Martingale information of the implied volatility smile

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Based on joint works with S. de Marco, C. Hillairet and M. Keller-Ressel.

Statement of the problem

Consider an (arbitrage-free) implied volatility smile for a given maturity.
There exists an underlying stock price process S that generates it.

We wish to answer the following two questions:

(I) Can S describe a defaultable asset?

(II) Is S a true martingale?

ANSWER:

Statement of the problem

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ANSWER: this can **ONLY** be detected in

(I) the left wing of the smile (small strikes).

(II) the right wing of the smile (large strikes).

PART I: MASS AT THE ORIGIN

Joint work with C. Hillairet and S. De Marco

The mass at zero case: the left wing literature

Theorem (Roger Lee, 2004)

Let S be a non-negative martingale and denote $q^* := \sup\left\{q \geq 0 : \mathbb{E}(S_T^{-q}) < \infty\right\}$. Then the left wing of the implied volatility smile behaves as

$$\limsup_{x \downarrow -\infty} I^2(x) T/|x| = \psi(q^*) \in [0, 2],$$

where $\psi(z) \equiv 2 - 4\left(\sqrt{z(z+1)} - z\right)$.

Remark: The $\limsup$ can sometimes be turned into a genuine limit (Benaim-Friz).

Theorem (Archil Gulisashvili, 2010)

Let S be a non-negative martingale, then

$$I(x) = \sqrt{\frac{|x|}{T}\psi\left(\frac{\log \mathrm{P}(x)}{x} - 1\right)} + \mathcal{O}(\cdots), \quad \text{as } x \downarrow -\infty.$$

Note: if $\mathbb{P}(S_T = 0) > 0$, then $q^* = 0$ and the left slope is equal to its maximal value 2. Gulisashvili's proof does not hold in that case.

Main result

Theorem (de Marco, Hillairet, Jacquier, 2014), (Gulisashvili, 2015)

Let S be a non-negative martingale, and denote $\mathrm{p} := \mathbb{P}(S_T = 0)$ and $\mathrm{q} := \mathcal{N}^{-1}(\mathrm{p})$.

- If $\mathrm{p} = 0$, then $\lim_{x\downarrow-\infty}\left(I(x) - \sqrt{2|x|/T}\right) = -\infty$;
- If $\mathrm{p} > 0$, then, as $x \downarrow -\infty$,

$$I(x) = \sqrt{\frac{2|x|}{T}} + \frac{\mathrm{q}}{\sqrt{T}} + \frac{\mathrm{q}^2}{2\sqrt{2T|x|}} + \sqrt{2\pi}\mathrm{e}^{\mathrm{q}^2/2}\int_{-\infty}^{x}[\mathbb{P}(S_T \leq S_0\mathrm{e}^y) - \mathrm{p}]\mathrm{d}y + \mathcal{O}(\cdots)$$

Remarks:

- Phase transition at $\mathrm{p} = 1/2$;
- Gulisashvili's formula actually holds, but with $\mathcal{O}(1)$.

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Remarks:

- Phase transition at $\mathrm{p} = 1/2$;
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Corollary: assume $\mathrm{p} > 0$.

- if $\mathbb{P}(S_T \le S_0 \mathrm{e}^x) - \mathrm{p} = \mathcal{O}(|x|^{-1/2})$, then $I(x) = \sqrt{\frac{2|x|}{T}} + \frac{\mathrm{q}}{\sqrt{T}} + \mathcal{O}(|x|^{-1/2})$;
- if $\mathbb{P}(S_T \le S_0 \mathrm{e}^x) - \mathrm{p} = \mathcal{O}(\mathrm{e}^{\varepsilon x})$, then $I(x) = \sqrt{\frac{2|x|}{T}} + \frac{\mathrm{q}}{\sqrt{T}} + \frac{\mathrm{q}^2}{2\sqrt{2T|x|}} + \Phi(x)$, with $\limsup \sqrt{2T|x|}\Phi(x) \le 1$;

Comparison with stochastic volatility models

$$I(x) = \sqrt{\frac{2|x|}{T}} + \frac{\mathrm{q}}{\sqrt{T}} + \frac{\mathrm{q}^2}{2\sqrt{2T|x|}} + \frac{\sqrt{2\pi}\mathrm{e}^{\mathrm{q}^2/2}}{S_0\mathrm{e}^x}\int_0^{S_0\mathrm{e}^x}[F(y) - F(0)]\mathrm{d}y + \mathcal{O}(\cdots)$$

- **Stein-Stein model:** $\mathrm{d}S_t = S_t|\sigma_t|\mathrm{d}W_t$ and $\mathrm{d}\sigma_t = \kappa(\theta - \sigma_t)\mathrm{d}t + \xi\mathrm{d}W_t^\perp$;

$$I(x,T) = \sqrt{\frac{\gamma_1|x|}{T}} + \frac{\gamma_2}{\sqrt{T}} + \mathcal{O}(|x|^{-1/2}), \quad \text{as } x \downarrow -\infty,$$

  with $\gamma_1 \in (0,2)$.
- **Heston:** $\mathrm{d}\sigma_t^2 = \kappa(\theta - \sigma_t^2)\mathrm{d}t + \xi\sigma_t\mathrm{d}B_t$;

$$I(x,T) = \sqrt{\frac{\gamma_1|x|}{T}} + \frac{\gamma_2}{\sqrt{T}} + \frac{\gamma_3\log(|x|)}{\sqrt{|x|}} + \mathcal{O}(|x|^{-1/2}), \quad \text{as } x \downarrow -\infty,$$

  with $\gamma_1 \in (0,2)$.
- **Uncorrelated Hull-White:** $\mathrm{d}\sigma_t = \sigma_t(\nu\mathrm{d}t + \xi\mathrm{d}W_t^\perp)$;

$$I(x) = \sqrt{\frac{2|x|}{T}} - \frac{\log|x| + \log\log|x|}{2T\xi\sqrt{T}} + \mathcal{O}(1), \quad \text{as } x \downarrow -\infty.$$

Example. The CEV model: $\mathrm{d}S_t = \sigma S_t^{1+\beta}\mathrm{d}W_t$

S is a true (non-negative) martingale if and only if $\beta \le 0$; When $\beta \in [-1/2, 0)$,

$$\mathbb{P}(S_T \in \mathrm{d}s) = -\frac{s_0^{1/2} s^{-2\beta-3/2}}{\sigma^2 \beta T} \exp\left(-\frac{s_0^{-2\beta} + s^{-2\beta}}{2\sigma^2\beta^2 T}\right) I_{-\nu}\left(\frac{s_0^{-\beta} s^{-\beta}}{\sigma^2\beta^2 T}\right)\mathrm{d}s,$$

$$\mathbb{P}(S_T = 0) = 1 - \Gamma\left(-\nu, \frac{s_0^{-2\beta}}{2\sigma^2\beta^2 T}\right).$$

As $s \downarrow 0$, one obtains $\mathbb{P}(S_T \in \mathrm{d}s) \sim const \times s^{2|\beta|-1}\mathrm{d}s$, which explodes at the origin when $\beta \in (-1/2, 0)$, and tends to a constant when $\beta = -1/2$.

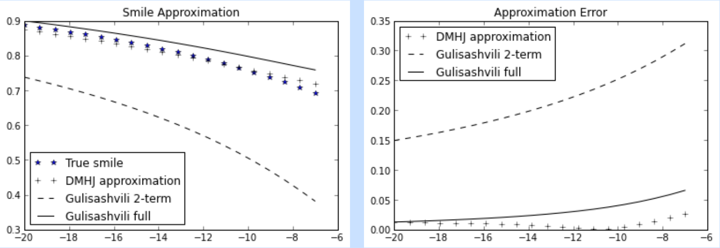

Figure: $s_0 = 0.1$, $T = 6.13$, $\beta = -0.4$, $\sigma = 0.1$, which implies $\mathrm{p} \approx 0.00059$.

Example. The SABR model

(Joint work with A. Gulisashvili and B. Horvath).

$$\mathrm{d}S_t = Y_t S_t^{\beta} \mathrm{d}W_t, \qquad \mathrm{d}Y_t = \nu Y_t \mathrm{d}Z_t, \qquad \mathrm{d}\langle W, Z\rangle_t = \rho \mathrm{d}t,$$

with $\beta \in (0,1)$, $\nu > 0$, $\rho \in (-1,1)$. One can show that

$$\mathbb{P}(S_t = 0) = \int_0^{\infty} \mathbb{P}\left(\widetilde{S}_r = 0\right) \mathbb{P}\left(\int_0^t Y_s^2 \mathrm{d}s \in \mathrm{d}r\right) \mathrm{d}r,$$

where $\widetilde{S}$ is the unique strong solution to $\mathrm{d}\widetilde{S}_t = \widetilde{S}_t^{\beta} \mathrm{d}W_t$.

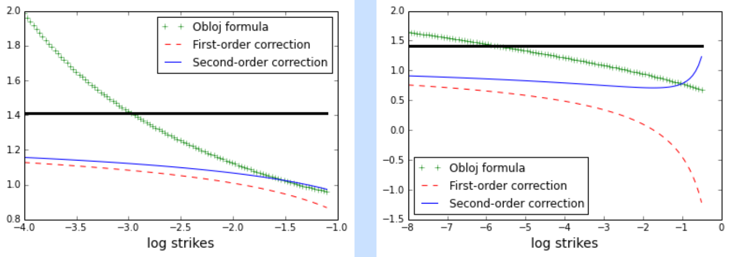

Figure: Parameters: $(\nu, \beta, \rho, S_0, Y_0, T) = (0.3, 0, 0, 0.35, 0.05, 10)$ for the left plot, and $(\nu, \beta, \rho, S_0, Y_0, T) = (0.6, 0.6, 0, 0.08, 0.015, 10)$ for the right graph. Obłój's expansion violates this upper bound. Large-time mass: 28.3% (left), 3.1% (right).

Financial implications: smile symmetries

- Absence of symmetry: If $\mathrm{p} = 0$, then the smile cannot be symmetric.
- Variance swap prices are infinite: using

$$\frac{1}{2}\mathbb{E}\left(\langle \log(S)\rangle_T\right) = \int_0^{S_0} \frac{\mathrm{P}(K)}{K^2}\mathrm{d}K + \int_{S_0}^{\infty} \frac{\mathrm{C}(K)}{K^2}\mathrm{d}K$$

and $\lim_{K\downarrow 0} \frac{\mathrm{P}(K)}{K} = \mathbb{P}(S_T = 0)$.

- Gamma swap prices are not impacted (to that extent) by potential default.

PART II: STRICT LOCAL MARTINGALES AND DUALITY

Joint work with M. Keller-Ressel

Detecting strict local martingales

Consider a one-dimensional diffusion

$$\mathrm{d}S_t = \sigma(S_t)\mathrm{d}W_t, \qquad S_0 > 0.$$

Proposition (Engelbert et al., Mijatović-Urusov...)

(i) $S_t > 0$ almost surely for all $t > 0$ if and only if $\int_0^1 \frac{z^2 \mathrm{d}z}{\sigma^2(z)} = \infty$;

(ii) S is a strict local martingale if and only if $\int_1^\infty \frac{z^2 \mathrm{d}z}{\sigma^2(z)} < \infty$.

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Goal here: develop an alternative test, based on the observed implied volatility smile.

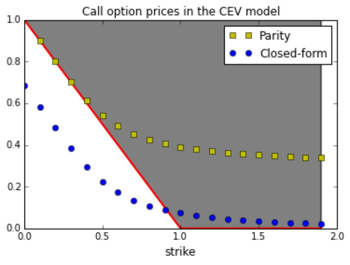
Strict local martingales, option prices, implied volatility

Consider the strict local martingale (CEV) process $\mathrm{d}S_t = S_t^2 \mathrm{d}W_t$.

$$\mathrm{C}_S(K) := \mathbb{E}(S_T - K)_+ = S_0\Big(\mathcal{N}(\kappa - \delta) - \mathcal{N}(-\delta) + \mathcal{N}(\delta) - \mathcal{N}(\kappa + \delta)\Big) - K\left(\mathcal{N}(\kappa + \delta) - \mathcal{N}(\delta - \kappa) + \frac{n(\kappa + \delta) - n(\kappa - \delta)}{\delta}\right),$$

$$\mathrm{P}_S(K) := \mathbb{E}(K - S_T)_+ = S_0 K \sqrt{T}\left(\zeta_+ \mathcal{N}(\zeta_+) + n(\zeta_+) - \zeta_- \mathcal{N}(\zeta_-) - n(\zeta_-)\right),$$

where $\delta := \frac{1}{S_0\sqrt{T}}$, $\quad \kappa := \frac{1}{K\sqrt{T}}$, $\quad \zeta_\pm := \frac{1}{\sqrt{T}}\left(\pm\frac{1}{S_0} - \frac{1}{K}\right)$.



Set-up: $(S, \mathbb{Q})$: market model without arbitrage opportunities (NFLVR). $S_0 = 1$.

Notations: $K = \mathrm{e}^x$.

Consequences and remarks:

- Martingale defect: $\mathrm{m} := 1 - \mathbb{E}^{\mathbb{Q}}(S_T) > 0$.
- Put-Call parity fails, in particular $\mathrm{C}_S(x) - \mathrm{P}_S(x) = 1 - \mathrm{e}^x - \mathrm{m}$
- Bounds for C_S: $(1 - \mathrm{e}^x - \mathrm{m})_+ \le \mathrm{C}_S(x) \le 1 - \mathrm{m}$.

Link with no-arbitrage theory:

- Consider a Call option valued at $(1 - \mathrm{e}^x - \mathrm{m})_+$. Choose $x \le \log(1 - \mathrm{m})$, and construct the portfolio Long Call, short Stock and $\mathrm{m} + \mathrm{e}^x$ cash.
  Payoff: $\mathrm{m} + (\mathrm{e}^x - S_T)_+ > 0$.
  Resolution of the 'paradox': the short position in S implies that the portfolio is unbounded from below, and hence not admissible in the sense of NFLVR.

Pricing with collateral

Theorem: Cox-Hobson (2005)—simplified

Let G be a positive convex function satisfying $\limsup_{s\uparrow\infty} s^{-1}G(s) = \alpha$ and $G(s) \leq (s-\mathrm{e}^x)_+$. The fair price of a European Call option is $\mathbb{E}^{\mathbb{Q}}(S_T - \mathrm{e}^x)_+ + \alpha\mathrm{m} =: \mathrm{C}_S^\alpha(x)$.

Note: α represents the amount of collateral the option seller needs to post.
Furthermore, $\lim_{x\uparrow\infty} C_S^\alpha(x) = \alpha\mathrm{m}$ and $\lim_{x\uparrow\infty}(P_S^\alpha(x) - \mathrm{e}^x) = \mathrm{m} - 1$.

Theorem: Madan-Yor (2006)—fully collateralised price $\alpha = 1$

For any sequence of stopping times $(\tau_n)_{n\geq 0}$,

$$\mathrm{C}_S^{\mathrm{MY}}(x) := \lim_{n\uparrow\infty} \mathbb{E}^{\mathbb{Q}}\left(S_{T\wedge\tau_n} - \mathrm{e}^x\right)_+ = (1-\mathrm{e}^x)_+ + \frac{1}{2}\mathbb{E}^{\mathbb{Q}}(\mathcal{L}_T^x) = \mathrm{C}_S(x) + \pi_T^S,$$

where $(\mathcal{L}_t^x)_{t\geq 0}$ denotes the local time of S at level e^x, and where the penalty term reads

$$\pi_T^S = \lim_{z\uparrow\infty} z\mathbb{Q}\left(\sup_{0\leq u\leq T} S_u \geq z\right) = 1 - \mathbb{E}^{\mathbb{Q}}(S_T) = \mathrm{m}.$$

A first duality result

Definition

$\mathbb{Q}$, $\mathbb{P}$: probability measures and T: fixed time horizon.
S: strictly positive local $\mathbb{Q}$-martingale; M: non-negative true $\mathbb{P}$-martingale.
$\tau := \inf\{t > 0 : M_t = 0\} > 0, \mathbb{P}$-almost surely.
We say that the pair $(S, \mathbb{Q})$ is **in duality** to $(M, \mathbb{P})$ if $\mathbb{Q} \ll_{\mathcal{F}_T} \mathbb{P}$ with

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_T} = M_T \qquad \text{and} \qquad S_t = \frac{1}{M_t} \quad \mathbb{P}\text{-a.s. on } \{t < \tau \wedge T\}.$$

Duality result

Let $(M, \mathbb{P})$ and $(S, \mathbb{Q})$ be market models in duality, then

$$\mathrm{m} = 1 - \mathbb{E}^{\mathbb{Q}}(S_T) = 1 - \mathbb{E}^{\mathbb{P}}(\mathbb{1}_{\{\tau > T\}}) = \mathbb{P}(\tau \le T) = \mathbb{P}(M_T = 0).$$

Furthermore $\mathrm{m} > 0$ if and only if $\mathbb{Q}$ is not equivalent to $\mathbb{P}$ on $\mathcal{F}_T$.

Example: Let $\mathrm{d}M = \sigma(M)\mathrm{d}W^{\mathbb{P}}$, and $\widetilde{\sigma}(y) \equiv y^2\sigma(1/y)$. Then $\mathrm{d}S = \widetilde{\sigma}(S)\mathrm{d}W^{\mathbb{Q}}$ and

$$\int_0^1 \frac{y\mathrm{d}y}{\widetilde{\sigma}^2(y)} = \int_1^\infty \frac{z\mathrm{d}z}{\sigma^2(z)} \qquad \text{and} \qquad \int_1^\infty \frac{y\mathrm{d}y}{\widetilde{\sigma}^2(y)} = \int_0^1 \frac{z\mathrm{d}z}{\sigma^2(z)}.$$

Put-Call Duality

Proposition: Call-Put relationships

Define $\mathrm{P}_M(x) := \mathbb{E}^{\mathbb{P}}(\mathrm{e}^x - M_T)_+$ and $\mathrm{C}_M(x) := \mathbb{E}^{\mathbb{P}}(M_T - \mathrm{e}^x)_+$. Let $(M, \mathbb{P})$ and $(S, \mathbb{Q})$ be market models in duality. For any $\alpha \in [0, 1]$,

$$\mathrm{C}_S^{\alpha}(x) = \mathrm{e}^x \mathrm{P}_M(-x) + (\alpha - 1)\mathrm{m} \quad \text{and} \quad \mathrm{P}_S(x) = \mathrm{e}^x \mathrm{C}_M(-x).$$

Implied volatility: existence

I_S^P: implied volatility corresponding to the Put price on S (under $\mathbb{Q}$).

I_S^α: implied volatility corresponding to the α-collateralised Call price on S (under $\mathbb{Q}$).

Theorem: Existence of implied volatilities

- I_S^P is well defined on $\mathbb{R}$.
- $I_S^1 \equiv I_S^P$.
- For $\alpha \in [0, 1)$, there exists x_α^* such that I_S^α is not well defined on $(-\infty, x_\alpha^*)$.
- Whenever $I_S^\alpha(x)$ is well defined, $I_S^\alpha(x) < I_S^P(x)$.

Implied volatility: asymptotic behaviour

Theorem: Asymptotic behaviour of the smile (not using duality)

Let S be a strict $\mathbb{Q}$-local martingale with $\mathrm{m} \in (0,1)$ and $\alpha \in (0,1)$. As $x \uparrow \infty$,

$$I_S^{\alpha}(x) = \sqrt{\frac{2x}{T}} + \frac{\mathcal{N}^{-1}(\alpha \mathrm{m})}{\sqrt{T}} + o(1), \quad \text{and} \quad I_S^{P}(x) = \sqrt{\frac{2x}{T}} + \frac{\mathcal{N}^{-1}(\mathrm{m})}{\sqrt{T}} + o(1).$$

Corollary

- If $\alpha = 0$, then $\lim_{x\uparrow\infty} \left(I_S^0(x) - \sqrt{\frac{2x}{T}} \right) = -\infty$;
- if $\mathrm{m} = 0$, for $\alpha \in [0,1]$, $\lim_{x\uparrow\infty} \left(I_S^p(x) - \sqrt{\frac{2x}{T}} \right) = \lim_{x\uparrow\infty} \left(I_S^{\alpha}(x) - \sqrt{\frac{2x}{T}} \right) = -\infty$.

Link with Benaim-Friz-Lee: $\mathrm{d}S = S^2 \mathrm{d}W$. $p^* := \sup\{p \geq 0 : \mathbb{E}(S_T^{1+p}) < \infty\} = 3$, so that $\limsup I(x)^2 T/x = \psi(p^*) < 2$. Benaim-Friz-Lee does not hold in the strict local martingale case.

Duality and implied volatility symmetry

Theorem: Smile symmetry

Let S be a positive strict local $\mathbb{Q}$-martingale in duality with the true $\mathbb{P}$-martingale M with mass at zero. Then, for all $x \in \mathbb{R}$, $I_S^p(x) = I_S^1(x) = I_M(-x)$. Furthermore, for any $\alpha \in (0,1)$, I_S^α cannot be symmetric.

Theorem: Smile asymptotics refined

S: positive strict local $\mathbb{Q}$-martingale. $G(x) := \mathbb{E}^{\mathbb{Q}}(S_T \mathbb{1}_{\{S_T \geq \mathrm{e}^x\}})$ and $\mathfrak{n} := \mathcal{N}^{-1}(\mathrm{m})$.

(i) If $G(x) = o(x^{-1/2})$ as x tends to ∞, then, with $0 \leq \limsup_{x\uparrow\infty} \Psi(x) \leq 1$

$$I_S^p(x) = I_S^1(x) = \sqrt{\frac{2x}{T}} + \frac{\mathfrak{n}}{\sqrt{T}} + \frac{\mathfrak{n}^2}{2\sqrt{2Tx}} + \frac{\exp(\frac{1}{2}\mathfrak{n}^2)}{\sqrt{2Tx}}\Psi(x), \qquad \text{as } x \uparrow \infty.$$

(ii) If $G(x) = \mathcal{O}(\mathrm{e}^{-\varepsilon x})$ as x tends to ∞, for some $\varepsilon > 0$, then

$$I_S^p(x) = I_S^1(x) = \sqrt{\frac{2x}{T}} + \frac{\mathfrak{n}}{\sqrt{T}} + \frac{\mathfrak{n}^2}{2\sqrt{2Tx}} + \Phi(x), \qquad \text{as } x \uparrow \infty,$$

where $\limsup_{x\uparrow\infty} \sqrt{2Tx}|\Phi(x)| \leq 1$.

Numerical example: $\mathrm{d}S = S^{1+\beta}\mathrm{d}W$

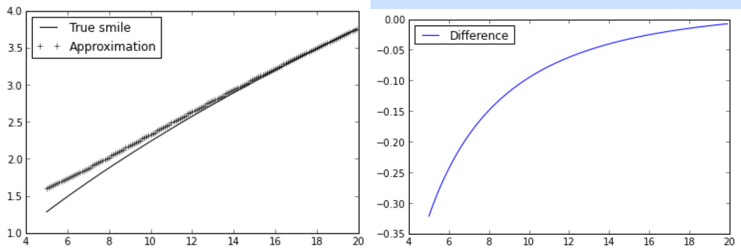


Figure: $(S_0, \beta, \sigma, T) = (1, 2.4, 10\%, 1)$. The horizontal axis represents the log strikes; the left figures represent the true value of $x \mapsto I_S^1(x)$ (solid line) and its approximation (crosses). The right graph represents the error between the true value and its approximation.

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$$\sup_{(t,x)\in\mathcal{R}_{T,\tilde{x}}} |P_S(x) - P_{S^n}(x)| = \sup_{(t,x)\in\mathcal{R}_{T,\tilde{x}}} |\mathbb{E}^{\mathbb{Q}}(\mathrm{e}^x - S_T)_+ - \mathbb{E}^{\mathbb{Q}}(\mathrm{e}^x - S_T^n)_+| \leq \mathrm{e}^{\tilde{x}}\mathbb{Q}(\tau_n \geq T),$$

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  where S^n is the stopped true martingale (along the localising sequence).
- Still...warning tool for extrapolation issues: for local-stochastic volatility models (Guyon-Henry-Labordère), arbitrage-free regularisation of SABR..