On surprising relations between Europeans and Americans

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Section 1

Introduction

Goal of the talk

- introduce a way towards tractable American option problems in multivariate frameworks.
- beat the curse of (factor process) dimension for some American (Bermudian) option problems.
- analyze surprising relations between American and European options.
- show several new results and open problems.

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A motivating example

Let $X = (X_t)_{0 \le t \le T}$ be, for instance, the Black-Merton-Scholes model in its pricing measure. Denote by u_g the solution of the European value function with bounded payoff function g and interest rate r,

$$u_g(t,x) = \mathbb{E}_{t,x}[e^{-r(T-t)} g(X_T)], \tag{1}$$

for $0 \le t \le T$, where $\mathbb{E}_{t,x}$ denotes expectation conditional on $X_t = x$.

Of course the stochastic process $\left(e^{-rs} u_g(s, X_s)\right)_{t \le s \le T}$ is a martingale with respect to $P_{t,x}$.

Furthermore, define $v_g(x)$ through the one-dimensional minimization problem

$$v_g(x) = \inf_{t \in [0,T]} u_g(t,x) = u_g(t^*(x), x)\,. \tag{2}$$

Assume the minimizer $t^*(x)$ is defining a continuous map in x, and define the *continuation region at time t*

$$C_t = \{x \in \mathbb{R}_{\geq 0} \,|\, t^*(x) > t\}\,.$$

Then, surprisingly, on the continuation region

$$C := \cup_{0\le t\le T} \{t\} \times C_t$$

the European value function u_g *coincides* with the value function of an American option problem with payoff v_g, namely

$$u_g(t,x) = \sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}_{t,x}[e^{-r(\tau-t)} v_g(X_\tau)] \quad \text{for} \quad x \in C_t\,, \tag{3}$$

and $0 \le t \le T$, where $\mathcal{T}_{t,T}$ denotes the set of all stopping times τ with $t \le \tau \le T$.

The proof is simple: Since $v_g(x) \leq u_g(s,x)$ for all $0 \leq s \leq T$ and x, and since the martingale property yields $e^{-r\tau} u_g(\tau, X_\tau) = \mathbb{E}_{t,x}[e^{-rT} g(X_T) \mid \mathcal{F}_\tau]$ due to the optional stopping theorem, one gets

$$\mathbb{E}_{t,x}[e^{-r(\tau-t)} v_g(X_\tau)] \leq \mathbb{E}_{t,x}[e^{-r(\tau-t)} u_g(\tau, X_\tau)] = u_g(t,x)$$

for any $\tau \in \mathcal{T}_{t,T}$.

On the other hand equality is achieved by the first exit time $\tau^* \geq t$ of X from C conditional on X starting in $C(t)$, which by continuity of $t^*(x)$ and X satisfies $v_g(X_{\tau^*}) = u_g(\tau^*, X_{\tau^*})$ provided $X_t \in C_t$.

Hence τ^* is optimal and $u_g(t,x)$ is indeed the value function of an American option problem with payoff v_g, as claimed.

First conclusions

1. There exists a large class of tractable American option problems that can be solved as efficiently as European option problems, namely those where the payoff function is of the form $v_g(x) = \inf_{0 \leq t \leq T} u_g(t, x) = u_g(t^*(x), x)$ for a European value function $u_g(t, x)$ and where the infimum is attained along a connected contour $(x, t^*(x))_{x \geq 0}$.
2. The above derivation goes through whenever X is a strong Markov process with continuous trajectories, possibly with values in some high-dimensional, or even infinite-dimensional, space. The crucial ingredient is the continuity of t^* with respect to state variables.
3. Even beyond continuous trajectories the above consideration seems worthwhile, since it leads to non-trivial lower and upper bounds for the American option problem.

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Section 2

Setting and structures

Setting

Let X be a Markov process with state space E, defined on a filtered measurable space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T})$, and let $\{\mathbb{P}_{t,x}\}_{(t,x)\in[0,T]\times E}$ be a strong Markov family for X.

Let $g : E \to \mathbb{R}$ be a bounded continuous function and define $u_g(t,x)$ and $v_g(x)$ such as in (1) and (2):

$$u_g(t,x) = \mathbb{E}_{t,x}[g(X_T)], \tag{4}$$

and

$$v_g(x) = \inf_{t\in[0,T]} u_g(t,x) = u_g(t^*(x),x)\,. \tag{5}$$

for $x \in E$ and $0 \le t \le T$.

We include interest rates into the factors of the general Markov process and can therefore simplify formulas (1) and (2) by setting $r = 0$.

By definition of u_g and v_g, and by the strong Markov property, the following holds for any $(t, x) \in [0, T] \times E$ and any $\tau \in \mathcal{T}_{t,T}$:

$$\begin{aligned} v_g(X_\tau) &\leq u_g(\tau, X_\tau) \\ &= \mathbb{E}_{\tau, X_\tau}\left[g(X_T)\right] \\ &= \mathbb{E}_{t,x}\left[g(X_T) \mid \mathcal{F}_\tau\right], \qquad \mathbb{P}_{t,x}\text{-a.s.} \end{aligned} \tag{6}$$

Taking expectations and supremum over stopping times yields

$$\sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_{t,x}\left[v_g(X_\tau)\right] \leq u_g(t, x)$$

for all $x \in E$ and $0 \leq t \leq T$.

Hence we *always* obtain a non-trivial upper bound.

Define furthermore the *continuation region at time t*

$$C_t = \{x \in \mathbb{R}_{\geq 0} \mid \inf t^*(x) > t\}.$$

and the continuation region

$$C := \cup_{0 \leq t \leq T} \{t\} \times C_t .$$

We *assume* that the supremum is attained by stopping times if for $x \in C_t$, $0 \leq t \leq T$ there is τ such that

$$\tau \in t^*(X_\tau), \quad \mathbb{P}_{t,x}\text{-a.s.}, \tag{7}$$

where we define the set of minimizers

$$\begin{aligned} t^*(x) &= \operatorname{argmin}\{u_g(t,x)\colon 0 \leq t \leq T\} \\ &= \{t \in [0,T] : u_g(t,x) \leq u_g(s,x) \text{ for all } s \in [0,T]\} . \end{aligned}$$

Indeed, then $u_g(\tau, X_\tau) = v_g(X_\tau)$ almost surely with respect to $P_{t,x}$ for $x \in C_t$ and $0 \leq t \leq T$, hence

$$v_g(t,x) := \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_{t,x}[v_g(X_\tau)] \geq \mathbb{E}_{t,x}[v_g(X_\tau)] = \mathbb{E}_{t,x}[u_g(\tau, X_\tau)] = u_g(t,x).$$

An example 1

Let us take the Bachelier model $X_t = x + B_t$ and a polynomial payoff

$$g(x) = ax^4 + bx^2\,,\ a > 0$$

then

$$u(t,x) = 3a(T-t)^2 + (6ax^2 + b)(T-t) + ax^4 + bx^2$$

and

$$\partial_t u(t,x) = -(6(T-t)a + 6ax^2 + b)\,,\ \partial_t^2 u(t,x) = 6a\,.$$

Whence we can expect minima at $t = T + x^2 + \frac{b}{6a}$ since $a > 0$.

An example 2

The extremal values lie in $[0, T]$ if

$$0 \leq T + x^2 + \frac{b}{6a} \leq T\,,$$

i.e. $x^2 \leq -\frac{b}{6a}$ and $-T - \frac{b}{6a} \leq x^2$. This leads to three cases:

- If $x^2 \geq -\frac{b}{6a}$, then $t^*(x) = T$.
- If $x^2 \leq -T - \frac{b}{6a}$, then $t^*(x) = 0$.
- If $-T - \frac{b}{6a} \leq x^2 \leq -\frac{b}{6a}$, then $t^*(x) = T + x^2 + \frac{b}{6a}$.

An example 3

The payoff v_g reads as follows.

- If $t^*(x) = T$, then $v_g(x) = g(x) = ax^4 + bx^2$.
- If $t^*(x) = 0$, then $v_g(x) = ax^4 + bx^2 + 3aT^2 + (6ax^2 + b)T$.
- If $t^*(x) = T + x^2 + \frac{b}{6a}$, then

$$v_g(x) = ax^4 + bx^2 + 3a(x^2 + \frac{b}{6a})^2 - (6ax^2 + b)(x^2 + \frac{b}{6a}).$$

For this payoff we have an explicit solution of the American option problem, namely u on the continuation region, which in turn is explicitly described by t^*.

Quantifying the gap

In the spirit of Davis-Glasserman-Karatzas-Rogers dual representation for American options we can prove the following generic formula

$$\mathbb{E}_{t,x}\left[v_g(X_\tau) - u_g(\tau, X_\tau)\right] \leq$$
$$\leq v(t,x) - u_g(t,x) \leq \mathbb{E}_{t,x}\left[\sup_{t \leq w \leq T} v_g(X_w) - u_g(w, X_w)\right]$$

for all stopping times $\tau \in \mathcal{T}_{t,T}$, $x \in E$ and $0 \leq t \leq T$. In other words, the upper bound is determined by correcting u_g through the value function (a non-positive value!) of a path dependent look back option, which prices how badly trajectories can fail to pass through the point, where $v_g(x)$ is attained as minimum of $t \mapsto u_g(t,x)$.

Usually this yields very precise upper bounds; additionally the natural lower bound by stopping when leaving C is quite impressive, too.

Open and solved Problems

Several open problems, which have been partially analyzed or solved, can be formulated:

- investigate the range of $g \mapsto v_g$ for multivariate affine or polynomial processes and characterize tractable American option problems. Notice that the one dimensional optimization problem to search for $t^*(x)$ is particularly easy to solve (joint work with Blanka Horvath, Martin Larsson).
- investigate the solutions of the minimization problem $\|h - v_g\| \to \min$, i.e. search for a European payoff such that $h \approx v_g$ in an appropriately discretized function space for generic multivariate Markov processes. This yields a low complexity, deterministic alternative to price and hedge American options in a multivariate framework (joint work with Oleg Reichmann).
- investige the properties of $g \mapsto v_g$ in a continuous time Markov chain setting (joint work with Christa Cuchiero, Archil Gulisashvilli, and Walter Schachermayer)

Open and solved Problems

- investigate the range of $g \mapsto v_g$ and interpret g as cheapest European option dominating an American option with payoff v_g. This can be viewed from a linear programming perspective (Søren Christensen, Mathias Lenga and Jan Kallsen, 2015).
- precise analysis of the Black-Scholes case with a result that the American Put cannot be represented by a certain class of European payoffs (Benjamin Jourdain, Claude Martini, 2001-2002).
- investigate the relationships to other free boundary problems, where a transformation to linear problems is possible (joint work with Martin Larsson, Marvin Müller).

Section 3

An inverse problem: the inf-functional for semigroups

The inf-functional for semigroups

In this last section we present some results on the map $g \mapsto v_g$, which can be formulated for any (time-homogenous) Feller semigroup P, i.e.

$$v_g := \inf_{0 \le t \le T} P_{T-t} g := \mathcal{I}(g)$$

for $g \in C_0(E)$.

The functional aludes to two well-known functionals from functional analysis:

- Elias Stein's max-functionals for analytic semigroups $g \mapsto \sup_{t \geq 0} P_t |g|$.
- The maximum process for martingales of type $(P_{T-t} g(X_t))_{0 \le t \le T}$.

In both cases quite important L^p-inequalities are known (Stein's maximal inequalities and BDG inequalities, respectively), which relate terminal values with running maxima. Here we ask a different questions, namely to describe the range of $g \mapsto v_g$.

Some topological properties

We suppose X is a Feller process (not necessarily continuous) and denote its Feller semigroup by $P: C_0(E) \to C_0(E)$. One can easily prove the following list of properties:

- The payoff profile v_g is continuous and vanishing at ∞ for every $g \in C_0(E)$. Furthermore $v_g \leq g$ holds true.
- The map $g \mapsto v_g$ is Lipschitz continuous with respect to the sup-norm, and – under certain additional assumptions on the Feller semigroup (anlyticity on some L^p, for $1 < p \leq \infty$) – Lipschitz continuous with respect to L^p-norms.
- The map v is monotone and positive homogeneous: for $g_1 \leq g_2$ we have $v_{g_1} \leq v_{g_2}$, and $v_{\lambda g} = \lambda v_g$ for $g \in C_0(E)$ and $\lambda \geq 0$.

Some topological properties

- The map is super-additive: $v_{g_1+g_2} \geq v_{g_1} + v_{g_2}$.
- If $\|g\|_\infty \to \infty$, then $\|v_g\|_\infty \to \infty$ (Hadamard condition).
- If X is a $\mathbb{P}_{t,x}$-martingale on $[t, T]$ for every $(t, x) \in [0, T] \times E$, and g is convex, then $v_g = g$.
- If X is a $\mathbb{P}_{t,x}$-martingale on $[t, T]$ for every $(t, x) \in [0, T] \times E$, and g is concave, then $v_g = u_g(T, \cdot)$.

Rademacher's theorem

Theorem

Let $V \subset C_0(E)$ be a finite-dimensional subspace and consider finitely many points $\{x_i : i = 1, \ldots, N\} \subset E$. Then the map $V \to \mathbb{R}^N$, $g \mapsto (v_g(x_i))_{i=1,\ldots,N}$ is almost everywhere differentiable with respect to Lebesgue measure on the finite dimensional space V.

Proof.

The proof is an application of Rademacher's theorem on differentiability of Lipschitz functions on finite dimensional spaces. □

Differentiability points

A more subtle property can be formulated with respect to differentiability of the map $g \mapsto v_g$:

Theorem

Fix $x \in E$ and assume that for every pair $0 \leq t_1 < t_2 \leq T$ there is $f \in C_0(E)$, such that $P_{t_1} f(x) \neq P_{t_2} f(x)$. Then the map $g \mapsto v_g(x)$ is Fréchet-differentiable if and only if the infimum of $t \mapsto \mathbb{E}_{t,x}[g(X_T)]$ on $[0, T]$ is taken at a unique point $t^*(x) \in [0, T]$.

(Unique points $t^*(x)$, where the infimum is attained, usually translate immediately into continuity of the map $x \mapsto t^*(x)$.)

The case card$(E) < \infty$

This innocent case is of particular importance, since it would allow for the derivation of several important generic results by approximation.

We expect that $g \mapsto v_g$ is surjective for many important Markov processes on state space E, but cannot yet prove it. However, several results are known, which are listed in the sequel.

Even though the problem can be formulated purely in terms of $n \times n$ intensity matrices $\mathcal{A}$ on $\mathbb{R}^n$, whether the map

$$\mathbb{R}^n \ni g \mapsto \Big(\inf_{0 \leq t \leq T} (\exp(\mathcal{A}t)g)(i) \Big)_{i=1,\ldots,n} \in \mathbb{R}^n$$

is surjective or even a homeomorphism, it is a delicate, up to our knowledge unsolved, problem.

Differentiability points in the finite case

Theorem

Let E be finite and X a continuous time (time-homogenous) Markov process on it and assume that for every $x \in E$ and for every pair $0 \leq t_1 < t_2 \leq T$ there is $f \in C_0(E)$, such that $P_{t_1} f(x) \neq P_{t_2} f(x)$. Then the set of points $g \in C_0(E)$ (which is just a finite-dimensional space), where $\mathcal{I}$ is differentiable is an open set of full measure. Let $g \in C_0(E)$ be a differentiability point of I, then

$$\big(D\mathcal{I}(g) \cdot h\big)(x) = \big(P_{t^*(x)} h\big)(x)$$

for $x \in E$.

(Even though P_t is of course invertible, notice that it is unclear whether $D\mathcal{I}$ is in general.)

Theorem (Archil Gulisashvilli, JT)

Let E be a two point set and X a symmetric Markov process on it, then $g \mapsto v_g$ is a homeomorphism.

(The direct proof is lengthy and cumbersome but one can prove that – at differentiability points – the first derivative is regular and its determinant is uniformly bounded from below. Together with the next theorem, this yields an easy proof.)

Invertibility of $D\mathcal{I}$

Theorem (Christa Cuchiero, JT)

Let E be finite. Assume that $|\det D\mathcal{I}(g)| \geq \alpha > 0$ for all differentiability points $g \in C_0(E)$, then $\mathcal{I}$ is surjective.

Proof.

This is an application of a version of the inverse function theorem for Lipschitz maps on finite dimensional spaces (together with the Hadamard condition). □

(Conjecture: we do strongly believe and numerical evidence shows it, too, that the assumption can be replaced by mere symmetry of the semigroup or can even be fully left away.)

Corollary

If the operator norm of the Markov chain's generator A is less than $1/T$, then the condition of the theorem is fullfilled and $\mathcal{I}$ is surjective.

Section 4

Conclusions

Conclusions

- we analyze a generic (i.e. essentially dimension-free) technique to construct tractable American option problems in a framework of tractable factor processes. Numerical techniques work quite well but no justification so far.
- it is conjectured, but still open, that all payoffs are close to payoffs in the range of the inf-functional, which would in turn mean that for all payoffs close tractable American option problems can be constructed.
- it is essential to prove surjectivity in a finite state continuous time Markov chain setting independent of the number of states and the norm of the generator. We are still not there ...