LYAPUNOV FUNCTIONS AS PORTFOLIO-GENERATORS

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1. INTRODUCTION

Back in 1999, Erhard Robert FERNHOLZ introduced
a construction that was both
(i) remarkable, and
(ii) remarkably easy to prove.

He showed that for a certain class of so-called "functionally-generated" portfolios, it is possible to express the wealth they generate, discounted by (denominated in terms of) the total market capitalization, solely in terms of the individual companies' *market weights* – and to do so in a pathwise manner, that *does not involve stochastic integration.*

This fact can be proved by an application of ITÔ's rule. Once the result is known, its proof can be assigned as a moderate exercise in a Master's level stochastic calculus course.

The discovery paved the way for finding simple, structural conditions on *large* equity markets – that involve more than one stock, and typically thousands – under which it is possible to outperform the market portfolio (w.p.1).

Put a little differently: conditions under which (strong) arbitrage relative to the market portfolio is possible.

Bob FERNHOLZ showed also how to implement this outperformance by simple portfolios – which can be constructed solely in terms of observable quantities, without any need to estimate parameters of the model or to optimize.

Although well-known, celebrated, and quite easy to prove, FERNHOLZ's construction has been viewed over the past 15+ years as somewhat "mysterious".

In this talk, and in the work on which the talk is based, we hope to help make the result a bit more celebrated and perhaps a bit less mysterious, via an interpretation of portfolio-generating functions as LYAPUNOV functions for the vector process of relative market weights.

We will try to settle then a question about functionally-generated portfolios that has been open for 10 years.

The aim of this talk is to offer an interpretation of FERNHOLZ's Generating Functions as LYAPUNOV functions.

Functions $G : \mathbf{\Delta}^n_+ \to (0, \infty)$, in other words, defined on (an open neighborhood of) the lateral face of the strictly positive unit simplex, such that the process

$$G(\mu(t)), \quad 0 \le t < \infty$$

is a supermartingale under an appropriate probability measure (a very liberal interpretation of the term "Lyapunov function").

. Here $\mu(t) = (\mu_1(t), \cdots, \mu_n(t))'$ is the vector of market weights of the individual companies in the market; that is, the ratios

$$\mu_i(t) = \frac{X_i(t)}{X_1(t) + \cdots + X_n(t)}, \qquad i = 1, \cdots, n$$

of their own capitalizations $X_i(t)$, divided by the capitalization of the entire market.

We posit that this interpretation, and this interpretation ALONE, leads to the possibility of arbitrage relative to the market portfolio under suitable conditions.

Then as a second, and in many ways more decisive, step, the construction of FERNHOLZ (1999, 2002) identifies a very specific portfolio – generated by the function $G : \mathbf{\Delta}^n_+ \to (0, \infty)$ itself, or by a suitable "shift" of this function – which implements the strong arbitrage *on this SAME range of time-horizons.*

2. THE SETTING

We place ourselves on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a right-continuous filtration $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \le t < \infty}$ that satisfies $\mathcal{F}(0) = \{\emptyset, \Omega\}$ mod. $\mathbb{P}$.

. We consider continuous semimartingales (sums of local martingales and processes of finite variation)

$$S_i(\cdot) = M_i(\cdot) + B_i(\cdot)\,, \qquad i = 1, \cdots, n$$

with $S_i(0) = 0$. These will play the rôle of *asset returns,* and their stochastic exponentials

$$X_i(\cdot) := X_i(0) \cdot \mathcal{E}(S_i)(\cdot)\,, \quad \mathrm{d}X_i(t) = X_i(t)\,\mathrm{d}S_i(t)\,; \quad i = 1, \cdots, n$$

will play the rôle of *asset capitalizations.*

The real constants $X_1(0), \cdots, X_n(0)$ are strictly positive, so these capitalizations are strictly positive, continuous semimartingales.

With these processes thus constructed, we introduce the various assets' *relative weights*

$$\mu_i(t) = \frac{X_i(t)}{X_1(t) + \cdots + X_n(t)}\,, \qquad i = 1\,, \cdots\,, n$$

just as before. These are positive, continuous semimartingales in their own right. NO OTHER ASSUMPTION.

. We denote by $\mathfrak{I} = \mathscr{P}(\mathcal{S})$ the collection of progressively measurable vector processes $\pi = (\pi_1, \cdots, \pi_n)'$ which are integrable with respect to the vector semimartingale $\mathcal{S} = (S_1, \cdots, S_n)'$.

. We interpret these processes as *investment rules,* and denote by $V^\pi(\cdot)$ their corresponding "value" or "wealth" process, namely

$$\frac{\mathrm{d}V^\pi(t)}{V^\pi(t)} = \sum_{i=1}^{n} \pi_i(t)\,\frac{\mathrm{d}X_i(t)}{X_i(t)} = \sum_{i=1}^{n} \pi_i(t)\,\mathrm{d}S_i(t)\,, \qquad V^\pi(0) = 1\,.$$

To wit: $V^\pi(\cdot)$ is the stochastic exponential of $\int_0^\cdot \langle \pi(t), \mathrm{d}\mathcal{S}(t)\rangle$.

In this scheme of things, $\pi_i(t)$ is the proportion of wealth $V^\pi(t)$ invested at time t in the i th asset; the proportion

$$\pi_0(t) := 1 - \sum_{i=1}^{n} \pi_i(t)$$

is kept under the mattress ("invested in a bank account").

. We call an investment rule $\pi(\cdot)$ *long-only,* if it satisfies

$$\pi_0(\cdot) \geq 0\,, \quad \pi_1(\cdot) \geq 0\,, \cdots, \pi_n(\cdot) \geq 0\,.$$

The investment rule $\kappa(\cdot) \equiv (0, \cdots, 0)'$ keeps the entire wealth under the mattress:

$$\kappa_0(\cdot) \equiv 1\,, \quad V^\kappa(\cdot) \equiv 1\,.$$

. We call an investment rule $\pi(\cdot)$ *portfolio,* if it satisfies

$$\sum_{i=1}^{n} \pi_i(\cdot) \equiv 1 \, .$$

We denote the class of portfolios by $\mathfrak{P}$, and by $\mathfrak{P}_+$, the class of long-only portfolios.

- The *Market Portfolio*

$$\mu(\cdot) = \big(\mu_1(\cdot), \cdots, \mu_n(\cdot)\big)' \in \mathfrak{P}_+$$

is an important long-only portfolio. It generates wealth proportional to the total market capitalization:

$$V^{\mu}(\cdot) = \frac{X_1(t) + \cdots + X_n(t)}{X_1(0) + \cdots + X_n(0)} \, .$$

- It is easy to check that the performance of an arbitrary *portfolio* $\pi(\cdot)$, when measured relative to the performance of the market – a very natural "discounting" – has the dynamics

$$\mathrm{d}\left(\frac{V^{\pi}(t)}{V^{\mu}(t)}\right) = \left(\frac{V^{\pi}(t)}{V^{\mu}(t)}\right) \sum_{i=1}^{n} \frac{\pi_i(t)}{\mu_i(t)}\, \mathrm{d}\mu_i(t)\,, \qquad \forall \;\; \pi(\cdot) \in \mathfrak{P}\,.$$

Please note the formal analogy with the dynamics of the wealth process from a couple of slides ago:

$$\mathrm{d}V^{\pi}(t) = V^{\pi}(t) \sum_{i=1}^{n} \frac{\pi_i(t)}{X_i(t)}\, \mathrm{d}X_i(t)\,, \qquad \forall \;\; \pi(\cdot) \in \mathfrak{I}\,.$$

3. NUMÉRAIRE AND ARBITRAGE

Definition: We shall say that an investment rule $\pi(\cdot)$ is *Arbitrage relative to another rule* $\rho(\cdot)$*,* over a fixed time-horizon $[0, T]$, if

$$\mathbb{P}\big(V^{\pi}(T) \geq V^{\rho}(T)\big) \;=\; 1\,, \qquad \mathbb{P}\big(V^{\pi}(T) > V^{\rho}(T)\big) \;>\; 0\,.$$

We call this relative arbitrage *strong,* if in fact
$\mathbb{P}\big(V^{\pi}(T) > V^{\rho}(T)\big) \;=\; 1\,.$ □

. If we take $\rho(\cdot) \equiv \kappa(\cdot) \equiv (0, \cdots, 0)'$ in this definition, we recover the classical notion of arbitrage (relative to cash).

Definition: We shall say that a given investment rule $\nu(\cdot) \in \mathfrak{I}$ has the *Numéraire Property,* if the ratio

$V^{\pi}(\cdot) \big/ V^{\nu}(\cdot)$ is a supermartingale, for every $\pi(\cdot) \in \mathfrak{I}\,.$ □

Properties: It can be shown that, if a given investment rule $\nu(\cdot) \in \mathfrak{I}$ has the numéraire property, then

(i) it has also the relative-log-optimality and growth-optimality properties;

(ii) no arbitrage can exist relative to it, over any given time-horizon; and

(iii) the ratio

$$V^{\pi}(\cdot) \big/ V^{\nu}(\cdot)$$

is actually *a local martingale,* for every $\pi(\cdot) \in \mathfrak{I}$.

. Besides, for any two investment rules $\nu_1(\cdot)$, $\nu_2(\cdot)$ with the numéraire property, we have

$$V^{\nu_1}(\cdot) \equiv V^{\nu_2}(\cdot) .$$

FOR CERTAIN RESULTS WE SHALL NEED THE ASSUMPTION THAT **"NP HOLDS"**, i.e., THAT THERE EXISTS AN INVESTMENT RULE $\nu(\cdot) \in \mathfrak{I}$ WITH THE NUMÉRAIRE PROPERTY.

Equivalently, with the property

$$B_i(\cdot) = \sum_{j=1}^{n} \int_0^{\cdot} \nu_j(t) \, \mathrm{d}\langle M_i, M_j \rangle(t) , \qquad i = 1, \cdots , n .$$

. This assumption does not proscribe (relative) arbitrages; it can coëxist very happily with them.

. But it DOES proscribe "Unbounded Increasing Profits": investment rules $\pi(\cdot)$ whose wealth process $V^\pi(\cdot)$ is non-decreasing and satisfies $\mathbb{P}(V^\pi(T) > 1) > 0$ for every $T \in (0, \infty)$.

. An entire theory for Finance can be built around the assumption that "NP Holds". But this is another story; see the survey/monograph by K. & Kardaras (2015).

4. CONCAVE AND LYAPUNOV FUNCTIONS

We consider a function $G : \mathscr{U} \to (0, \infty)$ defined and of class $\mathcal{C}^2$ in some neighborhood $\mathscr{U} \subset \mathbb{R}^n$ of the "lateral face" of the unit simplex; that is, of the set

$$\mathbf{\Delta}_+^n := \left\{ (x_1, \cdots, x_n)' \in (0, \infty)^n : \sum_{i=1}^n x_i = 1 \right\}.$$

THEOREM 1: *Suppose that NP holds, and that there exist a function* G *as above, and a constant* $\eta > 0$ *, such the process*

$$\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \int_0^T \Big(- D_{ij}^2 G\big(\mu(t)\big)\Big) \, \mathrm{d}\langle \mu_i, \mu_j \rangle(t) \; - \; \eta \, T \, , \quad 0 \le T < \infty \tag{1}$$

is non-decreasing. Then arbitrage is possible with respect to the market portfolio over any time horizon $[0, T]$ *of sufficiently long, finite duration, namely*

$$T \; > \; T_* \; := \; \frac{G\big(\mu(0)\big)}{\eta} \, .$$

Please note that the threshold $T_* = G(\mu(0))/\eta$ is at its lowest, when the initial market-weight configuration is at a site "where G attains its smallest value on $\mathbf{\Delta}_+^n$. " These are the most "propitious" sites from which relative arbitrage can be launched. We'll return to this point.

Proof: Let us consider any time-horizon $[0, T]$ of finite length $T > 0$, a real number, over which arbitrage with respect to the market portfolio is *NOT* possible.

We invoke now the existence of an investment rule with the NP property, and use a deep and celebrated result of DELBAEN & SHACHERMAYER (1994) to show that: There exists on $\mathcal{F}(T)$ an equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$, under which the market weights $\mu_1(\cdot \wedge T), \cdots, \mu_n(\cdot \wedge T)$ are (local) martingales.

- We invoke also the ITÔ decomposition

$$G(\mu(\cdot)) = G(\mu(0)) + \sum_{i=1}^{n} \int_0^{\cdot} D_i G(\mu(t))\, \mathrm{d}\mu_i(t) - \Gamma^G(\cdot)\,.$$

The assumption of (1) implies, in particular, that the continuous process

$$\Gamma^G(\cdot) := -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_0^{\cdot} D^2_{ij} G\big(\mu(t)\big)\, \mathrm{d}\langle \mu_i, \mu_j \rangle(t)$$

is nondecreasing. Therefore, $G(\mu(\cdot \wedge T))$ is a local $\mathbb{Q}-$supermartingale, thus also a true $\mathbb{Q}-$supermartingale, as it is positive.

. In other words, G plays the rôle of a LYAPUNOV function along the trajectories of the vector process $\mu(\cdot \wedge T)$ of market weights (a very liberal interpretation...).

On the other hand, for the bounded-from-below process

$$N^G(\cdot) := \sum_{i=1}^{n} \int_0^{\cdot} D_i G\big(\mu(t)\big)\, \mathrm{d}\mu_i(t)$$

$$= G(\mu(\cdot)) - G(\mu(0)) + \Gamma^G(\cdot) \geq -G(\mu(0)),$$

we note that $N^G(\cdot \wedge T)$ is then a $\mathbb{Q}-$local martingale, thus also a $\mathbb{Q}-$supermartingale, under this new equivalent measure.

In particular,

$$\mathbb{E}^{\mathbb{Q}}\big[N^G(T)\big] \leq 0$$

holds.

We claim that the event

$$\mathcal{A} := \left\{ -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_0^T D_{ij}^2 G\big(\mu(t)\big)\, \mathrm{d}\langle \mu_i, \mu_j \rangle(t) \;\leq\; G(\mu(0)) \right\}$$

of $\mathcal{F}(T)$, has then positive probability: $\mathbb{P}(A) > 0$.

. Indeed, it suffices to argue $\mathbb{Q}\,(\mathcal{A}) > 0$, as the two probability measures are equivalent on $\mathcal{F}(T)$. Now the set-inclusion

$$\{N^G(T) \leq 0\} \subseteq \mathcal{A} = \{\Gamma^G(T) \leq G(\mu(0))\},$$

a consequence of the identity from the previous slide

$$N^G(T) = G(\mu(T)) - G(\mu(0)) + \Gamma^G(T),$$

suggests that it suffices to show $\mathbb{Q}(N^G(T) \leq 0) > 0$.

But this follows easily by contradiction: For if we had $\mathbb{Q}\big(N^G(T) > 0\big) = 1$, then $\mathbb{E}^{\mathbb{Q}}\big[N^G(T)\big] > 0$ would hold, contradicting our earlier conclusion (end of previous slide)

$$\mathbb{E}^{\mathbb{Q}}\big[N^G(T)\big] \leq 0.$$

- We are now ready to complete the argument – once again, by contradiction.

Suppose there existed a real number

$$T > \frac{G\big(\mu(0)\big)}{\eta}$$

as in the statement of the Theorem, such that no arbitrage with respect to the market portfolio $\mu(\cdot)$ is possible over $[0, T]$.

From what we have just shown, the event

$$\mathcal{A} = \left\{ -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_0^T D_{ij}^2 G\big(\mu(t)\big)\, \mathrm{d}\langle \mu_i, \mu_j \rangle(t) \le G\big(\mu(0)\big) \right\}$$

from the previous slide would have then positive $\mathbb{P}-$probability. But in conjunction with the condition (1) of the Theorem, this would imply

$$\eta\, T \le G(\mu(0)),$$

contradicting our supposition right above.

DISCUSSION: A MORE RECENT RESULT

Bob FERNHOLZ showed recently (May 2015) that: If, for some function $G : \mathscr{U} \to (0, \infty)$ and real constant $\eta > 0$ as in Theorem 1 we have, not only the non-decrease of the process

$$\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_0^T \Big(- D_{ij}^2 G\big(\mu(t)\big)\Big) \, \mathrm{d}\langle \mu_i, \mu_j \rangle(t) \, - \, \eta \, T \,, \quad T \in (0, \infty), ;$$

but also, for some real constant $h \geq 0$ with $G(\mu(\cdot)) \geq h$, the additional "homogeneous-support-theorem"-like condition

$$\mathbb{P}\Big(G\big(\mu(\cdot)\big) \text{ visits } (h, h+\varepsilon) \text{ during } [0, T] \Big) > 0 \,, \quad \forall \ (T, \varepsilon) \in (0, \infty)^2 \,;$$

then the arbitrage of Theorem 1 can be realized over ANY time-horizon $[0, T]$ with $T \in (0, \infty)$.

IDEA: If you can arrive "quickly" with positive probability at some point in the state space which is "propitious" for relative arbitrage, then you already have realized short-term relative arbitrage.

5. FUNCTIONAL GENERATION OF PORTFOLIOS

OK, so we have found conditions under which the market can be outperformed. BUT HOW?

. BY MEANS OF WHAT INVESTMENT RULE?

. CAN THIS BE DONE BY A PORTFOLIO?

. A LONG-ONLY PORTFOLIO, PERHAPS?

• Here FERNHOLZ's argument comes into full force. Given a function $G : \mathscr{U} \to (0, \infty)$ defined and of class $\mathcal{C}^2$ as before, he introduces the portfolio generated by G, namely, for $i = 1, \cdots, n$,

$$\pi_i^G(t) := \mu_i(t) \left[D_i \log G\big(\mu(t)\big) + 1 - \sum_{j=1}^{n} \mu_j(t)\, D_j \log G\big(\mu(t)\big) \right]. \tag{2}$$

This portfolio is long-only, if G is convave (FERNHOLZ (2002)).

. In order to construct this portfolio on "day" t, we need the market weights $\mu_i(t)$, $i = 1, \cdots, n$ for that day – *and nothing else.*

We can solve now the linear SDE with random coëfficients

$$\mathrm{d}\left(\frac{V^{\pi^G}(t)}{V^{\mu}(t)}\right) = \left(\frac{V^{\pi^G}(t)}{V^{\mu}(t)}\right) \sum_{i=1}^{n} \frac{\pi_i^G(t)}{\mu_i(t)} \,\mathrm{d}\mu_i(t)$$

for the "discounted" wealth generated by this "Functionally-Generated" $\pi^G(\cdot)$, and obtain the so-called "Master Equation":

$$\log\left(\frac{V^{\pi^G}(T)}{V^{\mu}(T)}\right) = \log\left(\frac{G(\mu(T))}{G(\mu(0))}\right) + \sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^T \frac{\left(-D_{ij}^2 G(\mu(t))\right)}{2\,G(\mu(t))}\,\mathrm{d}\langle\mu_i,\mu_j\rangle(t)\,. \tag{3}$$

This is proved by yet another application of ITÔ's rule... .
Just a little more "determined" this time around.

Please note the appearance of only a LEBESGUE-STIELTJES (L-S) integral on the right-hand side of (3).

Stochastic integrals just do not appear; they have been excised. This is a *Pathwise Expression*.

IDEA: Now
(i) if the generating function G is bounded away from both zero and infinity; and
(ii) if the L-S integral on the right-hand side, which is non-decreasing, actually increases all the way to infinity;

. then the right-hand side of the equation (3) on the previous slide will be strictly positive for all T sufficiently large – and thus $\pi^G(\cdot)$ will be strong arbitrage relative to the market over all such time-horizons.

THEOREM 2: *Given a concave function* $G : \mathscr{U} \to (0, \infty)$ *defined and of class* $\mathcal{C}^2$ *as before, we impose the non-decrease assumption of (1) on the process*

$$\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_0^T \Big(- D_{ij}^2 G\big(\mu(t)\big)\Big) \,\mathrm{d}\langle \mu_i, \mu_j \rangle(t) \, - \, \eta \, T \,, \quad 0 \le T < \infty \tag{4}$$

for some real $\eta > 0$. *We consider also the "shifts"*

$$G^c(\cdot) \, := \, c + G(\cdot) \quad \text{for} \quad 0 \le c < \infty \,.$$

Then, for each real number T *as in Theorem 1, namely with*

$$T \, > \, T_* \, := \, \frac{G\big(\mu(0)\big)}{\eta} \,,$$

the long-only portfolio $\pi^{G^c}(\cdot) \in \mathfrak{P}_+$ *generated as in (2) by this "shifted" function, is STRONG arbitrage relative to the market on* $[0, T]$*, for all* $c > 0$ *sufficiently large.* □

Please note the "deafening silence" regarding the NP; plays no rôle here. Also, the fact that the range is exactly the same as before.

Proof: For $c \in (0, \infty)$, the "shifted" generating function $G^c(\cdot)$ is bounded away from both zero and infinity: For some real constant K, we have

$$0 < c \leq G^c(x) \leq c + K < \infty\,, \qquad \forall \;\; x \in \mathbf{\Delta}^n_+.$$

As a result of these bounds and of the assumption (4) from the previous slide, the process on the right-hand side of the analogue

$$\log\left(\frac{V(T\,;\pi^{G^c})}{V(T\,;\mu)}\right) = \log\left(\frac{c + G(\mu(T))}{c + G(\mu(0))}\right)$$
$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^T \frac{\big(-D^2_{ij}G(\mu(t))\big)}{2\,(c + G(\mu(t)))}\,\mathrm{d}\langle\mu_i,\mu_j\rangle(t)$$

of the Master Equation (3), is bounded from below by

$$\log\left(\frac{c}{c+G(\mu(0))}\right) + \frac{\eta\, T}{c+K} = \frac{\eta\, T - \Phi(c)}{c+K}$$

for each given $T \in (0,\infty)$, where

$$\Phi(c) := \big(c{+}K\big)\, \log\left(1{+}\frac{G\big(\mu(0)\big)}{c}\right) \longrightarrow G\big(\mu(0)\big)\,, \quad \text{as} \;\; c \to \infty\,.$$

Thus, for every real number $T > T_*$ we can select $c_* \in (0,\infty)$ large enough so that $\eta\, T > \Phi(c)$ holds for all $c > c_*$.

. Substituting above, this gives the strong arbitrage property of $\boldsymbol{\pi}^{G^c}(\cdot)$ relative to the market portfolio, as a consequence of the $\mathbb{P}-$a.e. comparison

$$\frac{V(T\,;\pi^{G^c})}{V(T\,;\mu)} \geq \exp\left(\frac{\eta T - \Phi(c)}{c+K}\right) > 1\,. \qquad \square$$

TO RECAPITULATE: This argument
(a) uses completely elementary methods,
(b) does not need to assume that the NP holds, and
(c) establishes STRONG arbitrage, not just arbitrage.

- I like to call the identity of (3), namely

$$\log\left(\frac{V^{\pi^G}(T)}{V^{\mu}(T)}\right) = \log\left(\frac{G(\mu(T))}{G(\mu(0))}\right) + \sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^T \frac{\big(-D^2_{ij}G(\mu(t))\big)}{2\,G(\mu(t))}\,\mathrm{d}\langle \mu_i,\mu_j\rangle(t)\,,$$

the *Master Equation*. It generates examples of strong arbitrage relative to the market portfolio "by the bushel".

I will present just one of them right below but, as you can sense, there is in principle no end to what you can come up with using this formula.

6. AN ENTROPIC GENERATING FUNCTION

Let us consider now the classical GIBBS *Entropy function*

$$H(x) := \sum_{i=1}^{n} x_i \log \big(1/x_i\big)\,, \qquad x \in \mathbf{\Delta}_+^n\,.$$

This concave function takes values in $(0, \log n]$, and decreases to zero as x tends to one of the unit vectors $\mathfrak{e}_1, \cdots, \mathfrak{e}_n$.

The corresponding process $\Gamma^H(\cdot)$ as in (1) is an important quantity in Stochastic Portfolio Theory, the *cumulative excess growth of the market portfolio*

$$\Gamma^H(\cdot) \equiv \frac{1}{2}\sum_{i=1}^{n}\int_0^{\cdot} \frac{\mathrm{d}\langle\mu_i\rangle(t)}{\mu_i(t)} = \frac{1}{2}\sum_{i=1}^{n}\int_0^{\cdot} \mu_i(t)\,\mathrm{d}\langle \log \mu_i \rangle(t)\,.$$

A trace-like quantity; measures the market's cumulative "relative variation" – stock-by-stock, then "averaged" according to market weight.

As generating function we shall take the shift of the entropy

$$H^c(x) := c + H(x),$$

by some real constant $c > 0$.

Via the recipe of (2), this function generates the *Entropy-Weighted portfolio*

$$\pi_i^{H^c}(t) = \frac{\mu_i(t)}{c + H(\mu(t))} \left[c + \log\left(\frac{1}{\mu_i(t)}\right)\right], \quad i = 1, \cdots, n,$$

all of whose weights are strictly positive.

If for some $\eta > 0$ we have the non-decrease of the process

$$\Gamma^H(T) - \eta\, T \;=\; \frac{1}{2} \sum_{i=1}^{n} \int_0^T \frac{\mathrm{d}\langle \mu_i \rangle(t)}{\mu_i(t)} - \eta\, T\,, \qquad 0 \le T < \infty\,, \tag{5}$$

then Theorems 1, 2 show that strong arbitrage is possible relative to the market portfolio over any time-horizon $[0, T]$ with

$$T \;>\; \frac{H(\mu(0))}{\eta}\,,$$

and that this strong arbitrage is implemented, for each such T, by the above portfolio $\boldsymbol{\pi}^{H^c}(\cdot)$ for $c > 0$ sufficiently large.

Here is a plot of the cumulative excess growth $\Gamma^H(\cdot)$ for the U.S. equities market over most of the twentieth century.

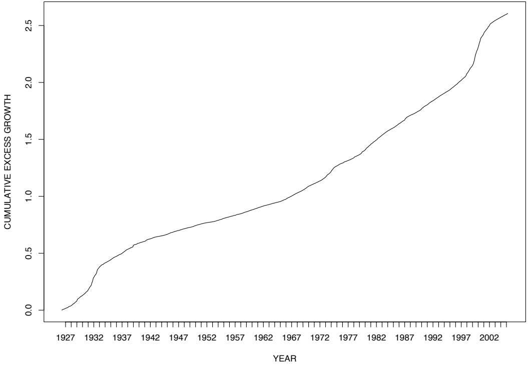

Figure 1 : Cumulative Excess Growth $\Gamma^H(\cdot)$ for the U.S., 1926 - 1999.

Can be estimated in what B. Dupire calls "tradable" manner, via (3):

$$\log\left(\frac{V(T\,;\pi^{H^c})}{V(T\,;\mu)}\right) = \log\left(\frac{c+H(\mu(T))}{c+H(\mu(0))}\right) + \int_0^T \frac{\mathrm{d}\Gamma^H(t)}{c+H(\mu(t))}\,.$$

7. AN INTRIGUING QUESTION

Suppose that the cumulative excess growth satisfies the non-decrease condition on the process

$$\frac{1}{2}\sum_{i=1}^{n}\int_0^T \frac{\mathrm{d}\langle\mu_i\rangle(t)}{\mu_i(t)} - \eta\, T\,, \qquad 0 \le T < \infty$$

of (5), for some real constant $\eta > 0$.

We just established that (strong) arbitrage exists then with respect to the market portfolio over sufficiently long time-horizons $[0, T]$, *namely, with*

$$T \;>\; T_* \;:=\; \frac{H\big(\mu(0)\big)}{\eta}\;.$$

Does arbitrage exist also over arbitrary time-horizons? □

This question was posed by FERNHOLZ & K. (2005).
It stayed open for 10+ years.

Needless perhaps to say, this same question can be asked regarding the non-decrease property of

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^T \Big(- D^2_{ij} G\big(\mu(t)\big)\Big)\, \mathrm{d}\langle \mu_i, \mu_j\rangle(t) \,-\, \eta\, T\,, \quad \forall \;\; 0 \le T < \infty$$

for ANY generating function as in Theorems 1 and 2.

. But in the spirit of Mike HARRISON's exhortation, let us "concentrate on the simplest problem we cannot do".

In a few cases of considerable independent interest, the answer to the this question is known and is affirmative.

I will present them briefly below.

• In another very recent development, however, Johannes RUF constructed a beautiful and very interesting example, which shows that the answer CANNOT BE AFFIRMATIVE IN GENERAL.

CASE I: Strictly Non-Degenerate Covariation Structure, Coupled with Strong Diversity.

Suppose there exists a real $\varepsilon > 0$, such that the process

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \int_0^{\cdot} \xi_i(t)\, \xi_j(t)\, \mathrm{d}\langle M_i, M_j \rangle(t) \,-\, \varepsilon \int_0^{\cdot} \|\xi(t)\|^2 \, \mathrm{d}t$$

is non-decreasing for every predictable, locally square-integrable vector process $\xi(\cdot)$. Impose also for some $\delta \in (0,1)$ the strong diversity

$$\max_{1 \le i \le n} \mu_i(t) \le 1 - \delta\,, \qquad 0 \le t < \infty\,.$$

FERNHOLZ, K. & KARDARAS (2005) show that these two conditions
(i) are compatible with each other;
(ii) imply the condition of (5) with $\eta = (\delta\, \varepsilon)/2$, i.e., the non-decrease of

$$\frac{1}{2} \sum_{i=1}^{n} \int_0^T \frac{\mathrm{d}\langle \mu_i \rangle(t)}{\mu_i(t)} \,-\, \eta\, T\,, \qquad 0 \le T < \infty\,;$$

(iii) lead to strong arbitrage relative to the market over ANY horizon $[0, T]$ with $T \in (0, \infty)$.

Discussion: Diversity helps identify a portfolio, which takes negative positions (long in stock $i = 1$, short in the market) and *underperforms* the market portfolio.

. Then we use this portfolio as a "seed", to create long-only portfolios that *outperform* the market – swamping the short positions in a sea of market portfolio, while retaining the essential portfolio characteristics that lead to the outperformance.

Osterrieder & Rheinländer (2006) have a different, abstract approach to a similar result: Existence – though not identification – of arbitrage relative to cash under conditions of diversity. They use an auxiliary probability (Föllmer-) measure $\mathbb{Q}$, under which capitalizations are martingales and which satisfies $\mathbb{P} \ll \mathbb{Q}$, but not the other way round.

. In the same spirit are the results in Ruf & Runggaldier (2013), Chau & Tankov (2013).

CASE II: Itô Processes with VS Covariation Structure.
Let us consider now the case (Fernholz & K. (2005), Picková (2014))

$$B_i(\cdot) = \int_0^{\cdot} \beta_i(t)\,\mathrm{d}t\,, \qquad M_i(\cdot) = \sum_{k=1}^{n} \int_0^{\cdot} \sigma_{ik}(t)\,\mathrm{d}W_k(t)$$

with $W_1, \cdots, W_n$ independent Brownian motions and

$$\sigma_{ik}(t) = \frac{C\,\delta_{i,k}}{\sqrt{\mu_i(t)}}\,, \qquad 1 \le i, k \le n \tag{6}$$

of the so-called *Volatility-Stabilized* type, for some constant $C > 0$. In this case the individual stocks' cumulative relative variations are

$$\langle \log \mu_i \rangle(T) = C^2 \int_0^T \Big(\frac{1}{\mu_i(t)} - 1 \Big)\,\mathrm{d}t\,, \qquad i = 1, \cdots, n\,,$$

therefore

$$\int_0^T \mu_i(t)\,\mathrm{d}\langle \log \mu_i \rangle(t) \;=\; C^2 \int_0^T \Big(1 - \mu_i(t)\Big)\,\mathrm{d}t\,.$$

Examples of this type are typically NOT diverse, meaning that this last integrand gets on occasion to be very close to zero for any given individual stock $i = 1, \cdots, n$.

However, the SUM of all these quantities, to wit, the market's cumulative excess growth, not only dominates a straight line with positive slope: IT IS a straight line with positive slope:

$$\Gamma^H(T) \;=\; \frac{1}{2}\sum_{i=1}^{n}\int_0^T \mu_i(t)\,\mathrm{d}\langle \log \mu_i \rangle(t) \;=\; \frac{C^2}{2}\big(n-1\big)\,T \;\equiv\; \eta\, T$$

(whence the appellation "stabilization by volatility").

- For considerable generalizations of this kind of model, Adrian BANNER and Daniel FERNHOLZ (2008) show that strong arbitrage relative to the market is possible over ANY horizon $T \in (0, \infty)$.
- Also very recent work by Christa CUCHIERO et al. (2015).

Another intricate construction... . The portfolio that implements this strong arbitrage, is strictly speaking, NOT Functionally Generated.

It employs, rather, *a concatenation of two functionally generated portfolios:* starting with the portfolio $\pi_i^G(\cdot)$ generated as in (2) by the function

$$G(x_1, \cdots, x_n) = \sum_{i=1}^{n} f(x_i)\,, \qquad \text{with}$$

$$f(y) := \int_{\log(1/y)}^{\infty} e^{-y}\, r^c\, \mathrm{d}r\,, \quad 0 < y \leq 1 \qquad \text{and} \qquad f(0) := 0$$

for an appropriate real constant $c > 1$, this portfolio switches at an appropriate stopping time $\mathscr{T}$ to the market $\mu(\cdot)$.

. The switch is made in order to "lock in" the outperformance that creates the strong arbitrage; and the rôle of the condition (6) is to guarantee that the time $\boldsymbol{\tau}$ does occur with certainty before the end of the horizon, that is, $\mathbb{P}(\mathscr{T} \leq T) = 1$.

8. THE RECENT EXAMPLE OF JOHANNES RUF

THEOREM (Johannes RUF, 2015): *There exists a time-homogeneous* ITÔ *diffusion* $\mu(\cdot)$ *with values in* $\mathbf{\Delta}_+^n$ *and* LIPSCHITZ*–continuous dispersion matrix, such that the cumulative excess growth process*

$$\Gamma^H(\cdot) := \frac{1}{2}\sum_{i=1}^{n}\int_0^{\cdot}\frac{\mathrm{d}\langle\mu_i\rangle(t)}{\mu_i(t)}$$

is strictly increasing, and its slope uniformly bounded from below.

But with respect to which arbitrage over time-horizons $[0, T]$ *of sufficiently short duration is NOT possible.* □

Yet another intricate construction... .

IDEA OF CONSTRUCTION: Suppose that the probability measure $\mathbb{Q}$ solves the martingale problem for a diffusion with state-space

$$\overline{\mathbf{\Delta}}_\flat^3 := \left\{ (x_1, x_2)' \in [0,1]^2 : \sum_{i=1}^{2} x_i \leq 1 \right\} \subset \mathbb{R}^2,$$

such that (properties of a FÖLLMER measure):

- Each component $\mu_1(\cdot)$ and $\mu_2(\cdot)$ is a $\mathbb{Q}-$martingale; and thus so is $\mu_3(\cdot) := 1 - \mu_1(\cdot) - \mu_2(\cdot)$.

- The $\mathbb{Q}-$probability of the event "$(\mu_1(\cdot), \mu_2(\cdot))'$ does NOT hit the boundary of $\mathbf{\Delta}_\flat^3$ before time T" is strictly positive, for ANY given real number $T > 0$.

- There is some real number $\delta > 0$ such that, $\mathbb{Q}-$a.s., the process $(\mu_1(\cdot), \mu_2(\cdot))'$ does NOT hit the boundary of $\mathbf{\Delta}_\flat^3$ before time δ. And

- $\Gamma^H(t) - \frac{1}{2}\, t\,, \quad 0 \leq t < \infty \quad$ is non-decreasing.

We take now an arbitrary but fixed real number $T > 0$ and construct a NEW probability measure $\mathbb{P}$, by conditioning $\mathbb{Q}$ on the event
"$(\mu_1(\cdot), \mu_2(\cdot))'$ does NOT hit the boundary of $\mathbf{\Delta}^3_\flat$ before time T ".

The covariance structure is not affected by this conditioning, so
$\Gamma^H(t) - \frac{1}{2}\, t\,, \quad 0 \le t < \infty$ is non-decreasing.
continues to hold a.e. under this new measure as well;
and the two measures $\mathbb{P}$ and $\mathbb{Q}$ agree on $\mathcal{F}(\delta)$.

This then yields the result: NO ARBITRAGE IS POSSIBLE UNDER THE NEW MEASURE $\mathbb{P}$ ON $[0, T]$, FOR $0 < T \le \delta$.
(because no arbitrage is possible under $\mathbb{Q}$, and $\mathbb{P} \sim \mathbb{Q}$ on $\mathcal{F}(T)$).

. The construction is inspired by a very interesting paper of Dan Stroock (1971), and by results in Dan Fernholz & K. (2010).

- It raises yet another OPEN QUESTION:

If, as we have shown, arbitrage is possible in this example over time-horizons $[0, T]$ with

$$T > 2\,H(\mu(0))$$

but impossible over time-horizons $[0, T]$ with

$$0 < T \leq \delta\,,$$

then what is the threshold

$$T_* := \inf\Big\{\, T \in (0, \infty) \;:\; \exists \;\; \pi(\cdot) \in \mathfrak{I} \;\text{ s.t. }\; \mathbb{P}\big(V^{\pi}(T) \geq V^{\mu}(T)\big) = 1 \quad \text{and} \quad \mathbb{P}\big(V^{\pi}(T) > V^{\mu}(T)\big) > 0 \,\Big\}\,?$$

How about replacing $\mathfrak{I}$ in this definition by $\mathfrak{P}$? or by $\mathfrak{P}_+$?
What if one insists on strong relative arbitrage ?

THE DISPERSION MATRIX: We start by defining

$$c \,=\, 1 - \frac{1}{\sqrt{2}} \;, \qquad r \,=\, \frac{c}{3} \,, \qquad \varrho \,=\, 2\,r$$

and

$$R(x) \;:=\; \sqrt{(x_1 - c)^2 + (x_2 - c)^2} \;.$$

With this notation, we define the entries of the dispersion-matrix-valued function

$$\begin{aligned} \sigma_{1,1}(x) &= 1 - \frac{(x_1 - c)^2}{r^2 \vee (R^2(x) \wedge \varrho^2)} \\ \sigma_{2,2}(x) &= 1 - \frac{(x_2 - c)^2}{r^2 \vee (R^2(x) \wedge \varrho^2)} \\ \sigma_{1,2}(x) = \sigma_{2,1}(x) &= \frac{-\,(x_1 - c)(x_2 - c)}{r^2 \vee (R^2(x) \wedge \varrho^2)} \;. \end{aligned}$$

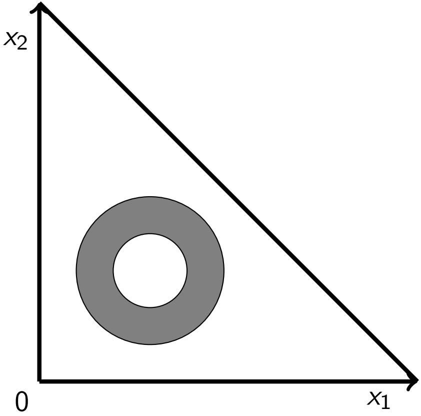

Figure 2 : $\Delta^3_\flat$ as $(x_1, x_2)-$plane. The center of the two circles is the point $(c, c) = (1 - 1/\sqrt{2}, 1 - 1/\sqrt{2})$ and their radii are r and ϱ. On the "lifesaver" (shaded area), the diffusion $(\mu_1(\cdot), \mu_2(\cdot))'$ only moves "outwards."

DYNAMICS ON THE LIFESAVER: We start from $x = (x_1, x_2)'$ on the lifesaver (shaded area) $r \leq R(x) \leq \varrho$. Then

$$\sigma_{1,1}(x) = 1 - \frac{(x_1 - c)^2}{R^2(x)}, \qquad \sigma_{2,2}(x) = 1 - \frac{(x_2 - c)^2}{R^2(x)}$$

$$\sigma_{1,2}(x) = \sigma_{2,1}(x) = -\frac{(x_1 - c)(x_2 - c)}{R^2(x)}.$$

At this point, ITÔ's formula yields the totally deterministic dynamics for the radial part of the diffusive motion

$$\mathrm{d}R(\mu(t)) = \mathrm{d}t$$

on the lifesaver, as well as

$$\mathrm{d}\Gamma^H(t) \geq \frac{1}{2}\,\mathrm{d}t$$

. Thus, if started at a location x on the lifesaver, the diffusion needs time to hit the boundary of $\mathbf{\Delta}_\flat^3$ of at least

$$\delta = \varrho^2 - R^2(x) > 0. \qquad \square$$

9. SOURCES FOR THIS TALK

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THANK YOU FOR YOUR ATTENTION