Malliavin Calculus:
Absolute continuity and regularity

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Isonormal Gaussian process

A *Gaussian space* is a (complete) probability space together with a Hilbert space of centered real valued Gaussian random variables defined on it. We speak about Gaussian spaces by means of a coordinate space.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, H a Hilbert space, and $W : H \to L^2[(\Omega, \mathcal{F}, P); \mathbb{R}]$ a linear isometry. Then W is called *isonormal Gaussian process* if $W(h)$ is a centered Gaussian random variable for all $h \in H$.

A Gaussian space is called *irreducible* if $\mathcal{F}_H = \mathcal{F}$. In the sequel we shall work with irreducible Gaussian spaces equipped with one isonormal Gaussian process.

Calculation rules

We denote the Malliavin derivative by $D : \mathcal{D}^{1,2} \to L^2 \otimes H$ and have that

$$D(FG) = GDF + FDG$$

for $F, G, FG \in \mathcal{D}^{1,2}$ if the right hand side is square integrable. The Skorohod integral is denoted by δ and we have the following rule, which is the dual version of the previous Leibnitz rule:

$$\delta(Fu) = F\delta(u) - \langle u, DF \rangle$$

for $F \in \mathcal{D}^{1,2}$ and $u, Fu \in \mathrm{dom}_{1,2}(\delta)$ if the right hand side is square integrable.

A one-dimensional theorem

Let F be a random variable in $\mathcal{D}^{1,2}$ and suppose that $\frac{DF}{||DF||_H^2}$ is Skorohod integrable. Then the law of F has a continuous and bounded density f with respect to the Lebesgue measure λ given by

$$f(x) = E\left[1_{\{F>x\}}\delta\left(\frac{DF}{||DF||_H^2}\right)\right]$$

for real x.

Proof

We consider $\psi(y) = 1_{[a,b]}(y)$ for $a < b$ and $\phi(y) := \int_{-\infty}^{y} \psi(x)dx$. Since $\phi(F) \in \mathcal{D}^{1,2}$ we obtain

$$\langle D(\phi(F)), DF \rangle_H = \psi(F)\,||DF||_H^2$$

which allows to compute $\psi(F)$. By integration by parts

$$E(\psi(F)) = E\left(\left\langle D(\phi(F)), \frac{DF}{||DF||_H^2} \right\rangle_H\right) =$$
$$= E\left(\phi(F)\,\delta\left(\frac{DF}{||DF||_H^2}\right)\right)$$

which leads to

$$P(a \le F \le b) = E\left(\int_{-\infty}^{F} \psi(x)dx\, \delta\left(\frac{DF}{||DF||_H^2}\right)\right) =$$

$$= \int_a^b E\left(1_{\{F>x\}}\, \delta\left(\frac{DF}{||DF||_H^2}\right)\right) dx$$

by Fubini's theorem.

The Gagliardo-Nirenberg inequality

It holds that

$$||f||_{L^{\frac{N}{N-1}}} \leq \prod_{i=1}^{N} ||\partial_i f||_{L^1}^{\frac{1}{N}}$$

for $f \in C_0^\infty(\mathbb{R}^m)$ and $N \geq 2$.

How to detect densities

Let μ be a finite measure on $\mathbb{R}^N$ and assume that there are constants c_i for $i = 1, \dots, N$ such that

$$\left| \int_{\mathbb{R}^m} \partial_i \phi(x) \mu(dx) \right| \leq c_i ||\phi||_\infty$$

for all $\phi \in C_0^\infty(\mathbb{R}^m)$, then μ is absolutely continuous with respect to the Lebesgue measure.

Proof

We show the case for $N \geq 2$: we shall show that the density of μ belongs to $L^{\frac{N}{N-1}}$ for $N > 1$. We take a Dirac sequence ψ_ϵ for $\epsilon > 0$ and a sequence of smooth bump functions $0 \leq c_M \leq 1$ with

$$c_M(x) = \begin{cases} 1 \text{ for } ||x|| \leq M \\ 0 \text{ for } ||x|| \geq M+1 \end{cases}$$

where we assume that the partial derivatives are bounded uniformly with respect to M. Then the measures $c_M\,(\psi_\epsilon * \mu)$ have densities $p_{M,\epsilon}$ belonging to $C_0^\infty(\mathbb{R}^N)$.

Proof

We apply the Gagliardo-Nirenberg inequality and have to estimate additionally

$$\begin{aligned} ||\partial_i p_{M,\epsilon}||_{L^1} &\leq \int_{\mathbb{R}^N} c_M(x) \left|((\partial_i \psi_\epsilon) * \mu)\right| (dx) \\ &\quad + \int_{\mathbb{R}^N} \left|\partial_i c_M(x)\right| (\psi_\epsilon * \mu)(dx) \\ &\leq \int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} \psi_\epsilon(x-y) \nu_i(dy) \right| dx \\ &\quad + \int_{\mathbb{R}^N} \left|\partial_i c_M(x)\right| (\psi_\epsilon * \mu)(dx) \end{aligned}$$

where ν_i denotes the signed finite measure on $\mathbb{R}^N$ induced by $\phi \mapsto \int \partial_i \phi \mu(dx)$ for $\phi \in C_b^\infty(\mathbb{R}^N)$. This expression is bounded by a constant independent of M and ϵ by Fubini's theorem.

Proof

The unit ball of $L^{\frac{N}{N-1}}$ is weakly compact, so we find a weak limit of $c_M\left(\psi_\epsilon * \mu\right)$ in $L^{\frac{N}{N-1}}$: on the one hand

$$\int_{\mathbb{R}^N} g(x) c_M(x)\left(\psi_\epsilon * \mu\right)(dx) \to \int_{\mathbb{R}^N} g(x)\mu(dx)$$

for $g \in L^\infty(\mathbb{R}^N)$ as $M \to \infty$ and $\epsilon \to 0$. However, since there exists a weak limit p we obtain

$$\int_{\mathbb{R}^N} g(x)\mu(dx) = \int_{\mathbb{R}^N} g(x)p(x)dx$$

which is the desired result.

Localization

We denote $F \in \mathcal{D}^{1,p}_{loc}$ for some $p \geq 1$, if there exists a sequence $(\Omega_n, F_n)_{n\geq 0}$, where Ω_n is a measurable set and $F_n \in \mathcal{D}^{1,p}$ for $n \geq 0$ such that

$$\Omega_n \uparrow \Omega \text{ almost surely,}$$
$$F_n 1_{\Omega_n} = F 1_{\Omega_n} \text{ almost surely.}$$

The Malliavin Covariance atrix

Take now a random vector $F := (F^1, ..., F^N)$, which belongs to $\mathcal{D}^{1,1}_{loc}$ componentwise. We associate to F the Malliavin (covariance) matrix γ_F, which is a non-negative, symmetric random matrix:

$$\gamma(F) := \gamma_F := (\langle DF^i, DF^j \rangle_H)_{1 \leq i,j \leq N}.$$

From regular invertibility of this matrix we shall obtain the basic condition on the existence of a density.

Main theorem

Let F be a random vector satisfying

1. $F^i \in \mathcal{D}^{2,4}_{loc}$ for all $i = 1, ..., N$.
2. The matrix γ_F is invertible almost surely.

Then the law of F is absolutely continuous with respect to the Lebesgue measure.

Proof

We shall assume $F^i \in \mathcal{D}^{2,4}$ for each $i = 1, ..., N$ first. We fix a test function $\phi \in C_0^\infty(\mathbb{R}^N)$, then by the chain rule $\phi(F) \in \mathcal{D}^{2,4}$, consequently

$$D(\phi(F)) = \sum_{i=1}^{N} \frac{\partial \phi}{\partial x_i}(F)\, DF^i,$$

$$\left\langle D(\phi(F)), DF^j \right\rangle_H = \sum_{i=1}^{N} \frac{\partial \phi}{\partial x_i}(F)\, \gamma_F^{ij}$$

and therefore by invertibility

$$\frac{\partial \phi}{\partial x_i}(F) = \sum_{j=1}^{N} \left\langle D(\phi(F)), DF^j \right\rangle_H (\gamma_F^{-1})^{ij}.$$

Proof

In the sequel we have to apply a localization argument: consider the compact subset $K_m \subset GL(N)$ of matrices σ with $|\sigma^{ij}| \leq m$ and $|\det(\sigma)| \geq \frac{1}{m}$ for $i, j = 1, ..., m$. We can define $\psi_m \in C_0^\infty(M_N(\mathbb{R}))$ with $\psi_m \geq 0$, $\psi_m|_{K_m} = 1$ and $\psi_m|_{GL(N)\setminus K_{m+1}} = 0$, which is easily possible since K_m is an exhaustion of $GL(N)$ by compact sets such that $K_m \subset (K_{m+1})^\circ$. Now we can integrate reasonably the above equation

$$E(\psi_m(\gamma_F)\frac{\partial\phi}{\partial x_i}(F)) = \sum_{j=1}^{N} E(\psi_m(\gamma_F)\left\langle D(\phi(F)), DF^j\right\rangle_H(\gamma_F^{-1})^{ij}).$$

Proof

Remark that $\psi_m(\gamma_F) DF^j \, (\gamma_F^{-1})^{ij} \in \mathrm{dom}_{1,2}(\delta)$, since $\psi_m(\gamma_F) \, (\gamma_F^{-1})^{ij}$ is a bounded random variable (it equals the inversion rational function applied to γ_F times a smooth function with compact support applied to γ_F, but $\gamma_F \in \mathcal{D}^{2,4}$) and

$$E\left(\left(\psi_m(\gamma_F)(\gamma_F^{-1})^{ij}\right)^2 \left\langle DF^j, DF^j \right\rangle_H\right) < \infty.$$

Consequently we can apply integration by parts to arrive at

$$E(\psi_m(\gamma_F)\frac{\partial\phi}{\partial x_i}(F)) = E(\phi(F)\sum_{j=1}^{N} \delta\left(\psi_m(\gamma_F)\, DF^j \, (\gamma_F^{-1})^{ij}\right))$$

$$\leq ||\phi||_\infty E\left(\left|\sum_{j=1}^{N} \delta\left(\psi_m(\gamma_F)\, DF^j \, (\gamma_F^{-1})^{ij}\right)\right|\right).$$

Proof

Hence we obtain that for any $A \in \mathcal{B}(\mathbb{R}^N)$ with zero Lebesgue measure

$$\int_{F^{-1}(A)} \psi_m(\gamma_F) dP = 0$$

holds true, but as $m \to \infty$ – via property 2 of the assumptions – $\int_{F^{-1}(A)} dP = 0$. Therefore $F_* P \ll \lambda$.

In general – for $F \in \mathcal{D}^{2,4}_{loc}$ – we calculate for F_n and obtain the result by the property that $F_n^{-1}(A) \to F^{-1}(A)$.

Detection of smooth densities

Let μ be a finite measure on $\mathbb{R}^N$ and $A \subset \mathbb{R}^N$ open. Assume that there are constants c_α for a multiindex α such that

$$\left| \int_{\mathbb{R}^N} \partial_\alpha \phi(x) \mu(dx) \right| \leq c_\alpha ||\phi||_\infty$$

for all $\phi \in C_b^\infty(\mathbb{R}^N)$ with compact support in A, then the restriction of μ to A is absolutely continuous with respect to the Lebesgue measure and the density is smooth.

Main theorem in the smooth case

Let F be a random vector satisfying

1. $F^i \in \mathcal{D}^\infty$ for all $i = 1, ..., N$.
2. The matrix γ_F is invertible almost surely and $\frac{1}{\det(\gamma_F)} \in L^{\infty-0}$.

Then the law of F is absolutely continuous with respect to Lebesgue measure and the existing density is smooth.

A more geometric point of view

Let $g : \Omega \to \mathbb{R}^N$ be a random variable with well-defined covariance matrix $\gamma(g)$, then we can define for any vector $z \in \mathbb{R}^N$ the covering vector field $Z \in L^2(\Omega, \mathcal{F}, P) \otimes H$ via

$$\langle Dg^j, Z \rangle_H = z^j.$$

Apparently one solution is given by

$$Z = \sum_{i=1}^{N} Dg^i \left(\gamma(g)^{-1} z\right)^i,$$

since for $j = 1, \ldots, N$

$$\langle Dg^j, Z \rangle_H = \sum_{i=1}^{N} \langle Dg^j, Dg^i \rangle_H \left(\gamma(g)^{-1} z\right)^i = z^j.$$

A more geometric point of view

Hence the previously calculated solutions are in fact lifts of vectors to covering vector fields on the given Gaussian space. Usually Z can be chosen to be Skorohod-integrable, whence integration by parts will work. This leads to the following theorem:

Let $F \in \mathcal{D}^\infty$ and $\frac{1}{\det(\gamma_F)} \in L^{\infty-0}$, then for any multiindex $\alpha \in \mathbb{N}^N$ we obtain for all $\phi \in C_b^\infty(\mathbb{R}^N)$.

$$E\left(\partial_\alpha \phi(F)\right) = E\left(\phi(F) Q_\alpha\right)$$

by integration by parts for some random variable $Q_\alpha \in \mathcal{D}^\infty$ (independent of ϕ).

We do the proof by induction: for $\alpha = 0$ there is nothing to show. Let us assume now that it holds for $|\alpha| < k$ and we choose some β of order k. Without restriction we assume that $\partial_\alpha = \partial_\beta \partial_1$, whence

$$\begin{aligned} E\left(\partial_\beta \partial_1 \phi(F)\right) &= E\left(\partial_1 \phi(F) Q_\beta\right) \\ &= E\left(\langle D\phi(F), Z\rangle Q_\beta\right) \\ &= E\left(\phi(F)(Q_\beta \delta(Z) - \langle DQ_\beta, Z\rangle)\right), \end{aligned}$$

where Z is a covering vector field for e_1. This proves the statement for ∂_α and completes the induction.

Proof for the smooth case

Choose $\phi_\xi(x) = \exp(\langle \xi, x \rangle)$, then

$$||\xi||^k |E(\exp(\langle \xi, F \rangle))| \leq |E(\exp(\langle \xi, F \rangle) Q_k) \leq E(|Q_k|) < \infty,$$

which means that the characteristic function of g tends to zero as $\xi \to \infty$ faster than any polynomial in the Fourier variable ξ. This in turn means that there is a smooth density with bounded derivatives of all orders.

With the same methodology one can show that the density is in fact Schwarz.