Malliavin Calculus: Absolute continuity and regularity

Josef Teichmann

ETH Zürich

Oxford 2011

Isonormal Gaussian process

A *Gaussian space* is a (complete) probability space together with a Hilbert space of centered real valued Gaussian random variables defined on it.

We speak about Gaussian spaces by means of a coordinate space.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, H a Hilbert space, and $W : H \to L^2[(\Omega, \mathcal{F}, P); \mathbb{R}]$ a linear isometry. Then W is called *isonormal Gaussian process* if $W(h)$ is a centered Gaussian random variable for all $h \in H$.

A Gaussian space is called *irreducible* if its σ-algebra is generated by the elements of the distinguished Hilbert space. In the sequel we shall mainly work with irreducible Gaussian spaces equipped with one isonormal Gaussian process.

Calculation rules

We denote the closed operator Malliavin derivative by $D : \mathcal{D}^{1,2} \to L^2 \otimes H$ and have that

$$D(FG) = GDF + FDG$$

for $F, G, FG \in \mathcal{D}^{1,2}$ if the right hand side is square integrable. The Skorohod integral is denoted by δ and we have the following rule, which is the dual version of the previous Leibnitz rule:

$$\delta(Fu) = F\delta(u) - \langle u, DF \rangle$$

for $F \in \mathcal{D}^{1,2}$ and $u, Fu \in \mathrm{dom}_{1,2}(\delta)$ if the right hand side is square integrable.

A one-dimensional theorem

Let F be a random variable in $\mathcal{D}^{1,2}$ and suppose that $\frac{DF}{||DF||_H^2}$ is Skorohod integrable. Then the law of F has a continuous and bounded density f with respect to the Lebesgue measure λ given by

$$f(x) = E\left[1_{\{F>x\}}\delta\left(\frac{DF}{||DF||_H^2}\right)\right]$$

for real x.

Proof

We consider $\psi(y) = 1_{[a,b]}(y)$ for $a < b$ and $\phi(y) := \int_{-\infty}^{y} \psi(x)dx$. Since $\phi(F) \in \mathcal{D}^{1,2}$ we obtain

$$\langle D(\phi(F)), DF \rangle_H = \psi(F) \, ||DF||_H^2$$

which allows to compute $\psi(F)$. By integration by parts

$$E(\psi(F)) = E\left(\left\langle D(\phi(F)), \frac{DF}{||DF||_H^2} \right\rangle_H\right) =$$
$$= E\left(\phi(F) \, \delta\left(\frac{DF}{||DF||_H^2}\right)\right)$$

which leads to

$$P(a \leq F \leq b) = E\left(\int_{-\infty}^{F} \psi(x)dx\, \delta\left(\frac{DF}{||DF||_H^2}\right)\right) =$$

$$= \int_a^b E\left(1_{\{F>x\}}\, \delta\left(\frac{DF}{||DF||_H^2}\right)\right) dx$$

by Fubini's theorem.

The Gagliardo-Nirenberg inequality

It holds that

$$||f||_{L^{\frac{N}{N-1}}} \leq \prod_{i=1}^{N} ||\partial_i f||_{L^1}^{\frac{1}{N}}$$

for $f \in C_0^\infty(\mathbb{R}^m)$ and $N \geq 2$.

How to detect densities

Let μ be a finite measure on $\mathbb{R}^N$ and assume that there are constants c_i for $i = 1, ..., N$ such that

$$\left| \int_{\mathbb{R}^m} \partial_i \phi(x) \mu(dx) \right| \leq c_i ||\phi||_\infty$$

for all $\phi \in C_0^\infty(\mathbb{R}^m)$, then μ is absolutely continuous with respect to the Lebesgue measure.

Proof

We show the case for $N \geq 2$: we shall show that the density of μ belongs to $L^{\frac{N}{N-1}}$ for $N > 1$. We take a Dirac sequence ψ_ϵ for $\epsilon > 0$ and a sequence of smooth bump functions $0 \leq c_M \leq 1$ with

$$c_M(x) = \begin{cases} 1 \text{ for } ||x|| \leq M \\ 0 \text{ for } ||x|| \geq M+1 \end{cases}$$

where we assume that the partial derivatives are bounded uniformly with respect to M. Then the measures $c_M\left(\psi_\epsilon * \mu\right)$ have densities $p_{M,\epsilon}$ belonging to $C_0^\infty(\mathbb{R}^N)$.

Proof

To apply the Gagliardo-Nirenberg inequality we have to estimate

$$\begin{aligned}||\partial_i p_{M,\epsilon}||_{L^1} &\leq \int_{\mathbb{R}^N} c_M(x) \left|((\partial_i \psi_\epsilon) * \mu)\right|(dx) \\ &\quad + \int_{\mathbb{R}^N} |\partial_i c_M(x)| \, (\psi_\epsilon * \mu)(dx) \\ &\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \psi_\epsilon(x-y) |\nu_i|(dy) dx \\ &\quad + \int_{\mathbb{R}^N} |\partial_i c_M(x)| \, (\psi_\epsilon * \mu)(dx)\end{aligned}$$

where ν_i denotes the signed finite measure on $\mathbb{R}^N$ induced by $\phi \mapsto \int \partial_i \phi \mu(dx)$ for $\phi \in C_0^\infty(\mathbb{R}^N)$. This expression is bounded by a constant independent of M and ϵ by Fubini's theorem.

Proof

The unit ball of $L^{\frac{N}{N-1}}$ is weakly compact, so we find a weak limit of $c_M(\psi_\epsilon * \mu)$ in $L^{\frac{N}{N-1}}$: on the one hand

$$\int_{\mathbb{R}^N} g(x) c_M(x) (\psi_\epsilon * \mu)(dx) \to \int_{\mathbb{R}^N} g(x)\mu(dx)$$

for $g \in L^\infty(\mathbb{R}^N)$ as $M \to \infty$ and $\epsilon \to 0$ and $\mu(\mathbb{R}^n) < \infty$. However, since there exists a weak limit $p \in L^1(\mathbb{R}^N)$ we obtain

$$\int_{\mathbb{R}^N} g(x)\mu(dx) = \int_{\mathbb{R}^N} g(x)p(x)dx$$

which is the desired result.

Localization

We denote $F \in \mathcal{D}^{1,p}_{loc}$ for some $p \geq 1$, if there exists a sequence $(\Omega_n, F_n)_{n\geq 0}$, where Ω_n is a measurable set and $F_n \in \mathcal{D}^{1,p}$ for $n \geq 0$ such that

$$\Omega_n \uparrow \Omega \text{ almost surely,}$$
$$F_n 1_{\Omega_n} = F 1_{\Omega_n} \text{ almost surely.}$$

The Malliavin Covariance Matrix

Take now a random vector $F := (F^1, ..., F^N)$, which belongs to $\mathcal{D}^{1,1}_{loc}$ componentwise. We associate to F the Malliavin (covariance) matrix γ_F, which is a non-negative, symmetric random matrix:

$$\gamma(F) := \gamma_F := (\langle DF^i, DF^j \rangle_H)_{1 \leq i,j \leq N}.$$

From regular invertibility of this matrix we shall obtain the basic condition on the existence of a density.

Main theorem

Let F be a random vector satisfying

1. $F^i \in \mathcal{D}^{2,4}_{loc}$ for all $i = 1, ..., N$.
2. The matrix γ_F is invertible almost surely.

Then the law of F is absolutely continuous with respect to the Lebesgue measure.

Proof

We shall assume $F^i \in \mathcal{D}^{2,4}$ for each $i = 1, ..., N$ first. We fix a test function $\phi \in C_0^\infty(\mathbb{R}^N)$, then by the chain rule $\phi(F) \in \mathcal{D}^{2,4}$, consequently

$$D(\phi(F)) = \sum_{i=1}^{N} \frac{\partial \phi}{\partial x_i}(F)\, DF^i,$$

$$\left\langle D(\phi(F)), DF^j \right\rangle_H = \sum_{i=1}^{N} \frac{\partial \phi}{\partial x_i}(F)\, \gamma_F^{ij}$$

and therefore by invertibility

$$\frac{\partial \phi}{\partial x_i}(F) = \sum_{j=1}^{N} \left\langle D(\phi(F)), DF^j \right\rangle_H (\gamma_F^{-1})^{ij}.$$

Proof

In the sequel we have to apply a localization argument: consider the compact subset $K_m \subset GL(N)$ of matrices σ with $|\sigma^{ij}| \leq m$ and $|\det(\sigma)| \geq \frac{1}{m}$ for $i, j = 1, \ldots, m$. We can define $\psi_m \in C_0^\infty(M_N(\mathbb{R}))$ with $\psi_m \geq 0$, $\psi_m|_{K_m} = 1$ and $\psi_m|_{GL(N)\setminus K_{m+1}} = 0$, which is easily possible since K_m is an exhaustion of $GL(N)$ by compact sets such that $K_m \subset (K_{m+1})^\circ$. Now we can integrate reasonably the above equation

$$E(\psi_m(\gamma_F)\frac{\partial \phi}{\partial x_i}(F)) = \sum_{j=1}^{N} E(\psi_m(\gamma_F)\big\langle D(\phi(F)), DF^j\big\rangle_H (\gamma_F^{-1})^{ij}).$$

Proof

Remark that $\psi_m(\gamma_F) DF^j \, (\gamma_F^{-1})^{ij} \in \mathrm{dom}_{1,2}(\delta)$, since $\psi_m(\gamma_F) \, (\gamma_F^{-1})^{ij}$ is a bounded random variable (it equals the inversion rational function applied to γ_F times a smooth function with compact support applied to γ_F, but $\gamma_F \in \mathcal{D}^{2,4}$) and

$$E\left(\left(\psi_m(\gamma_F)(\gamma_F^{-1})^{ij}\right)^2 \left\langle DF^j, DF^j \right\rangle_H \right) < \infty.$$

Consequently we can apply integration by parts to arrive at

$$\begin{aligned} E(\psi_m(\gamma_F) \frac{\partial \phi}{\partial x_i}(F)) &= E(\phi(F) \sum_{j=1}^{N} \delta \left(\psi_m(\gamma_F) \, DF^j \, (\gamma_F^{-1})^{ij} \right)) \\ &\leq ||\phi||_\infty E \left(\left| \sum_{j=1}^{N} \delta \left(\psi_m(\gamma_F) \, DF^j \, (\gamma_F^{-1})^{ij} \right) \right| \right). \end{aligned}$$

Proof

Hence we obtain that for any $A \in \mathcal{B}(\mathbb{R}^N)$ with zero Lebesgue measure

$$\int_{F^{-1}(A)} \psi_m(\gamma_F) dP = 0$$

holds true, but as $m \to \infty$ – via property 2 of the assumptions – $\int_{F^{-1}(A)} dP = 0$. Therefore $F_* P \ll \lambda$.

In general – for $F \in \mathcal{D}^{2,4}_{loc}$ – we calculate for F_n and obtain the result by the property that $F_n^{-1}(A) \to F^{-1}(A)$.

Detection of smooth densities

Let μ be a finite measure on $\mathbb{R}^N$ and $A \subset \mathbb{R}^N$ open. Assume that there are constants c_α for a multiindex α such that

$$\left| \int_{\mathbb{R}^N} \partial_\alpha \phi(x) \mu(dx) \right| \leq c_\alpha ||\phi||_\infty$$

for all $\phi \in C_b^\infty(\mathbb{R}^N)$ with compact support in A, then the restriction of μ to A is absolutely continuous with respect to the Lebesgue measure and the density is smooth.

Main theorem in the smooth case

Let F be a random vector satisfying

1. $F^i \in \mathcal{D}^\infty$ for all $i = 1, ..., N$.
2. The matrix γ_F is invertible almost surely and $\frac{1}{\det(\gamma_F)} \in L^{\infty-0}$.

Then the law of F is absolutely continuous with respect to Lebesgue measure and the existing density is smooth.

A more geometric point of view

Let $g : \Omega \to \mathbb{R}^N$ be a random variable with well-defined covariance matrix $\gamma(g)$, then we can define for any vector $z \in \mathbb{R}^N$ the covering vector field $Z \in L^2(\Omega, \mathcal{F}, P) \otimes H$ via

$$\langle Dg^j, Z \rangle_H = z^j.$$

Apparently one solution is given by

$$Z = \sum_{i=1}^{N} Dg^i \left(\gamma(g)^{-1} z\right)^i,$$

since for $j = 1, \ldots, N$

$$\langle Dg^j, Z \rangle_H = \sum_{i=1}^{N} \langle Dg^j, Dg^i \rangle_H \left(\gamma(g)^{-1} z\right)^i = z^j.$$

A more geometric point of view

Hence the previously calculated solutions are in fact lifts of vectors to covering vector fields on the given Gaussian space. Usually Z can be chosen to be Skorohod-integrable, whence integration by parts will work. This leads to the following theorem:

Let $F \in \mathcal{D}^{\infty}$ and $\frac{1}{\det(\gamma_F)} \in L^{\infty-0}$, then for any multiindex $\alpha \in \mathbb{N}^N$ we obtain for all $\phi \in C_b^{\infty}(\mathbb{R}^N)$.

$$E\left(\partial_{\alpha}\phi(F)\right) = E\left(\phi(F)Q_{\alpha}\right)$$

by integration by parts for some random variable $Q_{\alpha} \in \mathcal{D}^{\infty}$ (independent of ϕ).

We do the proof by induction: for $\alpha = 0$ there is nothing to show. Let us assume now that it holds for $|\alpha| < k$ and we choose some β of order k. Without restriction we assume that $\partial_\alpha = \partial_\beta \partial_1$, whence

$$\begin{aligned} E\left(\partial_\beta \partial_1 \phi(F)\right) &= E\left(\partial_1 \phi(F) Q_\beta\right) \\ &= E\left(\langle D\phi(F), Z\rangle Q_\beta\right) \\ &= E\left(\phi(F)(Q_\beta \delta(Z) - \langle DQ_\beta, Z\rangle)\right), \end{aligned}$$

where Z is a covering vector field for e_1. This proves the statement for ∂_α and completes the induction.

Proof for the smooth case

Choose $\phi_\xi(x) = \exp(\langle \xi, x \rangle)$, then

$$||\xi||^k |E(\exp(\langle \xi, F \rangle))| \leq |E(\exp(\langle \xi, F \rangle) Q_k) \leq E(|Q_k|) < \infty,$$

which means that the characteristic function of g tends to zero as $\xi \to \infty$ faster than any polynomial in the Fourier variable ξ. This in turn means that there is a smooth density with bounded derivatives of all orders.

With the same methodology one can show that the density is in fact Schwarz.