\alpha(t)} + \underbrace{\left\| \left(\sigma(t)^{-1}\right)' \theta(t)X(t) \right\|}_{\beta(t)}. \end{aligned}$$

As $\sigma(t)\sigma(t)'$ is uniformly positive definite and H(t) is continuous and strict positive on $[0, T]$, it follows that

$$\int_0^T \alpha^2(t)\, dt < \infty$$

for P-almost all $\omega \in \Omega$. As $\sigma(t)\sigma(t)'$ is uniformly positive definite, b and r are uniformly bounded and $X(t)$ is continuous on $[0, T]$, this yields

$$\int_0^T \beta^2(t)\, dt < \infty.$$

for P-almost all $\omega \in \Omega$.

With this and the relation $(a + b)^2 \leq 2(a^2 + b^2)$, we arrive at

$$\int_0^T \|\pi(t)X(t)\|^2 \leq 2\left(\int_0^T \alpha^2(t)\, dt + \int_0^T \beta^2(t)\, dt\right) < \infty$$

for P-almost all $\omega \in \Omega$. □

Lemma 2.65

Let $\pi(t), X(t), c(t)$ be as in the proof of part (2) of Theorem 2.63. If $X(t)$ solves the SDE

$$\begin{aligned} d(H(t)X(t)) &= H(t)X(t)\left(\pi(t)'\sigma(t) - \theta(t)'\right) dW(t) - H(t)c(t)\, dt \\ X(0) &= x \end{aligned}$$

then $X(t)$ is the wealth process corresponding to (π, c) with $X(t) = x$.

Proof idea

Let $\tilde{X}(t) := X(t)H(t)$. First assume that $X(t) > 0$ P-almost surely. Then we have

$$d\tilde{X}(t) = \Psi(t)'\, dW(t) - H(t)c(t)\, dt.$$

By the product rule, we have

$$dX(t) = (X(t)\,[r(t) + \pi(t)'(b(t) - r(t) \cdot \underline{1})] - c(t))\; dt + X(t)\pi(t)'\sigma(t)\, dW(t).$$

Hence, X(t) solves the wealth equation corresponding to $(\pi, c,)$ and due to $X(t) \geq 0$ and Lemma 2.64, we also have $(\pi, c) \in \mathcal{A}(x)$.

If we now assume, that for some $(t_0, \omega_0) \in [0, T] \times \Omega,\ X(t)$ attains the value of zero, then due to $H(t) > 0, c(t) \geq 0$ for all $t \in [0, T]$ and $B \geq 0$, we get

$$c(t, \omega_0) = 0 \text{ for all } t \geq t_0,$$
$$B(\omega_0) = 0.$$

Consequently, $X(t, \omega_0)$ retains the value of zero on $[t_0, T]$. This implies

$$\Psi(t, \omega_0) = 0,\ \pi(t, \omega_0) = 0 \text{ for all } t \in [t_0, T].$$

In this case, we have $dX(t) = 0$ for all $t \geq t_0$. Due to $X(t) = 0,\ \pi(t) = 0,\ c(t) = 0$ this then also coincides with the right-hand side of the wealth equation. □

Remark 2.66

(1) $1/H(t)$ is the wealth process corresponding to the pair

$$(\pi(t), c(t)) = \left(\sigma^{-1}(t)'\theta(t), 0\right)$$

with an initial wealth of $x := 1/H(0) = 1$ and a final wealth of $B := 1/H(T)$.

(2) Further it can be shown that the portfolio process π constructed in the proof of part (2) of Theorem 2.63 is the unique (up to indistinguishably with respect to P) portfolio process with $(\pi, c) \in \mathcal{A}(x)$ and $X(t) = B$ P-almost surely.

1 Introduction to optimal portfolios

In the continuous-time market model, the situation is that for a fixed initial capital $x > 0$, we want to find an admissible self-financing pair of portfolio and consumption process which yields a payment stream which should be as profitable as possible. The problem can now be splitted into a choice problem - which security should be bought, a problem of volume - how many units, and a time component - what to do at $t \in [0, T]$.
To evaluate a payment stream, we first define the concept of a *utility function*.

Definition: (Utility function)

(i) U is a *utility function* $:\Leftrightarrow U \in C^1((0,\infty);\mathbb{R})$, U strictly concave and $U'(0) := \lim_{x\to 0} U'(x) = \infty,\ U'(\infty) := \lim_{x\to\infty} U'(x) = 0$.

(ii) If $U \in C(([0,T] \times (0,\infty));\mathbb{R})$ and for all $t \in [0,T]$ the function $U(t,.)$ is a utility function, then U is also considered to be a utility function.

The functional J, to measure the utility of a payment stream, is now given by

$$J(x;\pi,c) := \mathbb{E}\Big[\int_0^T U_1(t,c(t))dt + U_2(X(T))\Big].$$

Here, U_1 and U_2 are utility functions defined like in (ii) and (i), respectively. Since a higher value for J is considered as good for the investor, the *continuous-time portfolio problem* is given by:

$$\max_{(\pi,c)\in\mathcal{A}'(x)} J(x;\pi,c), \text{ with} \tag{1}$$

$$\mathcal{A}'(x) = \{(\pi,c) \in \mathcal{A}(x) : \mathbb{E}\Big[\int_0^T U_1(t,c(t))^- dt + U_2(X(T))^-\Big] < \infty\}$$

2 Martingale Method

Main Idea of the Martingale Method

The key idea of the martingale approach is to decompose the original dynamic optimization problem into two simpler parts:

- **Static optimization problem:** Determine the optimal terminal wealth B^* by solving a static maximization problem over all attainable terminal wealth.
- **Representation problem:** Find a portfolio process π^* with corresponding wealth process $X^{\pi^*}(T) = B^*$.

Motivation of the Static Optimization Problem

We first simplify the dynamic portfolio optimization problem. We assume there is no consumption ($c = 0$) and consider utility only from the terminal wealth ($U_1 = 0$). Hence, the investor just trades a self-financing portfolio $(\pi, 0)$ with initial wealth $x > 0$. The simplified version of our portfolio problem now looks like this:

$$\max_{(\pi,0)\in\mathcal{A}'(x)} \mathbb{E}\big[U_2(X^\pi(T))\big].$$

Note, that we are still maximizing over a class of stochastic processes. Our next step is to change that so that time-dependence disappears and we end up with a static problem.

By market completeness, we know that every attainable terminal wealth B can be generated by some self-financing portfolio process. Therefore we can write our maximization problem as

$$\max_{B \in \mathcal{B}(x)} \mathbb{E}\big[U_2(B)\big],$$

where $\mathcal{B}(x)$ denotes the set of all final wealth achievable from initial capital x.

Solving the Static Optimization Problem

We apply the Lagrange method to solve the static optimization problem

$$\max_{B \in \mathcal{B}(x)} \mathbb{E}[U_2(B)] \quad \text{subject to} \quad \mathbb{E}[H(T)B] = x.$$

To do that, we define the Lagrangian function as

$$L(B, y) = \mathbb{E}\big[U_2(B)\big] - y\big(\mathbb{E}[H(T)B] - x\big),$$

where $y > 0$ is the Lagrange multiplier. By differentiating with respect to B, setting the derivative equal to zero and because U_2' can be inverted, we obtain

$$B = (U_2')^{-1}\big(yH(T)\big).$$

Now by differentiating with respect to y, setting the derivative zero and plugging in the result for B we get

$$0 = x - \underbrace{\mathbb{E}\big[H(T)\,(U_2')^{-1}\big(y\,H(T)\big)\big]}_{=:\,\chi(y)}.$$

If this equation is uniquely solvable for y, then we have found a possible candidate for the optimal final wealth. We define

$$I_2 := (U_2')^{-1} \quad \text{and} \quad Y(x) := \chi^{-1}(x)$$

and obtain the possible candidate for the optimal terminal wealth:

$$B^* = I_2\big(Y(x) \cdot H(T)\big) > 0.$$

The fact that this B^* is indeed optimal will be proven later.

Back to the General Optimization Problem

After solving the simplified model without consumption, we now return to the general case with consumption ($c \neq 0$) and utility from both consumption and terminal wealth ($U_1 \neq 0$). Analogue to the static case, we define for $y \in (0,\infty)$:

$$I_2(y) := (U_2')^{-1}(y)$$
$$I_1(t,y) := (U_1'(t,y))^{-1}(y)$$
$$\chi(y) := \mathbb{E}\left[\int_0^T H(t)\, I_1(t, yH(t))\, dt + H(T)\, I_2(yH(T))\right].$$

The function $\chi(y)$ satisfies the following properties which ensure the existence of its inverse function $Y(x) = \chi^{-1}(x)$:

- χ is continuous and strictly decreasing on $(0,\infty)$,
- $\chi(0) = \infty, \quad \chi(\infty) = 0.$

Lemma: Let U be a utility function, $y > 0,\ x < \infty$ and set $I := (U')^{-1}$. Then

$$U(I(y)) \geq U(x) + y(I(y) - x). \tag{2}$$

Theorem: (Optimal consumption and optimal wealth) Let $x > 0$ and $\chi(y) < \infty$ for all $y > 0$. Set $Y(x) := \chi^{-1}(x)$ and look at the portfolio problem (1). For

$$B^* := I_2(Y(x)\cdot H(T)) \qquad \text{"optimal terminal wealth"}$$
$$c^*(t) := I_1(t, Y(x)\cdot H(t)) \qquad \text{"optimal consumption"}$$

exists a self-financing portfolio process $(\pi^*(t))_{t\in[0,T]}$ such that $(\pi^*, c^*) \in \mathcal{A}'(x)$ solves the portfolio problem (1) and the corresponding wealth process fulfills
$X^{x,\pi^*,c^*}(T) = B^* \ \ \mathbb{P}-a.s.$

Proof: (idea)

(i) *Existence of π^*:* Since $I_1, I_2 > 0$ and by definition of B^*, c^*, it follows that

$$\mathbb{E}\Big[\int_0^T H(t)c(t)dt + H(T)B^*\Big] \overset{\text{Def.}}{=} \chi(Y(x)) = \chi(\chi^{-1}(x)) = x.$$

Now, the existence of a corresponding portfolio process π^* satisfying $(\pi^*, c^*) \in \mathcal{A}'(x)$ is given by Theorem 2.63 as well as $X^{x,\pi^*,c^*}(T) = B^* \ \ \mathbb{P}-a.s.$

(ii) *w.t.s. $(\pi^*, c^*) \in \mathcal{A}'(x)$:* We use (2) for $U_1(t, c^*(t))$ and $U_2(B^*)$ with $x = 1$, respectively, the fact that $a \geq b \geq 0 \Rightarrow a^- \leq b$, the triangle inequality and linearity to obtain

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T U_1(t, c^*(t))^- dt + U_2(B^*)\Big] \\
&\leq \mathbb{E}\Big[\int_0^T |U_1(t,1)| + Y(x)H(t)(c^*(t)+1) + |U_2(1))| + Y(x)H(T)(B^*+1)\Big] \\
&= \underbrace{|U_2(1))|}_{<\infty \text{ by def.}} + \underbrace{\int_0^T |U_1(t,1)|dt}_{<\infty\ \text{, cont. integrand}} + Y(x)\underbrace{\big(x + \mathbb{E}[H(T)] + \int_0^T \mathbb{E}[H(t)]dt\big)}_{<\infty,\ H\in L^1([0,T])} < \infty
\end{aligned}$$

(iii) *show optimality:* Consider an arbitrary $(\pi, c) \in \mathcal{A}'(x)$ and use (2) for $U_1(t, c^*(t))$ with $x = c(t)$ as well as $U_2(B^*)$ with $x = X^{x,\pi,c}$. With this, the definition of $J(x, \pi, c)$ and Theorem 2.63, we get:

$$J(x, \pi^*, c^*) = \mathbb{E}\Big[\int_0^T U_1(t, c^*(t))dt + U_2(B^*)\Big]$$
$$\geq J(x, \pi, c) + Y(x)\underbrace{\Big(\underbrace{\mathbb{E}\Big[\int_0^T H(t)c^*(t)dt + H(T)B^*\Big]}_{=x} - \mathbb{E}\Big[\int_0^T H(t)c(t)dt + H(T)X^{x,\pi,c}\Big]\Big)}_{\geq 0}.$$

■

Example: Logarithmic Utility

We consider the special case $U_1(t, x) = U_2(x) = \ln(x)$, for which $I_1(t, y) = I_2(y) = \frac{1}{y}$.
This leads to

$$\chi(y) = \mathbb{E}\bigg[\int_0^T H(t)\frac{1}{yH(t)}\,dt + H(T)\frac{1}{yH(T)}\bigg] = \frac{1}{y}(T+1), \qquad Y(x) = \chi^{-1}(x) = \frac{T+1}{x}.$$

Now for the optimal consumption and the optimal wealth, we get

$$c^*(t) = I_1(t, Y(x)H(t)) = \frac{x}{T+1}\frac{1}{H(t)}, \qquad B^* = I_2(Y(x)H(T)) = \frac{x}{T+1}\frac{1}{H(T)}.$$

In this special case we can even calculate the portfolio process explicitly which then leads to the following generalization.

Solution to the representation problem

Theorem: (solution of representation problem of the continuous-time portfolio problem) Let $x > 0$ and $\chi(y) < \infty$ for all $y > 0$. Let c^* and B^* be the optimal consumption and the optimal terminal wealth like before. Suppose there exists a continuously differentiable function

$$f \in C^{1,2}([0,T] \times \mathbb{R}^d), \qquad f(0, .., 0) = x,$$

such that for all $t \in [0, T]$

$$\frac{1}{H(t)}\,\mathbb{E}\bigg[\int_t^T H(s)c^*(s)\,ds + H(T)B^* \mid \mathcal{F}_t\bigg] = f(t, W_1(t), \dots, W_d(t)).$$

Then the optimal portfolio process is given by

$$\pi^*(t) = \frac{1}{X^{x,\pi^*,c^*}(t)}\,\sigma^{-1}(t)\,\nabla_x f(t, W_1(t), ..., W_d(t)), \qquad t \in [0, T].$$

Proof:(idea)

(1) Applying the multi-dimensional Itó-Formula to

$$\frac{1}{H(t)}\,\mathbb{E}\Big[\int_t^T H(s)c^*(s)\,ds + H(T)B^* \mid \mathcal{F}_t\Big] = f(t, W_1(t), \dots, W_d(t)).$$

leads to

$$f(t, W_1(t), \dots, W_d(t)) = f(0,..,0) + \sum_{i=1}^{d}\int_0^t f_{x_i}(s, W_1(s), \dots, W_d(s))\,dW_i(s)$$

$$+\int_0^t \left(f_t(s, W_1(s), \dots, W_d(s)) + \frac{1}{2}\sum_{i=1}^{d} f_{x_i x_i}(s, W_1(s), \dots, W_d(s))\right) ds.$$

(2) We also know

$$\begin{aligned} X^{x,\pi^*,c^*}(t) &= \frac{1}{H(t)}\mathbb{E}\Big[\int_t^T H(s)c^*(s)\,ds + H(T)B^* \mid \mathcal{F}_t\Big] \\ &= x + \int_0^t X^{x,\pi^*,c^*}(s)\,\pi^*(s)'\sigma(s)\,dW(s) \\ &\quad + \int_0^t \Big(\big(r(s) + \pi^*(s)'(b(s) - r(s)\mathbf{1})\big)X^{x,\pi^*,c^*}(s) - c(s)\Big)\,ds, \end{aligned}$$

where the last equation holds because (π^*, c^*) is an admissible, self-financing pair and therefore fulfills the wealth equation holds (integrating from 0 to t gives the expression).

(3) If we compare the integrands of the stochastic integrals of both representations we get the wanted result

$$\pi^*(t) = \frac{1}{X^{x,\pi^*,c^*}(t)}\,\sigma^{-1}(t)\,\nabla_W f(t, W(t)).$$

■

Corollary:(solution of the pure consumption and the pure terminal wealth maximization problem)

(1) For the optimal terminal wealth B^* of

$$\max_{(\pi,0)\in\mathcal{A}'(x)} \mathbb{E}\big[U_2(X^{x,\pi}(T))\big],$$

we get $B^* = I_2\big(Y(x)H(T)\big)$, where in the definition of $\chi(y)$ we set $I_1(t,y) \equiv 0$.

(2) For the optimal consumption process $c^*(t)$ of

$$\max_{(\pi,c)\in\mathcal{A}'(x)} \mathbb{E}\left[\int_0^T U_1(t, c(t))\,dt\right],$$

we get $c^*(t) = I_1\big(t, Y(x)H(t)\big)$, where in the definition of $\chi(y)$ we set $I_2(y) \equiv 0$.

This section will apply the martingale method for portfolio optimisation in the complete market, as introduced in the preceding section. In this section, we shall consider a portfolio problem in which the only financial instruments permitted for trade are option on stocks, as opposed to the original stocks.

General assumptions:
We use same notations as in Chapter 2, Section 3. The assumptions of of Theorem 2.63 on complete market are satisfied. Further we restrict ourselves in this case of constant market coefficients r, b, σ

Description of the market model:
Consider a market where a bond, d stocks and d options on these stocks are traded.We assume the price process of options is given by

$$f^{(i)}(t, P_1(t), ..., P_d(t)), \quad i = 1, ..., d, \quad f \in C^{1,2}.$$

Let

$$\varphi(t) = (\varphi_0(t), \varphi_1(t), ..., \varphi_d(t),$$

i.e

$$\int_0^t \varphi_0(s)\, dP(s),$$

$$\int_0^t \varphi_i(s)\, df^{(i)}(s, P_1(s), ..., P_d(s)),$$

be an admissible trading strategy in bond and options and we assume $\varphi(t)$ is $\mathcal{F}_t$-progressively measurable.

Motivation: In this section we want find the solution of the following problem

$$\underset{\varphi}{max}\, \mathbb{E}[U(X(T))], \tag{1}$$

where U is the **utility function** and $X(t)$ is the **wealth process**, given by

$$X(t) = \varphi_0(t)P_0(t) + \sum_{i=1}^{d} \varphi_i(t) f^i(t, P_1(t), ..., P_d(t)).$$

Option pricing:
Starting from the option's final payoff B, we determine its price by constructing a portfolio strategy whose terminal wealth replicates this payoff. The minimal initial cost of this replication strategy then yields the option price.

Portfolio optimization with stocks:
In portfolio optimization, we invest an initial wealth x according to a portfolio process $\pi^\star(t)$ to achieve a terminal wealth that maximizes utility; to do this, we first determine the optimal terminal payoff B^* and then look for a replication strategy for B^*.

Portfolio optimization with options:
In portfolio optimization with options, starting from an initial capital x, one first determines an optimal payoff B and then replicates it using a strategy in bonds and options, achieving the optimal terminal wealth $X^(T)$ Stocks are not held directly but are represented through the bond-and-option strategy.

Theorem 5.11 Let the Delta matrix $\Psi(t) = (\Psi_{ij}(t)),\ i,j = 1,\dots,d$ with

$$\Psi_{ij}(t) = f^{(i)}_{p_j}(t, P_1(t), \dots, P_d(t)) \tag{2}$$

be regular for all $t \in [0,T)$. Then the option portfolio problem (1) has an explicit solution:

1. Optimal terminal wealth B^* coincides with terminal wealth of corresponding stock portfolio problem.
2. Let $\xi(t) = (\xi_0(t), \dots, \xi_d(t))$ be optimal trading strategy for stocks, then optimal trading strategy for options is given by

$$\phi_0(t) = \frac{X(t) - \sum_{i=1}^d \phi_i(t) f^{(i)}(t, P_1(t), \dots, P_d(t))}{P_0(t)} \tag{3}$$

$$\overline{\phi}(t) = (\Psi(t)')^{-1} \cdot \overline{\xi}(t) \tag{4}$$

Proof idea:
By Proposition 3.19, we have

$$f^{(i)}(t, P_1(t), \dots, P_d(t)) = \sum_{j=0}^d \Psi_{ij}(t) \cdot P_j(t), \quad i = 1, \dots, d \tag{5}$$

with

$$\Psi_{i0} = \frac{f^{(i)}(t, P_1(t), \dots, P_d(t)) - \sum_{j=1}^d \Psi_{ij}(t) \cdot P_j(t)}{P_0(t)} \tag{6}$$

Furthermore by same proposition each row in Ψ is self-financing strategy, so

$$df^{(i)}(t, P_1(t), \dots, P_d(t)) = \sum_{j=0}^d \Psi_{ij}(t) \cdot dP_j(t), \tag{7}$$

Now let ϕ be an admissible and self-financing strategy in bond and *options*, then

$$dX(t) = \phi_0(t) \cdot P_0(t) + \sum_{i=1}^d \phi_i(t) df^{(i)}(t, P_1(t), \dots, P_d(t)) \tag{8}$$

Using (6) and (7), we obtain

$$dX(t) = \left(\phi_0(t) + \sum_{i=1}^d \phi_i(t)\Psi_{i0}(t)\right) dP_0(t) + \sum_{j=1}^d \left(\sum_{i=1}^d \phi_i(t)\Psi_{ij}(t)\right) dP_j(t) \tag{9}$$

$$= \xi_0(t) dP_0(t) + \sum_{j=1}^d \xi_j(t) dP_j(t) \tag{10}$$

Hence $\xi(t)$ is self-financing strategy for bond and *stocks*, and admissibility of $\phi(t)$ (since Ψ is replicating strategy) implies admissibility of $\xi(t)$ for their respective problems.

To prove second point, we consider optimal trading strategy of stock portfolio problem, which implies $X(T) = B^*$ *a.s.* and wealth process has form of (10). To obtain trading strategy in bond and options admitting same wealth process $X(t)$, we make ansatz that its wealth process has form (8). Repeating our steps we again arrive at (9). Comparing coefficients of dP_i-terms between (9) and (10) yield the desired form of last d components of $\phi(t)$, and regularity of Ψ asserts correctness of (4). Comparison between bond part of the strategy gives us

$$\xi_0(t) = \phi_0(t) + \sum_{i=1}^{d} \phi_i(t) \cdot \Psi_{i0}(t) \tag{11}$$

We have

$$\xi_0(t) = \frac{X(t) - \sum_{i=1}^{d} \xi_i(t) P_i(t)}{P_0(t)} = \frac{X(t) - \sum_{i=1}^{d} \left(\sum_{j=1}^{d} \phi_j(t) \Psi_{ji}(t) \right) P_i(t)}{P_0(t)} \tag{12}$$

Solving (11) for $\phi_0(t)$ and substituting $\xi_0(t)$ with (12) we get

$$\begin{aligned}
\phi_0(t) &= \frac{X(t) - \sum_{i=0}^{d} \sum_{j=1}^{d} \phi_j(t) \Psi_{ji}(t) P_i(t)}{P_0(t)} \\
&= \frac{X(t) - \sum_{j=1}^{d} \phi_j(t) \sum_{i=0}^{d} \Psi_{ji}(t) P_i(t)}{P_0(t)} \\
&\overset{(5)}{=} \frac{X(t) - \sum_{j=1}^{d} \phi_j(t) f^{(i)}(t, P_1(t), \dots, P_d(t))}{P_0(t)}
\end{aligned}$$

This shows $\phi(t) = (\phi_0(t), \phi_1(t), \dots, \phi_d(t))$ is self-financing. To show it is admissible it is enough to show stochastic integrals

$$\sum_{i=1}^{d} \int_0^t \phi_i(s) df^{(i)}(s, P_1(s), \dots, P_d(s))$$

are well-defined. But this follows by substituting df-term with (7), then using our derived relation between $\phi_i(t)$ and $\xi_i(t)$. Admissibility of $\xi(t)$ in stock portfolio problem then implies admissibility of $\phi(t)$ in option portfolio problem.

We have also shown that following option strategy $\phi(t)$ leads to the same utility as by using $\xi(t)$ in stock portfolio problem. We also cannot obtain higher utility in option portfolio problem, since that would induce the existence of stock strategy $\zeta(t)$ yielding higher expected utility than $\xi(t)$, which contradicts optimality of $\xi(t)$.

Remark 5.12. (1) Under given assumptions, the optimal final wealth only depends on utility functions, but not on the choice of tradeable securities.

(2) Optimal strategy depends heavily on traded options via delta matrix (more precisely, replication strategy for options).

Example:(Logarithmic utility) Let $U(x) = ln(x)$ and we consider the Black-Scholes model with $d = 1$. We known from the previous section, example "logarithmic utility"

$$\xi_1(t) = \frac{\pi^*(t) \cdot X(t)}{P_1(t)} = \frac{b - r}{\sigma^2} \cdot \frac{X(t)}{P_1(t)}$$

The optimal trading strategy is given in dependece of Ψ_1 as

$$\varphi_1(t) = \frac{b - r}{\sigma^2} \cdot \frac{X(t)}{\Psi_1(t) \cdot P_1(t)} \quad .$$

We define the optimal portfolio process

$$\pi_{opt}(t) := \frac{\varphi(t) \cdot f^{(1)}(t, P_1(t))}{X(t)}$$

and we obtain by using $\varphi_1(t)$

$$\pi_{opt}(t) = \pi_{stock}(t) \cdot \frac{f^{(1)}(t, P_1(t))}{f^{(1)}_{p_1}(t, P_1(t)) \cdot P_1(t)} \quad .$$

Proposition
We consider the Black-Scholes model with $d = 1$ and let $U(x) = ln(x)$, then we have

(1) $\pi_{opt}(t) = \pi_{stock}(t)$ for all $t \in [0, T] \Leftrightarrow f^{(1)}(t, P_1(t)) = k \cdot P_1(t)$ for a const. $k \in \mathbb{R} \setminus \{0\}$.

(2) For a European call option we have $\pi_{opt}(t) < \pi_{stock}(t)$ for all $t \in [0, T]$.

Proof.
For (1) we use the relation between π_{opt} and π_{stock} from the above Example.
For (2) we know in the Black-Scholes-Model, the European call option is given by $f^{(1)}(t, P_1(t)) = \Phi(d_1(t)) \cdot P_1(t) - \Phi(d_2(t)) \cdot e^{-r(T-t)} \cdot K$ and we have the factor

$$\frac{f^{(1)}(t, P_1(t))}{f^{(1)}_{p_1}(t, P_1(t)) \cdot P_1(t)} < 1$$

from the relation between π_{opt} and π_{stock}. We get the following inequality:

$$f^{(1)}(t, P_1(t)) = \Phi(d_1(t)) \cdot P_1(t) - \Phi(d_2(t)) \cdot e^{-r(T-t)} \cdot K < \Phi(d_1(t)) \cdot P_1(t) = f^{(1)}_{p_1}(t, P_1(t)) \cdot P_1(t).$$

Remark:
(1) In the Black–Scholes model, $\pi_{opt}(t)$ is constant if and only if the contingent claim's payoff is simply a multiple of the stock price (the degenerate case).
(2) If a European call option is used in the option portfolio problem, the optimal investment in the risky asset is always smaller than in the corresponding pure stock portfolio problem.

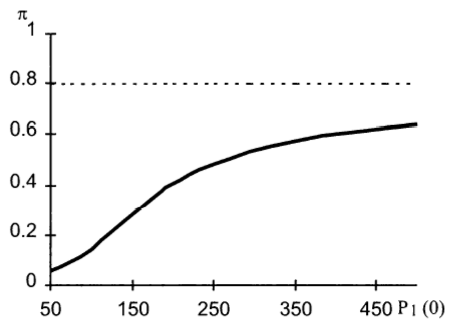

Figure 5.4. Portfolio with options

Example We now look at European call and market coefficients $r = 0, b = 0.05, \sigma = 0.25, T = 1, K = 100, P_0(0) = 1$. Figure 5.4 shows part of the wealth which is optimally invested in the stock or option problem respectively. More precisely, $\pi_{opt}(0)$ bold line and $\pi_{stock}(0)$ dotted line, are given as functions of the underlying stock price $P_1(0)$. We see:

- The deeper option is in the money (i.e. $P_1(0) > K$), the closer $\pi_{opt}(0)$ gets to optimal stock portfolio component $\pi_{stock}(0)$.
- The more option is out of the money ($P_1(0) < K$), the smaller $\pi_{opt}(0)$ gets.

Kristina Kovanusic, Ali Ünver and Solomija Domsha

ASYMPTOTIC RUIN PROBABILITIES AND OPTIMAL INVESTMENT

by J. Gaier, P. Grandits and W. Schachermayer

1 Model Setup

The *surplus process* of an insurance company is given by

$$R(t,x) = x + ct - \sum_{i=1}^{N(t)} X_i,$$

where $x \geq 0$ is the initial capital, $c \in \mathbb{R}$ the premium rate, $N(t)$ a Poisson process with intensity λ, and $X_i \geq 0$ are i.i.d. claim sizes with distribution function F. The claim number process $N(t)$ is independent of the claim sizes.

The company invests in a risky asset described by a geometric Brownian motion $S(t)$:

$$dS(t) = S(t)(a\,dt + b\,dW(t)),$$

where $a, b \in \mathbb{R}$ and $W(t)$ is a standard Brownian motion independent of R. Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by R and S, and write $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot \mid \mathcal{F}_t]$.

We denote by $W_{a,b}$ the Wiener process with drift a and volatility b, and by $K = (K(t))_{t\geq 0}$ an investment strategy. The *wealth process* is

$$Y(t,x,K) = R(t,x) + (K \cdot W_{a,b})(t),$$

where $(K \cdot W_{a,b})$ denotes the stochastic integral.

The *ruin probability* is

$$\Psi(x,K) = \mathbb{P}[\, Y(t,x,K) < 0 \text{ for some } t \geq 0\,],$$

and the *time of ruin* is

$$\tau(x,K) = \inf\{t \geq 0 : Y(t,x,K) < 0\}.$$

The class of admissible strategies is

$$\mathcal{K} = \{K : K \text{ is predictable and adapted to } \mathbb{F} \text{ and } \int_0^t K(s)^2 ds < \infty \text{ a.s. for all } t \geq 0\}.$$

The minimal ruin probability is

$$\Psi^*(x) = \inf_{K \in \mathcal{K}} \Psi(x,K).$$

Define

$$h(r) := \mathbb{E}[e^{rX}] - 1, \qquad r \geq 0.$$

Assume that $h(r) < \infty$ for all $r < r_\infty$ for some $r_\infty \in (0,\infty]$, and $\lim_{r\to\infty} h(r) = \infty$. The function h is continuous, increasing and convex, with $h(0) = 0$.

Remark 1. *The above model extends the classical Cramér–Lundberg model by allowing investment in a risky asset. In the classical case, the safety loading condition $c > \lambda\mathbb{E}[X]$ yields*

$$\Psi(x) \leq e^{-vx},$$

where $v > 0$ solves $\lambda h(r) = cr$.

2 Main Statement

Theorem 1. *Assume $b \neq 0$. Then*

$$\Psi^*(x) \leq e^{-\hat{r}x},$$

where $\hat{r} < \infty$ is the positive solution of

$$\lambda h(r) = cr + \frac{a^2}{2b^2}.$$

If $\mathbb{E}[X] < c/\lambda$ and $a \neq 0$, then $\hat{r} > v$, where v is the classical Lundberg exponent. Even without $\mathbb{E}[X] < c/\lambda$, we have $\hat{r} > 0$.

3 Proof of the Main Statement

For $x, r \geq 0$ and $K \in \mathcal{K}$, define

$$M(t, x, K, r) := e^{-r\,Y(t,x,K)}.$$

Lemma 1. *Let $a \neq 0$, $b \neq 0$. There exists a unique $0 < \hat{r} < r_\infty$ satisfying*

$$\lambda h(\hat{r}) = \frac{a^2}{2b^2} + c\hat{r}.$$

For this $\hat{r}$ and the constant strategy

$$\hat{K}(t) \equiv \frac{a}{b^2\hat{r}},$$

the process $M(t, x, \hat{K}, \hat{r})$ is a martingale with respect to $\mathbb{F}$.

Proof. Since h is continuous, increasing and convex, the stated equation has a unique solution.

Define

$$f(K, r) := \lambda h(r) - (Ka + c)r + \tfrac{1}{2}K^2 b^2 r^2.$$

Then $f(\hat{K}, \hat{r}) = 0$. For arbitrary $t \geq 0$,

$$\mathbb{E}[M(t, 0, \hat{K}, \hat{r})] = e^{-\hat{r}(c+\hat{K}a)t} e^{\lambda t h(\hat{r})} e^{(\hat{r}^2\hat{K}^2b^2/2)t} = e^{f(\hat{K},\hat{r})t} = 1.$$

Since $Y(t, x, \hat{K})$ has stationary independent increments,

$$\mathbb{E}_t[M(T, x, \hat{K}, \hat{r})] = e^{-\hat{r}Y(t,x,\hat{K})}\mathbb{E}[e^{-\hat{r}Y(T-t,0,\hat{K})}] = M(t, x, \hat{K}, \hat{r}).$$

Thus M is an $\mathbb{F}$-martingale. □

Remark 2. *For any $r < \hat{r}$, there exist two constant strategies*

$$K_{1,2}(r) = \frac{a}{b^2 r} \pm \sqrt{\Delta(r)},$$

where

$$\Delta(r) = \frac{2}{b^2r^2}\left(\frac{a^2}{2b^2} + cr - \lambda h(r)\right) \geq 0.$$

At $r = \hat{r}$, $\Delta(\hat{r}) = 0$, so $K_1(\hat{r}) = K_2(\hat{r}) = \hat{K}$.

We introduce the stopped processes

$$\tilde{M}(t, x, K, r) := M(t \wedge \tau(x, K), x, K, r), \qquad \tilde{Y}(t, x, K) := Y(t \wedge \tau(x, K), x, K).$$

Theorem 2. *Let $a \neq 0$, $b \neq 0$. For the constant strategy $\hat{K}(t) = a/(b^2\hat{r})$,*

$$\Psi(x, \hat{K}) \leq e^{-\hat{r}x}, \qquad x \geq 0.$$

Proof. Since $M(t, x, \hat{K}, \hat{r})$ is a nonnegative martingale, the stopped process $\tilde{M}$ is a martingale as well. Hence,

$$e^{-\hat{r}x} = \tilde{M}(0, x, \hat{K}, \hat{r}) = \mathbb{E}[\tilde{M}(t, x, \hat{K}, \hat{r})].$$

Splitting over the event of ruin before time t,

$$e^{-\hat{r}x} \geq \mathbb{E}[\tilde{M}(t, x, \hat{K}, \hat{r})\mathbf{1}_{\{\tau<t\}}].$$

Letting $t \to \infty$ and using monotone convergence,

$$e^{-\hat{r}x} \geq \mathbb{E}[\tilde{M}(\tau, x, \hat{K}, \hat{r}) \mid \tau < \infty] \cdot \mathbb{P}(\tau < \infty).$$

Thus

$$\Psi(x, \hat{K}) \leq \frac{e^{-\hat{r}x}}{\mathbb{E}[\tilde{M}(\tau, x, \hat{K}, \hat{r}) \mid \tau < \infty]}.$$

At ruin, $Y(\tau) < 0$, hence $\tilde{M}(\tau, x, \hat{K}, \hat{r}) = e^{-\hat{r}Y(\tau)} \geq 1$, yielding the claim. □

The main theorem follows immediately. In the classical case, the Lundberg exponent v solves

$$h(r) = \frac{c}{\lambda}r.$$

With investment, the corresponding exponent $\hat{r}$ solves

$$h(r) = \frac{c}{\lambda}r + \frac{a^2}{2\lambda b^2}.$$

The right-hand side is strictly larger when $a \neq 0$, implying $\hat{r} > v$ by convexity of h.

Example. If X is exponentially distributed with parameter θ, then

$$h(r) = \frac{\theta r}{1 - \theta r}, \qquad r \in [0, 1/\theta).$$

Solving $h(r) = \frac{c}{\lambda}r$ yields the classical exponent

$$\nu = \frac{\rho}{(\rho + 1)\theta}, \qquad \rho = \frac{c}{\lambda\theta} - 1.$$

With investment, the exponent becomes

$$\hat{r} = \nu + \left(\sqrt{\left(\frac{\nu + a^2/(2b^2c)}{2}\right)^2 + \frac{a^2}{2b^2c}\left(\frac{1}{\theta} - \nu\right)} - \frac{\nu + a^2/(2b^2c)}{2} \right),$$

which is strictly larger than ν when $a \neq 0$.

Handout: Asymptotic Ruin Probabilities and Optimal Investment - 2nd part

by Jonathan Demming, Nina Radostits & Kathrin Schrank

In this abstract we introduce the surprising result of asymptotic optimality of the constant investment strategy.

Asymptotic optimality meaning:

- Mostly a theoretical concept.
- For large inputs the strategy performs at worst a constant factor worse than any other possible strategy.
- "There is no more dramatic improvement possible."

Constant investment strategy:

- Holding a fixed quantity in the risky asset, independent of the current reserve.

We need the following assumption of the exponential tail distribution of the claim sizes:

Definition 1. *Let $0 < r < r_\infty$ be given. We say that X has a uniform exponential moment in the tail distribution for r, if the following condition holds true:*

$$\sup_{y \geq 0} \mathbb{E}\left[e^{-r(y-X)} \mid X > y\right] < \infty. \tag{1}$$

Theorem 2. *Assume that X has a uniform exponential moment in the tail distribution for $\hat{r}$. Then for each $K \in \mathcal{K}$, the process $\big(\tilde{M}(t,x,K,\hat{r})\big)$ is a uniformly integrable submartingale.*

Proof. The main steps are:
Using Ito's Lemma to the process M and rewriting it leads us to

$$\begin{aligned}\frac{dM(t,x,K,r)}{M(t-,x,K,r)} &= \left(-(c+K(t)a)r + \frac{1}{2}r^2b^2K(t)^2 + \lambda h(r)\right)dt \\ &\quad - rbK(t)\,dW(t) + \big(e^{rX_{N(t)}} - 1\big)\,dN(t) - \lambda\mathbb{E}\big[e^{rX_{N(t)}} - 1\big]dt.\end{aligned} \tag{2}$$

Therefore we can express the stopped process $\tilde{M}(t,x,K,\hat{r})$ as

$$\begin{aligned}&\tilde{M}(t,x,K,\hat{r}) - \tilde{M}(0,x,K,\hat{r}) \\ &= \underbrace{\int_0^{t\wedge\tau} M(s-,x,K,\hat{r}) f\big(K(s),\hat{r}\big)\,ds}_{\geq 0} - rb\underbrace{\int_0^{t\wedge\tau} M(s-,x,K,\hat{r})K(s)\,dW(s)}_{\text{local martingale}} \\ &\quad + \underbrace{\int_0^{t\wedge\tau} M(s-,x,K,\hat{r})\left(e^{\hat{r}X_{N(s)}} - 1\right)dN(s) - \mathbb{E}\big[e^{\hat{r}X} - 1\big]\int_0^{t\wedge\tau} M(s-,x,K,\hat{r})\lambda\,ds}_{\text{martingale}}.\end{aligned} \tag{3}$$

Hence $\tilde{M}(t,x,K,\hat{r})$ is a local submartingale.
Now we show that it is a true uniformly integrable submartingale, we use Assumption (1) and notation $\tilde{M}^* := \sup_{t\geq 0}|\tilde{M}(t)|$. Then

$$\mathbb{E}\big[\tilde{M}^*\big] \leq \mathbb{E}\big[\tilde{M}(\tau,x,K,\hat{r}) \mid \tau < \infty,\ Y(\tau-) > 0\big] \tag{4}$$

since $M(\tau, x, K, \hat{r})$ equals 1 on $\{\tau < \infty, Y(\tau-) = 0\}$ and $M(\tau, x, K, \hat{r}) \geq 1$ on $\{\tau < \infty, Y(\tau-) > 0\}$. Let $H(dt, dy)$ denote the joint probability distribution of τ and $Y(\tau-)$ conditioned on ruin occurs (through a jump). Then

$$\begin{aligned}\mathbb{E}\big[\tilde{M}^*\big] &\leq \mathbb{E}\big[\tilde{M}(\tau, x, K, \hat{r}) \mid \tau < \infty,\ Y(\tau-) > 0\big] \\ &= \int_0^\infty \int_0^\infty H(dt, dy) \int_y^\infty e^{-\hat{r}(y-z)} \frac{dF(z)}{\int_y^\infty dF(u)} \\ &\leq \sup_{y\geq 0} \int_y^\infty e^{-\hat{r}(y-z)} \frac{dF(z)}{\int_y^\infty dF(u)} < \infty,\end{aligned} \tag{5}$$

by assumption (1). Dominated Convergence then implies that $\tilde{M}(t, x, K, \hat{r})$ is a uniformly integrable submartingale. □

Lemma 3. *If X has a uniform exponential moment in the tail distribution for $\hat{r}$, then for arbitrary $K \in \mathcal{K}$ and $x \in \mathbb{R}_+$, the stopped wealth process $(\tilde{Y}(t, x, K))_{t\geq 0}$ converges almost surely on $\{\tau = \infty\}$ to ∞ for $t \to \infty$. In other words, either ruin occurs, or the insurer becomes infinitely rich.*

Proof. From Theorem 2 we know that $\tilde{M}(t, x, K, \hat{r})$ is a uniformly integrable submartingale. If we now apply Doob's Supermartingale Convergence Theorem to $-\tilde{M}$, the a.s existence of $\lim_{t\to\infty} \tilde{M}(t, x, K, \hat{r})$ follows. As a result, also the stopped wealth process $\tilde{Y}(t, x, K)$ converges a.s for $t \to \infty$.

As X is a positive random variable, there must exist $d > 0$ such that $\mathbb{P}[X > d] > 0$. Defining the events $E_n := \{X_n > d\}$ implies $p := \mathbb{P}[E_n^c] < 1$, and the events $\{E_j\}_{j=1}^\infty$ are mutually independent. Therefore, using the continuity of measures and the independence of the defined events we get

$$\mathbb{P}\left[\bigcup_{k=1}^\infty \bigcap_{n\geq k} E_n^c\right] \overset{\text{cont. of meas.}}{=} \lim_{k\to\infty} \mathbb{P}\left[\bigcap_{n\geq k} E_n^c\right] \overset{E_n^c \text{ are indep.}}{=} \lim_{k\to\infty} \prod_{n\geq k} \underbrace{\mathbb{P}[E_n^c]}_{p} = \lim_{k\to\infty} \underbrace{\left(\lim_{N\to\infty} p^{N-k+1}\right)}_{=0,\text{ since } 0\leq p<1} = 0. \tag{6}$$

Hence, $\mathbb{P}\left[\bigcap_{k=1}^\infty \bigcup_{n\geq k} E_n\right] = 1 - \underbrace{\mathbb{P}\left[\bigcup_{k=1}^\infty \bigcap_{n\geq k} E_n^c\right]}_{=0} = 1$. This means that, almost surely, a jump larger than d occurs infinitely often.

In contrast to this, the stochastic integral $K \cdot W_{a,b}$ is a.s. continuous. As a result the stochastic integral $K \cdot W_{a,b}$ cannot compensate the jumps of the compound Poisson process, which are greater than d and occur infinitely often a.s.. All in all, the stopped wealth process cannot converge to a nonzero finite value with positive probability. □

Theorem 4. *Assume that X has a uniform exponential moment in the tail distribution for $\hat{r}$. Then the ruin probability satisfies, for every admissible process $K \in \mathcal{K}$,*

$$\Psi(x, K) \geq Ce^{-\hat{r}x}, \tag{7}$$

where

$$C = \inf_{y\geq 0} \frac{\int_y^\infty dF(u)}{\int_y^\infty e^{-\hat{r}(y-z)} dF(z)} = \frac{1}{\sup_{y\geq 0} \mathbb{E}[e^{-\hat{r}(y-X)} | X > y]} > 0. \tag{8}$$

Proof. We use again that $\tilde{M}(t, x, K, \hat{r})$ is a uniformly integrable submartingale (Theorem 2). Applying Doob's Optional Sampling Theorem it follows that (using τ as a shorthand notation for $\tau(x, K)$)

$$e^{-\hat{r}x} = e^{-\hat{r}\hat{Y}(0,x,K)} = \tilde{M}(0, x, K, \hat{r}) \leq \mathbb{E}[\tilde{M}(\tau, x, K, \hat{r})]. \tag{9}$$

Then,

$$\begin{aligned}\mathbb{E}[\tilde{M}(\tau, x, K, \hat{r})] &= \mathbb{E}[\tilde{M}(\tau, x, K, \hat{r}) | \tau < \infty]\mathbb{P}[\tau < \infty] + \underbrace{\mathbb{E}[\lim_{t\to\infty} \tilde{M}(t, x, K, \hat{r}) | \tau = \infty]}_{=0 \text{ (Lemma 5)}} \mathbb{P}[\tau = \infty] \\ &= \mathbb{E}[\tilde{M}(\tau, x, K, \hat{r}) | \tau < \infty]\mathbb{P}[\tau < \infty].\end{aligned} \tag{10}$$

Plugging this into (9), we get

$$e^{-\hat{r}x} \leq \mathbb{E}[\tilde{M}(\tau, x, K, \hat{r})] = \mathbb{E}[\tilde{M}(\tau, x, K, \hat{r})|\tau < \infty] \underbrace{\mathbb{P}[\tau < \infty]}_{\Psi(x,K)}. \tag{11}$$

This is equivalent to

$$\Psi(x, K) \geq \frac{e^{-\hat{r}x}}{\mathbb{E}[\tilde{M}(\tau, x, K, \hat{r})|\tau < \infty]} \tag{12}$$

Using $\mathbb{E}\big[\tilde{M}^*\big] \leq \mathbb{E}\big[\tilde{M}(\tau, x, K, \hat{r}) \mid \tau < \infty,\ Y(\tau-) > 0\big] \leq \sup_{y\geq 0} \int_y^\infty e^{-\hat{r}(y-z)} \frac{dF(z)}{\int_y^\infty dF(u)}$ obtained in (4) and (5) and

$$\sup_{y\geq 0} \int_y^\infty e^{-\hat{r}(y-z)} \frac{dF(z)}{\int_y^\infty dF(u)} = \sup_{y\geq 0} \mathbb{E}[e^{-\hat{r}(y-X)}|X > y] \tag{13}$$

we get

$$\Psi(x, K) \geq \frac{e^{-\hat{r}x}}{\mathbb{E}[\tilde{M}(\tau, x, K, \hat{r})|\tau < \infty]} \overset{(13)}{\geq} e^{-\hat{r}x} \underbrace{\frac{1}{\sup\limits_{y\geq 0} \mathbb{E}[e^{-\hat{r}(y-X)}|X > y]}}_{=:C} = Ce^{-\hat{r}x}. \tag{14}$$

For C holds

$$C = \frac{1}{\sup_{y\geq 0} \mathbb{E}[e^{-\hat{r}(y-X)}|X > y]} = \frac{1}{\sup_{y\geq 0} \int_y^\infty e^{-\hat{r}(y-z)} \frac{dF(z)}{\int_y^\infty dF(u)}} = \inf_{y\geq 0} \frac{\int_y^\infty dF(u)}{\int_y^\infty e^{-\hat{r}(y-z)} dF(z)} > 0. \tag{15}$$

Hence, the statement is proven. $\square$