Flexible complete Models with stochastic volatility generalizing Hobson-Rogers

F. Hubalek, J. Teichmann, R. Tompkins

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1 Hobson-Rogers Model

Hobson and Rogers proposed several complete stochastic volatility model. Fix a time horizon $T > 0$. Given a price process $(P_t)_{0 \leq t \leq T}$, which is a positive, square integrable Ito process on a stochastic basis $(\Omega, \mathcal{F}_T, P)$ with one-dimensional Brownian motion $(B_t)_{0 \leq t \leq T}$, we introduce $Z_t := \ln(e^{-rt}P_t)$, where $r \geq 0$ denotes the interest rate. Then we assume for a positive parameter $\lambda > 0$ the 2-dimensional Markov process (Z_t, S_t)

$$
\begin{aligned}
dZ_t &= \mu(S_t)dt + \sigma(S_t)dB_t \\
dS_t &= dZ_t - \lambda S_t dt = (\mu(S_t) - \lambda S_t)dt + \sigma(S_t)dB_t, \\
Z_0 &= z, S_0 = s,
\end{aligned}
$$

for volatility and drift vector fields μ, σ which satisfy the usual Lipschitz assumptions and $\sigma(s) > 0$ for $s \in \mathbb{R}$. The process $(S_t)_{0 \leq t \leq T}$ is the first offset function of $(Z_t)_{0 \leq t \leq T}$ with parameter $\lambda > 0$.

Defining

$$\theta(s) = \frac{1}{2}\sigma(s) + \frac{\mu(s)}{\sigma(s)},$$

where we assume additionally that the measure Q_t on $\mathcal{F}_t$ given through

$$\frac{dQ_t}{dP} = \exp(-\int_0^t \theta(S_u)dB_u - \frac{1}{2}\int_0^t \theta(S_u)^2 du)$$

is well-defined for $0 \leq t \leq T$ and that $Q := Q_T$ is a probability measure equivalent to P on $\mathcal{F}_T$. Then the process $\widetilde{B_t} := B_t + \int_0^t \theta(S_u)du$ is a Q-Brownian motion and the stochastic differential equation reads as follows with respect to $(\widetilde{B_t})_{0 \leq t \leq T}$

$$\begin{aligned} dZ_t &= -\frac{1}{2}\sigma(S_t)^2 dt + \sigma(S_t)d\widetilde{B_t}, \\ dS_t &= -(\frac{1}{2}\sigma(S_t)^2 + \lambda S_t)dt + \sigma(S_t)d\widetilde{B_t}, \\ Z_0 &= z, S_0 = s \end{aligned}$$

for $0 \leq t \leq T$.

The discounted price process $(e^{-rt}P_t)_{0\leq t\leq T}$ is a Q-martingale and we can apply the classical no-arbitrage pricing arguments. In particular the market is complete since this is the only martingale measure equivalent to P. Under Q the price process satisfies

$$dP_t = rP_t dt + \sigma(S_t)P_t d\widetilde{B}_t$$

Therefore the price of a European claim, which is given by a measurable function with at most linear growth $q : \mathbb{R} \to \mathbb{R}$, is defined by

$$V(P_t, S_t, T-t) = e^{r(T-t)}E(q(P_T)|\mathcal{F}_t)$$

for $0 \leq t \leq T$ via the Markov property. If the Lie algebra spanned by the two vector fields

$$(z,s) \mapsto \begin{pmatrix} \sigma(s) \\ \sigma(s) \end{pmatrix}$$

$$(z,s) \mapsto \begin{pmatrix} -\frac{1}{2}\sigma(s)^2 - \frac{1}{2}\sigma(s)'\sigma(s) \\ -\frac{1}{2}\sigma(s)^2 - \lambda s - \frac{1}{2}\sigma(s)'\sigma(s) \end{pmatrix}$$

or equivalently

$$(z,s) \mapsto \begin{pmatrix} \sigma(s) \\ \sigma(s) \end{pmatrix} \text{ and } (z,s) \mapsto \begin{pmatrix} 0 \\ \lambda s \end{pmatrix}$$

spans the tangent space $\mathbb{R}^2$ pointwise on $\mathbb{R}_{>0} \times \mathbb{R}$ (which is the case for non-vanishing σ and $\lambda \neq 0$), then by Hörmander's "Sum of the Squares" we know that f is a smooth function on $\mathbb{R}_{>0} \times \mathbb{R} \times]0, T[$ and satisfies the boundary condition $f(p, s, 0) = q(p)$ for all $(p, s) \in \mathbb{R}_{>0} \times \mathbb{R}$.

One particular choice for σ is a smooth vector field $\sigma : \mathbb{R} \to \mathbb{R}$ such that

$$\sigma(s)^2 := \eta^2(1 + \epsilon s^2)$$

on some ball with large radius $R > 0$ and constant outside for fixed $\epsilon \geq 0$. Here η is referred to as minimal level of implied volatility and $\epsilon \geq 0$ denotes a parameter calibrating the influence of the first offset process $(S_t)_{t \geq 0}$ on the stochastic evolution of the price process. Furthermore the option price depends smootly on the parameters η, ϵ and λ on the respective intervals of definition. By standard methods we can find a version of the solution $(Z_t, S_t)_{0 \leq t \leq T}$ of the stochastic differential equation, which depends in a smooth way on the initial values and the parameters.

In order to fit a model – with a finite number of parameters – to market data, one needs to solve an inverse problem:

- solve the associated PDE often and look which outcomes fit the data best.

- choose a best choice and verify.

Such procedures are expensive and (sometimes) instable. The value of tractable formulas gets particularly visible.

In mathematical finance we model price behaviours by semi-martingales $(S_t^{(\epsilon)})_{t \geq 0}$, which often depend on additional parameters, here denoted by $\epsilon \geq 0$. We propose a method, which allows to calculate derivatives of the function $\epsilon \mapsto E(\phi(S_T^{(\epsilon)}))$ efficiently. To be more precise, we are able to prove that – under some technical assumptions – there exist random variables $\pi^{(n)}$ such that

$$\frac{\partial^n}{\partial \epsilon^n} E(\phi(S_T^{(\epsilon)})) = E(\phi(S_T^{(\epsilon)}) \pi^n).$$

This approach provides

- explicit algorithms how to calculate the weights, even if the functions in question are not real analytic.
- probabilistic approaches for approximations of model prices (Monte Carlo evaluations).
- the method works well in the hypo-elliptic context.

2 Partial Integration and Taylor expansion of Prices

Let $(\Omega, \mathcal{F}, P)$ be a probability space, which is generated by a one-dimensional Brownian motion $(B_t)_{0 \leq t \leq T}$ for some $T > 0$, i.e. $\mathcal{F} = \mathcal{F}_T$. For the reader who is familiar with Ito-integration, but does not feel comfortable with Malliavin Calculus, we list the following simple rules, which allow to follow all calculations which are done in the article:

1. The Malliavin derivative associates to random variables (in its domain of definition) $X \in \mathrm{dom}(D) \subset L^2(\Omega)$ a not necessarily adapted process

$$(D_s X)_{0 \leq s \leq T} \in L^2([0,T] \times \Omega).$$

The Malliavin derivative is a closed, densely defined, unbounded linear operator and the following rules hold,

$$D_s(1_\Omega) = 0$$

$$D_s(\int_0^T \sigma(s)dB_s) = \sigma(s)1_{[0,T]}(s),$$

$$D_s(\phi(X_1,\dots,X_n)) = \sum_{i=1}^n \frac{\partial}{\partial x^i}\phi(X_1,\dots,X_n)D_sX_i$$

for $X_i \in \mathrm{dom}(D)$, $i = 1,\dots,n$ and σ a square-integrable, deterministic function on $[0,T]$. ϕ is given as a C^1-function on $\mathbb{R}^n$.

2. The adjoint of the Malliavin derivative is the Skorohod integral δ, which associates to a not necessarily adapted process $(Y_s)_{0\le s\le T} \in \mathrm{dom}(\delta) \subset L^2([0,T]\times\Omega)$ a random variable $\delta(s \mapsto Y_s) \in L^2(\Omega)$. The Skorohod integral is a closed, densely defined, unbounded linear operator and the following basic partial integration formula holds true

$$E(X\delta(s \mapsto Y_s)) = E(\int_0^T (D_sX)Y_s ds)$$

on the respective domains. The most important, non-trivial assertion on Skorohod integration is the relation to Ito-inegration: namely, for all square-inegrable, predictable processes $(Y_s)_{0 \leq s \leq T}$ we obtain that $(Y_s)_{0 \leq s \leq T} \in \mathrm{dom}(\delta)$ and

$$\delta(s \mapsto Y_s) = \int_0^T Y_s dB_s.$$

3. By extension of the derivative operator D on L^p-spaces we obtain domains of definition $\mathcal{D}^{p,1} \subset L^p(\Omega)$. By definition of iterated derivatives on the respective domains we obtain domains of definition $\mathcal{D}^{p,n} \subset L^p(\Omega)$, where the Malliavin-derivative can be applied n times. Smooth random variables are those, which lie in the domain of each derivative operator in each L^p, i.e.

$$\mathcal{D}^{\infty} = \cap_{p \geq 1} \cap_{n \geq 0} \mathcal{D}^{p,n}.$$

A fortiori smooth random variables are closed under compoistion with smooth, polynomially bounded functions and allow Skorohod inegration up to arbitrary orders.

4. For Skorohod integrable process $(u_s)_{0 \leq s \leq T}$ and $F \in \mathcal{D}^\infty$ with $E(\int_0^T F^2 u_s^2 ds) < \infty$, the process $(Fu_s)_{0 \leq s \leq T}$ is Skorohod-integrable and

$$\delta(s \mapsto u_s F) = F\delta(s \mapsto u_s) - \int_0^T u_s D_s F ds$$

holds true.

5. The Malliavin covariance matrix is a real-valued random variable in the one-dimensional case,

$$\gamma(X) := \int_0^T (D_s X)^2 ds.$$

If $\gamma(X)$ is invertible almost surely, then X has a density with respect to Lebesgue's measure.

We shall deal with families of random variables $\epsilon \mapsto G_\epsilon$ such that

- for all $\epsilon \geq 0$ the random variable and all its derivatives with respect to ϵ are smooth, i.e. $\frac{\partial^k}{\partial \epsilon^k} G_\epsilon \in \cap_{p \geq 1} \cap_{n \geq 0} \mathcal{D}^{p,n}$ for $k \geq 0$, together with all Malliavin derivatives. The derivatives are taken with respect to the topology of $\mathcal{D}^\infty$, which is equivalent to the assertion that the maps $\epsilon \mapsto \eta(G_\epsilon)$ are smooth, for all continuous linear functionals $\eta : \mathcal{D}^\infty \to \mathbb{R}$.

We denote this space by $C^\infty(\mathbb{R}_{\geq 0}, \mathcal{D}^\infty)$. Notice in particular that this space is a (smooth) algebra of random variables, where the Skorohod integral and the Malliavin derivative are well-defined. In particular the constant curve $\epsilon \mapsto 1$ satisfies the requirements. Observe the following rules of differentiation:

- Malliavin derivatives and Skorohod integrals commute with derivatives with respect to ϵ.

- all Malliavin derivatives of derivatives with respect to ϵ are Skorohod integrable.

Definition 2.1 *A family* $(\epsilon \mapsto F_\epsilon) \in C^\infty(\mathbb{R}_{\geq 0}, \mathcal{D}^\infty)$*, which satisfies additionally that the (Malliavin) covariance matrix* $\gamma(F_\epsilon)$ *is almost surely invertible for* $\epsilon \geq 0$ *and at* $\epsilon = 0$ *(but not necessarily off* 0*)*

$$\frac{1}{\gamma(F_0)} \in \mathcal{D}^\infty = \cap_{p\geq 1} \cap_{n\geq 0} \mathcal{D}^{p,n},$$

is called a family with regular density*.*

We shall provide the following characteristic (and useful!) example for families with regular density: given a continuous Gaussian process $(S_t)_{t\geq 0}$ with

$$dS_t = (a(t) - \lambda S_t)dt + \sigma(t)dB_t$$

with continuous (deterministic), square integrable functions $a, b : \mathbb{R}_{\geq 0} \to \mathbb{R}$, we define the family $(F_\epsilon)_{\epsilon\geq 0}$ for a fixed number $T > 0$ and show by simple calculations that $(F_\epsilon)_{\epsilon\geq 0} \in C^\infty(\mathbb{R}_{\geq 0}, \mathcal{D}^\infty)$,

$$F_\epsilon := z - \frac{\eta^2}{2}\int_0^T (1 + \epsilon S_t^2)^2 dt + \eta \int_0^T (1 + \epsilon S_t^2) dB_t,$$

$$D_s F_\epsilon = \eta(1 + \epsilon S_s^2) -$$

$$-2\epsilon\eta^2 \int_s^T (1 + \epsilon S_t^2) S_t \exp(-\lambda(T - t))\sigma(t)dt +$$

$$+2\epsilon\eta \int_s^T S_t \exp(-\lambda(T - t))\sigma(t)dB_t,$$

$$D_s F_\epsilon|_{\epsilon=0} = \eta,$$

$$\frac{\partial}{\partial \epsilon} F_\epsilon = -\eta^2 \int_0^T (1 + \epsilon S_t^2) S_t^2 dt + \eta \int_0^T S_t^2 dB_t,$$

$$\gamma(F_\epsilon)|_{\epsilon=0} = \eta^2 T.$$

This example will be applied for the generalized Hobson-Rogers model (GHR).

A more sophisticated example is given by the following structure, which resembles a slightly modified version of the original Hobson-Rogers model (HR):

$$F_\epsilon := z - \frac{1}{2}\int_0^T \sigma(S_t)^2 dt + \int_0^T \sigma(S_t) dB_t,$$
$$dS_t = (-\frac{1}{2}\sigma(S_t)^2 - \lambda S_t)dt + \sigma(S_t)dB_t$$

with $\sigma(s) = \eta(1+\epsilon s^2)^{1/2}\exp(-\frac{\epsilon^2 s^2}{M})$ for some large constant M. Hence σ is C^∞-bounded and bounded and we obtain $(F_\epsilon)_{\epsilon\geq 0} \in C^\infty(\mathbb{R}_{\geq 0}, \mathcal{D}^\infty)$. The relevant derivatives at $\epsilon = 0$ read as follows.

$$D_s F_\epsilon|_{\epsilon=0} = \eta,$$
$$\frac{\partial}{\partial\epsilon} F_\epsilon|_{\epsilon=0} = -\int_0^T \frac{\eta^2 S_t^2|_{\epsilon=0}}{2} dt + \int_0^T \frac{\eta S_t^2|_{\epsilon=0}}{2} dB_t,$$
$$\gamma(F_\epsilon)|_{\epsilon=0} = \eta^2 T,$$

where $(S_t)_{t\geq 0}$ at $\epsilon = 0$ is particularly simple, namely a mean-reverting Gaussian process,

$$dS_t = (-\frac{1}{2}\eta^2 - \lambda S_t)dt + \eta dB_t.$$

Theorem 2.1 *Given a family* $(F_\epsilon)_{\epsilon\geq 0} \in C^\infty(\mathbb{R}_{\geq 0}, \mathcal{D}^\infty)$ *with regular density. Then there exist random variables* $\pi^n \in \mathcal{D}^\infty$ *such that for all* $\phi \in C_0^\infty(\mathbb{R})$

$$\frac{\partial^n}{\partial \epsilon^n} E(\phi(F_\epsilon))|_{\epsilon=0} = E(\phi(F_0)\pi^n)$$

holds true for $n \geq 0$.

Definition 2.2 *The random variable* $\pi^n \in \mathcal{D}^\infty$ *is called* n*th Malliavin weight for differentiation with respect to the parameter* ϵ.

Remark 2.1 *If* π^n *is a polynomial of integrated Gaussian polynomials, then the expected value* $E(\phi(F_0)\pi^n)$ *can be calculated in two steps: first an ordinary Gaussian integral applied to a polynomial on some* $\mathbb{R}^n$, *second the inegration of this result with respect to Lebesgue measure on* $[0, T]^m$. *Both procedures are numerically cheap and yield quick and good results even for complicated stochastic differential equations.*

The applications which we have in mind are certainly solutions of standard stochastic differential equations of the type

$$dZ_t^{x,\epsilon} = V(\epsilon, t, Z_t^{x,\epsilon})dt + V^1(\epsilon, t, Z_t^{x,\epsilon})dB_t,$$

where the initial value is given by a real vector $x \in \mathbb{R}^N$ and $(B_t)_{t\geq 0}$ denotes a 1-dimensional Brownian motion. If the vector fields V, V^1 are regular enough, for instance real analytic and C^∞-bounded, then we can take each coordinate of the solution process at a certain time $T > 0$ – viewed as a family of random variables with respect to $\epsilon \geq 0$ – is an element of $C^\infty(\mathbb{R}_{\geq 0}, \mathcal{D}^\infty)$.

We take the above example with $a = \lambda = 0$ and $b = 1$, $S_0 = 0$, and calculate the outcome for the first and second derivative with respect to ϵ.

$$E(\phi(F_\epsilon)) = E(\phi(z - \frac{\eta^2}{2}T + \eta B_T))$$

$$\frac{\partial}{\partial \epsilon}|_{\epsilon=0} E(\phi(F_\epsilon))$$

$$= E(\phi(z - \frac{\eta^2}{2}T + \eta B_T)\delta(\frac{\partial}{\partial \epsilon}|_{\epsilon=0} F_\epsilon \frac{\eta}{\eta^2 T}))$$

$$= \frac{1}{\eta T} E(\phi(z - \frac{\eta^2}{2}T + \eta B_T) B_T(-\eta^2 T + \eta \int_0^T B_t^2 dB_t)) +$$

$$+ \frac{1}{\eta T} E(\phi(z - \frac{\eta^2}{2}T + \eta B_T) \int_0^T (\eta B_s^2 + 2\eta \int_s^T B_t dB_t) ds)$$

$$= \frac{1}{T} E(\phi(z - \frac{\eta^2}{2}T + \eta B_T) B_T(-\eta T + \frac{B_T^3}{3} - \int_0^T B_t dt)) +$$

$$+ \frac{1}{T} E(\phi(z - \frac{\eta^2}{2}T + \eta B_T)(B_T^2 T - \frac{T^2}{2})),$$

which has a simple polynomials structure.

For the second derivative we proceed as follows: We observe that two ingredients for Skorohod integral can be well-calculated, namely

$$\frac{\partial}{\partial\epsilon}|_{\epsilon=0}\delta(\frac{\partial}{\partial\epsilon}F_\epsilon\frac{D_sF_\epsilon}{\gamma(F_\epsilon)}) = \delta(\frac{\partial^2}{\partial\epsilon^2}|_{\epsilon=0}F_\epsilon\frac{1}{\eta T})+$$

$$+\delta(\frac{\partial}{\partial\epsilon}|_{\epsilon=0}F_\epsilon\frac{D_s\frac{\partial}{\partial\epsilon}|_{\epsilon=0}F_\epsilon}{\eta^2 T})-$$

$$-\delta(\frac{\partial}{\partial\epsilon}|_{\epsilon=0}F_\epsilon\frac{\eta}{\eta^4T^2}\frac{\partial}{\partial\epsilon}|_{\epsilon=0}\gamma(F_\epsilon)),$$

and

$$\delta(\frac{\partial}{\partial\epsilon}|_{\epsilon=0}F_\epsilon\frac{\eta}{\eta^2T}\delta(\frac{\partial}{\partial\epsilon}|_{\epsilon=0}F_\epsilon\frac{\eta}{\eta^2T}))$$

as above. Again we shall obtain a simple polynomial structure.

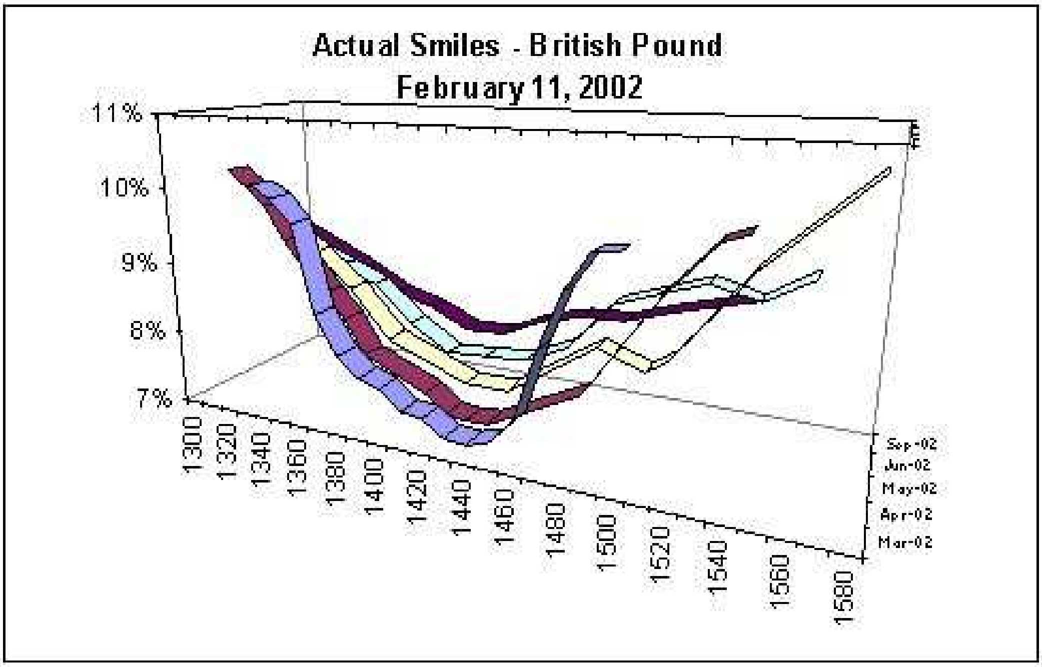

Figure 1: Market Data - Implied Volatilities

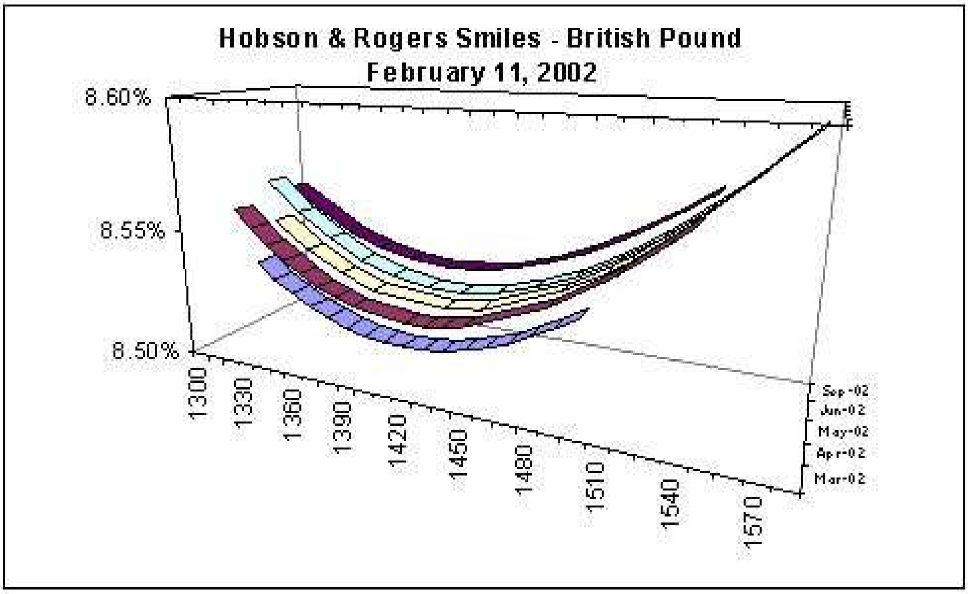

Figure 2: Smiles on microscopic scale

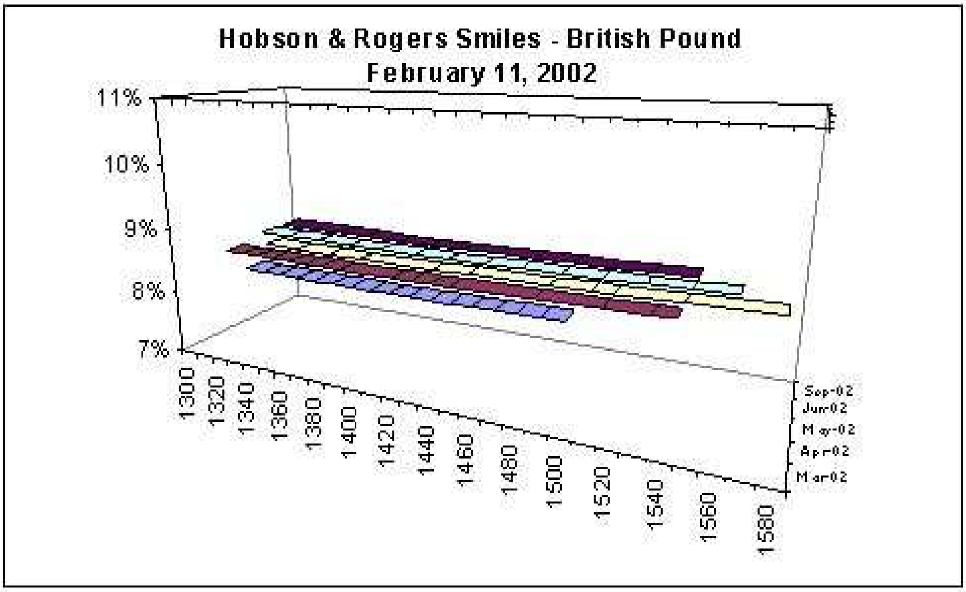

Figure 3: Smiles on appropriate scale

3 GHR-Model

The generalized Hobson-Rogers model reads in the martingale measure as follows.

$$dZ_t = -\frac{1}{2}\sigma_1(S_t)^2 dt + \sigma_1(S_t) d\widetilde{B}_t$$
$$dS_t = \mu(S_t)dt + \sigma_2(S_t) d\widetilde{B}_t$$
$$Z_0 = z, S_0 = s$$

with the following specification,

$$\sigma_1(s) = \eta(1 + \epsilon\beta s^2)$$
$$\sigma_2(s) = \chi\eta$$
$$\mu(s) = -\frac{\eta^2}{2} - \lambda s$$

for fixed $\epsilon \geq 0$. In contrast to the HR-model we are additionally given two positive parameters $\chi \geq 1$ and $\beta \in [0, \frac{1}{2}]$ (even though only the product $\epsilon\beta$ enters into the formulas).

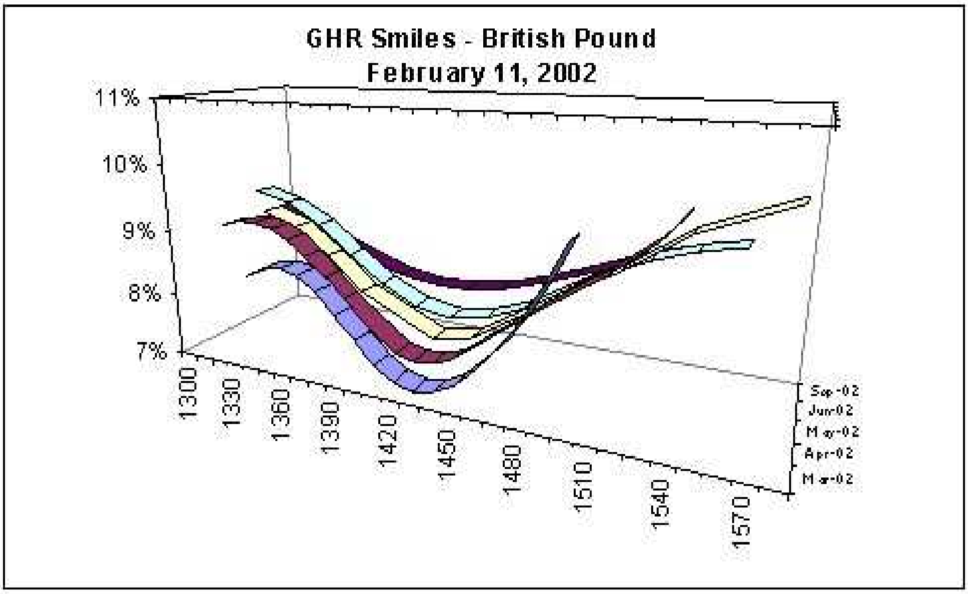

Figure 4: Smiles in GHR-model

With our method we obtain the previous data fit quickly...