Dividend Optimisation: A Behaviouristic Approach

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Abstract

In this paper, we study a dividend maximisation problem for a Brownian risk model as a surplus and a Markov-switching model describing the preference rate of an insurer. The preference rate can attain two values – a positive and a negative. The negative preference reflects the situation when the uncertainty prevails and the insurer shows more waiting tendency. In the times of the positive preference the insurer is in modus operandi.
We solve the problem of finding the optimal dividend payout strategy for the setting with a classical ruin concept as well as for the case of a Parisian ruin with an exponential delay. In the first case, the optimal strategy turns out to be of a barrier type for the positive preference rate and no dividends are paid in the times of the negative rate. The optimal barrier increases with increasing intensity to switch into the state with a negative preference and shows the inverse dependence on the counterpart intensity.
In the case of Parisian ruin, the optimal strategy depends on the parameter of the ruin delay and the severity of the negative preference rate. If the expected delay is relatively short the optimal strategy remains a barrier (even equal to zero) in the positive state with no dividends being paid in the negative state. If the expected delay is long, the optimal strategy in the negative state might change from not paying dividends to a band strategy. We give explicit expressions for the value functions and present conditions determining the type of the optimal strategy. Both problems are illustrated by examples.

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1 Introduction

Strategic optimisation of dividend policies is an important topic in economic and actuarial sciences. The distribution of dividends publicly displays the ability to earn profits and constitutes self-assurance and success of the company. Dividends are a way to redistribute profits to shareholders as a way to thank them for their support and to encourage additional investments. As dividends payments reduce the capital of the company, there is a trade-off between the maximisation of payouts and staying in business as long as possible. For this reason, timing of announcements and magnitude of dividend payments are crucial and influence strongly the

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benefit of a dividend payment for the company. For example, dividend change announcements receive stronger reaction from investors in recessions than in expansions.

A fundamental assumption in many early research articles on dividend strategies is that the percepted worth of immediate dividend payments exceeds the benefit from dividend distribution in the future. That is to say, there is a constant technical discounting on the time-value of dividend payments given through a positive preference rate $\delta > 0$. In this framework, dividend optimisation problems are well-studied (some examples connected to our model can be found in Asmussen and Taksar [6], Gerber and Shiu [14] and [15], and Hubalek and Schachermayer [17]; Avanzi and Gerber [7] and Avram et al. [8] are set in more general frameworks).
However, in reality, preference rates of companies change over time and are influenced by various exogenous and endogenous factors, such as a change in management, the influence of (im-)patient investors, additional regulatory requirements and, of course, drastic events such as market crashes. Recent behavioural studies also show that there is a relation of preference rates and cultural differences (see, for example, Wang et al. [31] and Breuer et al. [10]).

A second fundamental assumption in dividend optimisation problems is that the company goes out of business at the first time the surplus becomes negative. However, a company with a high credit score, which has been reliably repaying debt in the past, will likely have access to the financial means to get over a lean period. Therefore, the concept of ruin from the managerial perspective can, in practice, differ from the "classical" definition.

For a more realistic modelling of time preference, many recent articles introduce time-inconsistent discounting or preference functions. For example, Li et al. [21] and Zhu et al. [34] deal with dividend optimisation problems under a non-exponential, stochastic (quasi-)hyperbolic discounting. The approach of "hyperbolic" discounting is based on the assumption that the economic player imposes a higher subjective discounting on the near future than on the far future and is in line with the empirical findings of Wang et al. [31]. Another example provides the paper by Jiang and Pistorius [18], where an optimal dividend problem for a diffusion process is considered in a regime switching setting. In particular, the preference function takes an exponential form with positive discounting rates which depend on the (random) state of the economy. The changes of this state are determined by a continuous time Markov chain. Akyildirim et al. [1] analyse the problem of dividend maximisation in a diffusion model for which the preference rate switches back and forth between two positive states. They identify optimal thresholds for dividend distribution in the different states and analyse their sensitivity with respect to mean, volatility and, in particular, the jump rates in numerical examples. In contrast to [18], they additionally include the possibility to issue new shares. This is motivated by the fact that the changing costs of equity issuance (and the size of financial frictions in general) influence the dividend policy of firms.

Eisenberg and Krühner [13] consider the problem of minimal capital injections in a regime switching setting, where, in one of two regimes, the preference rate is allowed to be negative. An influential factor to the subjective preference rate of a manager or a company is the assessment of the current and future development of the economy. Since the future is never certain, we must consider the impact of uncertainty on time preference. The most likely effect of uncertainty or concern about future income and needs, given risk aversion, is trying to play it safe. This is done by preparing, partially or fully, for the worst eventuality with a sudden reduction in income or increase in needs. Such precautionary behaviour (or, if you like sophistication, such *minimum regret strategy*) should reduce the rate of time preference. In particular, uncertainty can be a reason to withhold earnings to either reinvest them into the company by building safety capital or to save them for dividend payments at a later, more advantageous time. This effect, which has also been observed by Wang et al. [31], corresponds to a negative preference rate and is, to our knowledge, not yet reflected in common preference rate models by optimising dividends.

The possibility of continuing business for a certain amount of time after the classical "ruin" event is reflected in the definition of Parisian ruin, proposed by Dassios and Wu [12]. In a model with Parisian ruin, the surplus is allowed to become negative for a certain (random or deterministic) amount of time. If the company recovers within this time, that is to say, if the surplus re-enters the positive half plane, it does not go out of business.
The cases of deterministic and exponential Parisian delays require different mathematical techniques and also differ in their economic tractability. Whilst the deterministic delays can be easily explained as a fixed time frame given to the insurer to come out of red numbers, the exponential delays, being memoryless, are harder to justify from the practical point of view. Mathematically, the deterministic delays add time dependence and increase the complexity of the considered problems, rarely allowing for explicit solutions. However, sometimes one can overcome the "deterministic curse". An example provide Wong and Cheung [32], who study Laplace transforms of Parisian ruin times with deterministic and stochastic ruin "clocks" for a renewal risk process with exponential jumps.
Loeffen et al. [20] express ruin probabilities of the Parisian ruin time with a deterministic clock for spectrally negative Lévy processes via scale functions. Landriault et al. [19] modify the model tackled in [20] by assuming that the Parisian delays are of a mixed Erlang nature. The authors show that this modification leads to more explicit results.
Renaud [26] considers the problem of dividend optimisation under a spectrally negative Lévy surplus and an exponential Parisian delay. There, it was possible to prove the optimal strategy to be a constant barrier.
Further results on dividend optimisation problems under Parisian ruin can be found, for instance, in Czarna and Palmowski [11], Xu et al. [33] and references therein.

In this paper, we examine the impact of negative preference rates and Parisian ruin on (optimal) dividend strategies for an insurance company. We consider an arithmetic Brownian motion surplus model in an independent Markov switching setting with two possible states of a positive and a negative preference rate. Payout strategies are considered in the setting of a classical ruin and under a Parisian ruin. In particular, we discover connections between the optimal payout strategies, switching intensities and the parameter of the exponential ruin clock in the case of Parisian ruin.

Choosing an exponentially distributed Parisian delay leads to a time independent value function, which allows us to get explicit results. Also, the considered model can be interpreted as a special case of the Omega model, introduced in Albrecher et al. [2] and further discussed, for example, in Gerber et al. [16] and in Albrecher and Lautscham [5]. Similar to the definition of the Parisian ruin framework, a company whose surplus is described by the Omega model does not go out of business when the surplus becomes negative. Instead, one introduces a bankruptcy rate function $\omega(x)$ describing the probability of ruin within $\mathrm{d}t$ time units for a given negative surplus x. Hence, the case of an exponential Parisian ruin clock with parameter γ corresponds to a constant bankruptcy rate $\omega(x) = \gamma^{-1}$, $x < 0$, in the Omega model.
Another interpretative link provide models with randomised observation times, see Albrecher et al. [3] and Albrecher et al. [4]. "Randomised observation" means that a manager or regulator can only monitor the surplus at discrete times. If the surplus is negative at an observation

time, bankruptcy is declared. In particular, if the number of observations is given by a Poisson process, i.e. the inter-observation times are identically exponentially distributed, having a positive surplus until the n-th observation means "no ruin" so far in the Parisian framework.

The remainder of the paper is organised as follows. In Section 2 we present the mathematical model constituting the basis of our analysis. Section 3 deals with the optimisation problem for a classical ruin setup. We define bipartite strategies where the insurer waits with the distribution of dividends until the preference rate is positive and the surplus lies above a certain barrier. For these "barrier" strategies we calculate explicitly the corresponding return functions and prove that the optimal strategy is also of this type. We analyse in detail the dependence of the value function and optimal barrier on the switching intensities and discover that the barrier and the return increase if more time is spent (on average) in the phases with negative preference. In Section 4, we analyse the optimal control problem for Parisian ruin. In this setting we define barrier and band strategies and find their return functions. We examine in detail the dependence of the optimal strategy on the average time the surplus is allowed to stay negative. In Section 5, we summarise our findings and comment on the economic implications. The appendix contains mathematical and technical details of some of the proofs for the interested reader.

2 The General Model

In this section, we present the mathematical formulation of the general model we are going to consider in both parts of the paper and introduce the crucial notation. All processes are defined on the probability space $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$, where $\mathcal{F} = \{\mathcal{F}_t\}$ is a right-continuous filtration. In the following sections, we use the common notation: $\mathbb{P}[\cdot|Y_0 = y] = \mathbb{P}_y[\cdot]$ and $\mathbb{E}[\cdot|Y_0 = y] = \mathbb{E}_y[\cdot]$ for any stochastic process $Y = \{Y_t\}$.
We assume that the surplus process of the insurance company under consideration is given by a Brownian motion with drift

$$X_t = x + \mu t + \sigma W_t,$$

where W is a standard Brownian motion, $\mu, \sigma > 0$ are constants. The insurance company is allowed to pay dividends. A dividend strategy is a non-decreasing process $D = \{D_t\}$ describing the accumulated payments until time t and leading to the post-dividend surplus

$$X_t^D = x + \mu t + \sigma W_t - D_t \quad \text{for all } t \le \tau,$$

where τ describes the ruin time of X^D. The concept of ruin will be defined in the corresponding sections below.
Aiming at maximising the expected discounted dividends, the following preference rate model will be taken as a basis. The company is assumed to live through a cycle of microeconomical changes in its preference rate process r. Let $r = \{r_t\}$ be a continuous-time $\mathcal{F}$-adapted Markov chain with two states $\delta_1 \le 0$ and $\delta_2 > 0$. The Markov chain switches with intensity $\lambda_1 > 0$ from δ_1 to δ_2 and $\lambda_2 \ge 0$ from δ_2 to δ_1, i.e. the generator matrix of r is given by

$$\begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}$$

The following result was first obtained in Pedler [24] and will be crucial for our derivations.

Lemma 2.1
Let $\Delta^+ := \frac{(\lambda_1+\delta_1+\lambda_2+\delta_2)+\sqrt{(\lambda_1+\delta_1-\lambda_2-\delta_2)^2+4\lambda_1\lambda_2}}{2}$, $\Delta^- := \frac{(\lambda_1+\delta_1+\lambda_2+\delta_2)-\sqrt{(\lambda_1+\delta_1-\lambda_2-\delta_2)^2+4\lambda_1\lambda_2}}{2}$
Then, it holds

$$\mathbb{E}_i\Big[\mathrm{e}^{-\int_0^t r_s\,\mathrm{d}s}\Big] = \frac{(\delta_i-\Delta^-)\mathrm{e}^{-\Delta^+ t}-(\delta_i-\Delta^+)\mathrm{e}^{-\Delta^- t}}{\Delta^+-\Delta^-}\,. \tag{1}$$

□

Note that the Markov chain r and the Brownian motion W are independent as they are adapted to the same filtration $\mathcal{F}$. Further, we define the value function as

$$V(x,i) = \sup_{D\in\mathcal{D}} V_D(x,i) = \sup_{D\in\mathcal{D}} \mathbb{E}_{x,i}\Big[\int_0^\tau \mathrm{e}^{-\int_0^t r_s\,\mathrm{d}s}\,\mathrm{d}D_t\Big]\,, \quad i\in\{1,2\}\,,\ x\ge 0\,, \tag{2}$$

where $V_D(x,i)$ denotes the return function of an admissible strategy D when starting in the regime with the preference rate δ_i and initial capital x, and $\mathcal{D}$ denotes the set of admissible strategies to be defined in every section separately.

3 Classical Ruin

In this section, we consider the problem given in (2) under the classical definition of the time of ruin. Thus, the company declares ruin at the moment the (post-dividend) surplus process becomes negative for the first time:

$$\tau = \inf\{t\ge 0:\ X_t^D<0\}\,.$$

We restrict our considerations to the set of admissible strategies $\mathcal{D}_c$ containing all strategies D with $D_0\ge 0$, D is cadlag and $X_t^D\ge 0$ for all $t\le\tau$.
The optimisation problem (2) will be tackled via solving the corresponding Hamilton–Jacobi–Bellman (HJB) equation for $x\ge 0$, $i,j\in\{1,2\}$ and $V(0,1)=0=V(0,2)$. The heuristically derived HJB is given by

$$\max\Big\{\frac{\sigma^2}{2}V''(x,i)+\mu V'(x,i)-(\lambda_i+\delta_i)V(x,i)+\lambda_i V(x,j),1-V'(x,i)\Big\}=0\,. \tag{3}$$

Before considering the HJB, we need to ensure that the value function in (2) is well-defined.

Proposition 3.1
The value function (2) *is well-defined if and only if*

$$\delta_1 > -\frac{\lambda_1\delta_2}{\lambda_2+\delta_2}\,. \tag{4}$$

Proof: • Assume first $\delta_1\le -\frac{\lambda_1\delta_2}{\lambda_2+\delta_2}$, then $\Delta^-\le 0$ and $\Delta^+>0$. The strategy $\tilde D_t=\xi t$ with $\xi=\frac{\mu}{2}$ is an admissible strategy and the corresponding ruin time $\tilde\tau$ of the post-dividend process $X^{\tilde D}$ fulfils $\mathbb{P}[\tilde\tau=\infty]>0$, compare Borodin and Salminen [9, p. 309]. Then, it holds

$$\begin{aligned} V(x,1)\ge V_{\tilde D}(x,1) &= \mathbb{E}_{x,1}\Big[\int_0^{\tilde\tau}\mathrm{e}^{-\int_0^t r_s\,\mathrm{d}s}\,\mathrm{d}\tilde D_t\Big] = \xi\mathbb{E}_x\Big[\int_0^{\tilde\tau}\mathbb{E}_1[\mathrm{e}^{-\int_0^t r_s\,ds}]\,\mathrm{d}t\Big] \\ &\ge \xi\mathbb{P}_x[\tilde\tau=\infty]\int_0^\infty \frac{(\delta_1-\Delta^-)\mathrm{e}^{-\Delta^+ t}-(\delta_1-\Delta^+)\mathrm{e}^{-\Delta^- t}}{\Delta^+-\Delta^-}\,\mathrm{d}t=\infty\,. \end{aligned}$$

• On the other hand, if (4) is fulfilled then $\Delta^-, \Delta^+ > 0$, $\delta_2 > \Delta^-$ and $\delta_2 < \Delta^+$. Let D be an arbitrary admissible strategy with the corresponding ruin time τ of X^D. Since D is by definition non-decreasing and $\int_0^t r_s \,\mathrm{d}s$ is continuous it holds via integration by parts

$$\int_0^\tau \mathrm{e}^{-\int_0^t r_s \mathrm{d}s} \,\mathrm{d}D_t = D_\tau \mathrm{e}^{-\int_0^\tau r_s \mathrm{d}s} + \int_0^\tau D_t r_t \mathrm{e}^{-\int_0^t r_s \mathrm{d}s} \,\mathrm{d}t - D_0 \,.$$

- We show first that the expression $D_\tau \mathrm{e}^{-\int_0^\tau r_s \,\mathrm{d}s}$ is well defined on $[\tau = \infty]$. Note that $\frac{\mathrm{e}^{aX_t}}{a} \geq X_t \geq D_t$ for any $a > 0$ and $t \geq 0$. Now we specifically define $a := \frac{-\mu+\sqrt{\mu^2+\sigma^2\Delta^-}}{\sigma^2} > 0$. The strong law of large numbers yields $\lim\limits_{t\to\infty} \frac{1}{t}\int_0^t r_s \mathrm{d}s = \frac{\delta_1}{\lambda_1} + \frac{\delta_2}{\lambda_2}$ and $\lim\limits_{t\to\infty} \frac{aX_t}{t} \to a\mu$ almost surely, see for instance Norris [23]. We conclude $\lim\limits_{t\to\infty} \big(aX_t - \int_0^t r_s \,\mathrm{d}s\big) = -\infty$ almost surely as $a\mu < \frac{\delta_1}{\lambda_1} + \frac{\delta_2}{\lambda_2}$ due to (4).

- Building expectations yields

$$\begin{aligned}
\mathbb{E}_{x,i}\Big[\int_0^\tau \mathrm{e}^{-\int_0^t r_s \mathrm{d}s} \,\mathrm{d}D_t\Big] &= \mathbb{E}_{x,i}\Big[D_\tau \mathrm{e}^{-\int_0^\tau r_t \mathrm{d}t}\Big] + \mathbb{E}_{x,i}\Big[\int_0^\tau D_t r_t \mathrm{e}^{-\int_0^t r_s \mathrm{d}s} \,\mathrm{d}t\Big] - D_0 \\
&\leq \mathbb{E}_{x,i}\Big[X_\tau \mathrm{e}^{-\int_0^\tau r_t \mathrm{d}t}\Big] + \mathbb{E}_{x,i}\Big[\int_0^\tau D_t \delta_2 \mathrm{e}^{-\int_0^t r_s \mathrm{d}s} \,\mathrm{d}t\Big] \\
&\leq \mathbb{E}_{x,i}\Big[\frac{\mathrm{e}^{aX_\tau - \int_0^\tau r_t \mathrm{d}t}}{a}\Big] + \int_0^\infty \frac{\delta_2}{a} \mathbb{E}_{x,i}\Big[\mathrm{e}^{aX_t - \int_0^t r_s \mathrm{d}s}\Big] \,\mathrm{d}t \,. \qquad (5)
\end{aligned}$$

- Note that for any $t > 0$ it holds via change of measure with the new measure $\mathbb{P}^*$, expectation $\mathbb{E}^*$ and the Radon-Nikodym derivative $\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}^*} = \mathrm{e}^{-a(X_t - x) + \frac{\sigma^2 a^2}{2}t + \mu a t}$:

$$\mathbb{E}_{x,i}\Big[\mathrm{e}^{aX_t - \int_0^t r_s \mathrm{d}s}\Big] = \mathrm{e}^{ax}\mathbb{E}^*_{x,i}\Big[\mathrm{e}^{\left(\frac{\sigma^2 a^2}{2} + \mu a\right)t - \int_0^t r_s \mathrm{d}s}\Big] = \mathrm{e}^{ax + \frac{\Delta^-}{2}t} \cdot \frac{(\delta_i - \Delta^-)\mathrm{e}^{-\Delta^+ t} - (\delta_i - \Delta^+)\mathrm{e}^{-\Delta^- t}}{\Delta^+ - \Delta^-} \,.$$

ensuring the existence of the integral in (5).

- It remains to consider $\mathbb{E}_{x,i}\Big[\frac{\mathrm{e}^{aX_\tau - \int_0^\tau r_t \,\mathrm{d}t}}{a}\Big]$. Since both processes X and r are Markovian the following holds true for any $t > 0$ and a given above:

$$\begin{aligned}
&\mathbb{E}_{x,i}\Big[\mathrm{e}^{aX_t - \int_0^t r_s \,\mathrm{d}s}\Big] \geq \mathbb{E}_{x,i}\Big[\mathrm{e}^{aX_t - \int_0^t r_s \,\mathrm{d}s} 1\!\!1_{[\tau \leq t]}\Big] \\
&\quad= \mathbb{E}_{x,i}\Big[\mathrm{e}^{aX_\tau - \int_0^\tau r_s \,\mathrm{d}s} \mathbb{E}\Big[\mathrm{e}^{a(X_t - X_\tau) - \int_\tau^t r_s \,\mathrm{d}s} 1\!\!1_{[\tau \leq t]} \Big| \mathcal{F}_\tau\Big]\Big] \\
&\quad= \mathbb{E}_{x,i}\Big[\mathrm{e}^{aX_\tau - \int_0^\tau r_s \,\mathrm{d}s} 1\!\!1_{[\tau \leq t]} \cdot \frac{(r_\tau - \Delta^-)\mathrm{e}^{-\left(\Delta^+ - \frac{\Delta^-}{2}\right)(t-\tau)} - (r_\tau - \Delta^+)\mathrm{e}^{-\frac{\Delta^-}{2}(t-\tau)}}{\Delta^+ - \Delta^-}\Big] \\
&\quad\geq \mathbb{E}_{x,i}\Big[\mathrm{e}^{aX_\tau - \int_0^\tau r_s \,\mathrm{d}s} 1\!\!1_{[\tau \leq t]}\Big] \frac{(\delta_2 - \Delta^-)\mathrm{e}^{-\left(\Delta^+ - \frac{\Delta^-}{2}\right)t} - (\delta_2 - \Delta^+)\mathrm{e}^{-\frac{\Delta^-}{2}t}}{\Delta^+ - \Delta^-} \,.
\end{aligned}$$

Dividing both sides of the above inequality by $\frac{(\delta_i - \Delta^-)\mathrm{e}^{-\left(\Delta^+ - \frac{\Delta^-}{2}\right)t} - (\delta_i - \Delta^+)\mathrm{e}^{-\frac{\Delta^-}{2}t}}{\Delta^+ - \Delta^-}$ and letting

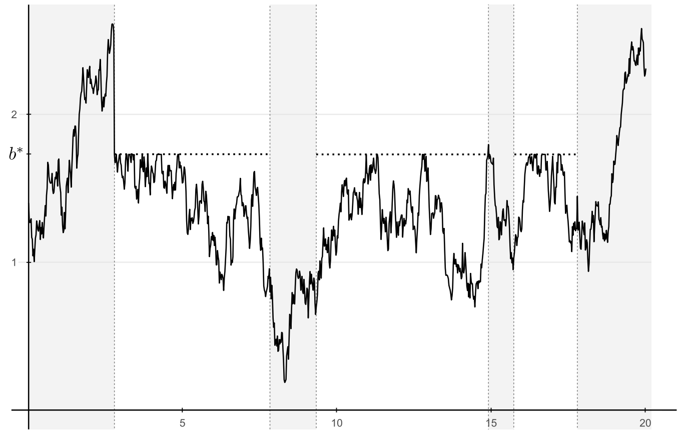

Figure 1: Ex-dividend process following a barrier strategy with barrier b^* (compare Example 3.15 below). The time intervals with $r_t = \delta_1$ are shown shaded in gray. The barrier b^* is represented by the dotted line.

t go to infinity, we get

$$\begin{aligned} e^{ax} &\geq \lim_{t\to\infty} \mathbb{E}_{x,i}\Big[e^{aX_\tau - \int_0^\tau r_s\,ds}\mathbb{1}_{[\tau\leq t]}\Big] \frac{(\delta_2-\Delta^-)e^{-\left(\Delta^+ - \frac{\Delta^-}{2}\right)t} - (\delta_2-\Delta^+)e^{-\frac{\Delta^-}{2}t}}{(\delta_i-\Delta^-)e^{-\left(\Delta^+ - \frac{\Delta^-}{2}\right)t} - (\delta_i-\Delta^+)e^{-\frac{\Delta^-}{2}t}} \\ &= \mathbb{E}_{x,i}\Big[e^{aX_\tau - \int_0^\tau r_s\,ds}\mathbb{1}_{[\tau<\infty]}\Big] \frac{\Delta^+ - \delta_2}{\Delta^+ - \delta_i}. \end{aligned}$$

This implies that $\mathbb{E}_{x,i}\big[e^{aX_\tau - \int_0^\tau r_s\,ds}\mathbb{1}_{[\tau<\infty]}\big]$ has an upper bound which is independent from D. In the same way one can show that $\mathbb{E}_{x,i}\big[e^{aX_\tau - \int_0^\tau r_s\,ds}\mathbb{1}_{[\tau=\infty]}\big] = 0$.

□

3.1 Barrier strategies and their return functions

In this section we consider a special type of strategies, called barrier strategies. The considered barriers are assumed to be positive constants. In models with no regime switching, a barrier strategy with a given barrier b pays out any amount exceeding b and does nothing if the surplus is below b.

Definition 3.2
In the present paper, we talk about a ***barrier strategy*** *with a barrier pair* (∞, b), $b > 0$, *if: no dividends are paid in the regime with a negative preference rate; whereas, we follow the classical barrier strategy for the barrier* b *during the positive rate phases.*

First, we introduce the notation and conventions to be used in the following explanations.

Notation 3.3

• *We let*

$$q_1 := \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2 \Delta^+}}{\sigma^2}, \quad q_2 := \frac{-\mu - \sqrt{\mu^2 + 2\sigma^2 \Delta^+}}{\sigma^2},$$
$$q_3 := \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2 \Delta^-}}{\sigma^2}, \quad q_4 := \frac{-\mu - \sqrt{\mu^2 + 2\sigma^2 \Delta^-}}{\sigma^2}, \quad q_5 := \frac{-\mu - \sqrt{\mu^2 + 2\sigma^2 (\lambda_1 + \delta_1)}}{\sigma^2}.$$

Note that $q_1 > 0 > q_2$ *are the solutions to the quadratic equation* $\frac{\sigma^2}{2} q^2 + \mu q = \Delta^+$ *and* $q_3 > 0 > q_4$ *solve* $\frac{\sigma^2}{2} q^2 + \mu q = \Delta^-$.

• *The functions* $\psi(x) := \mathrm{e}^{q_1 x} - \mathrm{e}^{q_2 x}$ *and* $\phi(x) := \mathrm{e}^{q_3 x} - \mathrm{e}^{q_4 x}$ *are strictly increasing solutions with* $\phi(0) = \psi(0) = 0$ *to the equations*

$$\frac{\sigma^2}{2}\psi''(x) + \mu \psi'(x) = \Delta^+ \psi(x)\,, \qquad \frac{\sigma^2}{2}\phi''(x) + \mu \phi'(x) = \Delta^- \phi(x)\,. \tag{6}$$

• *For convenience of notation we let* $A := \frac{-\lambda_1}{\Delta^+ - (\lambda_1+\delta_1)}$ *and* $B := \frac{-\lambda_1}{\Delta^- - (\lambda_1+\delta_1)}$. *It is easy to see that* $A < 0$ *and* $\frac{\lambda_2+\delta_2}{\lambda_2} > B > \frac{\lambda_1}{\lambda_1+\delta_1} > 1$.

• *In order to simplify the explanations, we write* (G_∞, F_b) *for the return function,* $V_{(\infty,b)}(x,i)$, *corresponding to a barrier pair* (∞, b), *where*

$$V_{(\infty,b)}(x,i) = \begin{cases} F_b(x) & : \text{if } x \in \mathbb{R}_+ \text{ and } i = 2 \\ G_\infty(x) & : \text{if } x \in \mathbb{R}_+ \text{ and } i = 1\,. \end{cases} \tag{7}$$

• *For a pair* H *of sufficiently smooth functions* $H(x,1)$ *and* $H(x,2)$, $x \geq 0$, *and* $i \neq j$ *we write:*

$$\mathcal{L}H(x,i) = \frac{\sigma^2}{2} H''(x,i) + \mu H'(x,i) - (\lambda_i + \delta_i) H(x,i) + \lambda_i H(x,j)\,.$$

The following properties of the functions ψ and ϕ will be crucial in derivation of the value function.

Lemma 3.4

1.) The second derivatives ϕ'' *and* ψ'' *are strictly increasing on* $(0,\infty)$, *have unique zeros* x_ψ, x_ϕ *respectively with* $0 < x_\psi < x_\phi$.

2.) Let $\tilde{\psi}(x) = \frac{\psi''(x)}{\psi'(x)}$ *and* $\tilde{\phi}(x) = \frac{\phi''(x)}{\phi'(x)}$. *Then* $\tilde{\psi}(x) - \tilde{\phi}(x) > 0$ *for all* $x > 0$.

3.) It holds $\tilde{\psi}'(x), \tilde{\phi}'(x) > 0$.

4.) Denote the solutions to Differential equations (6) *in dependence on* λ_2 *by* $\psi_{\lambda_2}(x)$ *and* $\phi_{\lambda_2}(x)$. *Then* $\tilde{\psi}_{\lambda_2}(x) = \frac{\psi''_{\lambda_2}(x)}{\psi'_{\lambda_2}(x)}$ *is increasing and* $\tilde{\phi}_{\lambda_2}(x) = \frac{\phi''_{\lambda_2}(x)}{\phi'_{\lambda_2}(x)}$ *is decreasing in* λ_2. *Furthermore,* $\tilde{\psi}_{\lambda_2}(x)$ *converges uniformly on compacts to a continuously differentiable function and* $\tilde{\phi}_{\lambda_2}(x)$ *converges uniformly on compacts to* $-\frac{2\mu}{\sigma^2}$ *as* λ_2 *goes to* $-\frac{\delta_2(\lambda_1+\delta_1)}{\delta_1}$.

Proof: See Appendix. □

A first observation concerning the return of a barrier strategy is made in the following Lemma.

Lemma 3.5
Let $V_{(\infty,b)}$ be the return function of the barrier strategy (∞, b), $b > 0$. Then,

$$V_{(\infty,b)}(x,1) = C_5(b)\mathrm{e}^{q_5(x-b)} + \frac{\lambda_1}{\lambda_1+\delta_1}V_{(\infty,b)}(x,2) + \frac{\mu\lambda_1}{(\lambda_1+\delta_1)^2}$$

on $[b,\infty)$ for some $C_5(b) \in \mathbb{R}$. In particular, $V_{(\infty,b)}$ is linearly bounded and solves the differential equation $\mathcal{L}(V_{(\infty,b)})(x,1) = 0$ for $x \geq b$.

Proof: Since the surplus exceeding b is immediately paid out during the regime with the positive preference rate δ_2, we have $V_{(\infty,b)}(x,2) = x - b + V_{(\infty,b)}(b,2)$. Starting at $(x,1)$ with $x > b$, i.e. with the preference rate δ_1 and initial surplus x, let T_1 denote the time of the first switch into the regime 2. Let $\theta = \inf\{t > 0 : X_t = b\}$ and get for $x > b$, then

$$V_{(\infty,b)}(x,1) = \mathbb{E}_{x,1}\Big[\mathrm{e}^{-\delta_1\theta}V_{(\infty,b)}(b,1)1\!\!1_{[\theta\leq T_1]} + \mathrm{e}^{-\delta_1 T_1}\big(X_{T_1} - b + V_{(\infty,b)}(b,2)\big)1\!\!1_{[\theta>T_1]}\Big]\,.$$

The claim follows using the formulas (2.0.1) and (1.2.8) from Borodin and Salminen [9, pp. 295,252]:

$$\mathbb{E}_{x,1}\big[\mathrm{e}^{-\delta_1\theta}1\!\!1_{[\theta\leq T_1]}\big] = \mathrm{e}^{q_5(x-b)} \quad \text{and} \quad \mathbb{E}_{x,1}[\mathrm{e}^{-\delta_1 T_1}1\!\!1_{[\theta>T_1]}] = -\frac{\lambda_1\mathrm{e}^{q_5(x-b)}}{\lambda_1+\delta_1} + \frac{\lambda_1}{\lambda_1+\delta_1}\,,$$
$$\mathbb{E}_{x,1}\big[\mathrm{e}^{-\delta_1 T_1}(X_{T_1} - b)1\!\!1_{[\theta>T_1]}\big] = \frac{\lambda_1(x-b)}{\lambda_1+\delta_1} + \frac{\mu\lambda_1}{(\lambda_1+\delta_1)^2} - \frac{\mu\lambda_1\mathrm{e}^{q_5(x-b)}}{(\lambda_1+\delta_1)^2}\,.$$

It is straight forward to show that the obtained function solves $\mathcal{L}(V_{(\infty,b)})(x,1) = 0$. □

Using standard martingale techniques, one can prove that the return function of a barrier strategy solves a system of differential equations. Lemma 3.5 in combination with the proof of Proposition 3.1 ensures existence of the related expectations:

Lemma 3.6
For a barrier strategy with a barrier $b > 0$, the corresponding return function $V_{(\infty,b)}$ on $[0,b]$ is the unique solution to the system of differential equations

$$\mathcal{L}(V_{(\infty,b)})(x,1) = 0 \quad \text{and} \quad \mathcal{L}(V_{(\infty,b)})(x,2) = 0 \tag{8}$$

with $V_{(\infty,b)}(0,1)(0) = V_{(\infty,b)}(0,2) = 0$, $V'_{(\infty,b)}(b,2) = 1$. We have $V_{(\infty,b)} = (G_\infty, F_b)$, with

$$\begin{aligned}
F_b(x) &= \begin{cases} C_2(b)\psi(x) + C_4(b)\phi(x)\,, & x\in[0,b)\,,\\ x - b + C_2(b)\psi(x)(b) + C_4(b)\phi(b)\,, & x\geq b\,,\end{cases}\\
G_\infty(x) &= \begin{cases} AC_2(b)\psi(x) + BC_4(b)\phi(x)\,, & x\in[0,b)\,,\\ C_5(b)\mathrm{e}^{q_5 x} + \frac{\lambda_1}{\lambda_1+\delta_1}F_b(x) + \frac{\lambda_1\mu}{(\lambda_1+\delta_1)^2}\,, & x\geq b\,,\end{cases}
\end{aligned} \tag{9}$$

where the coefficients are given by

$$\begin{aligned}
C_2(b) &= \frac{-B\big(\frac{\phi''(b)}{\phi'(b)} - q_5\big) - \frac{\lambda_1}{\lambda_1+\delta_1}q_5}{-B\psi'(b)\big(\frac{\phi''(b)}{\phi'(b)} - q_5\big) + A\psi'(b)\big(\frac{\psi''(b)}{\psi'(b)} - q_5\big)}, \quad C_4(b) = \frac{1-\psi'(b)C_2(b)}{\phi'(b)}\\
C_5(b) &= \frac{(A-B)\psi'(b)C_2(b) + B - \frac{\lambda_1}{\lambda_1+\delta_1}}{q_5\mathrm{e}^{q_5 b}}\,.
\end{aligned} \tag{10}$$

It holds $G_\infty \in \mathcal{C}^2(\mathbb{R}_+)$, $F_b \in \mathcal{C}^2(\mathbb{R}_+\backslash\{b\})$. There is a unique $b^* \in (x_\psi, x_\phi)$ leading to $F''_{b^*}(b^*) = 0$ and consequently to $F_{b^*} \in \mathcal{C}^2(\mathbb{R}_+)$ with $C_2(b^*) > 0$, $C_4(b^*) > 0$, $C_5(b^*) < 0$ and

$$(A - B)C_2(b^*)\psi'(b^*)\Big(q_5 - \frac{\psi''(b^*)}{\psi'(b^*)}\Big) + \Big(B - \frac{\lambda_1}{\lambda_1 + \delta_1}\Big)q_5 = 0 \tag{11}$$

Proof: See Appendix. □

Next, we prove that there is a certain barrier strategy, which not only solves the connected differential equation, but also the HJB equation (3).

Lemma 3.7
The return function $V_{(\infty,b^*)} = (G, F)$, corresponding to the barrier strategy with a barrier pair (∞, b^*) and leading to $G, F \in \mathcal{C}^2(\mathbb{R}_+)$ solves the HJB equation (3).

Proof: In order to prove that (G, F) solve the HJB equation (3) we need to show that $G'(x) > 1$ on $[0,\infty]$, $F'(x) > 1$ on $[0, b^*)$ and $\mu + (\lambda_2 + \delta_2)F(x) + \lambda_2 G(x) < 0$ on (b^*,∞).
1.) Recall first that due to Lemma 3.6 it holds that $b^* \in (x_\psi, x_\phi)$, $C_2(b^*), C_4(b^*) > 0$, $C_5(b^*) < 0$, $F'(b^*) = 1$, $F''(b^*) = 0$ and G is twice continuously differentiable and fulfils on $[b^*,\infty)$:

$$G'_\infty(x) = C_5(b^*)q_5 e^{q_5 x} + \frac{\lambda_1}{\lambda_1 + \delta_1} > \frac{\lambda_1}{\lambda_1 + \delta_1} > 1 \quad \text{and} \quad G''_\infty(x) = C_5(b^*)q_5^2 e^{q_5 x} < 0\ .$$

2.) On the other hand, representation (9) yields $F''(x) < 0$ on $[0, x_\psi]$ and $G''(x) < 0$ on $[x_\psi, b^*]$. Hence, if $F''(\hat{x}) = 0$ for some $\hat{x} \in (x_\psi, b^*]$ then $G''(x) < 0$ on $[\hat{x}, b^*]$. Assume $\hat{x} := \inf\{x > 0 :\ F''(x) = 0\} \in (x_\psi, b^*)$. Then, it obviously holds $F'''(\hat{x}) \ge 0$. Based on the differential equation for F and properties of F and G, we can conclude that

$$(\lambda_2 + \delta_2)F'(\hat{x}) - \lambda_2 G'(\hat{x}) \ge 0 \quad \text{and} \quad (\lambda_2 + \delta_2)F''(\hat{x}) - \lambda_2 G''(\hat{x}) > 0.$$

This means that on $(\hat{x}, \hat{x} + \varepsilon)$ for some $\varepsilon > 0$ it holds $(\lambda_2 + \delta_2)F'(x) - \lambda_2 G'(x) > 0$ implying that F'' and consequently $(\lambda_2 + \delta_2)F''(x) - \lambda_2 G''(x)$ are positive on $(\hat{x}, \hat{x} + \varepsilon)$. Hence, $(\lambda_2 + \delta_2)F'(x) - \lambda_2 G'(x)$ will stay positive on $(\hat{x}, b^*)$ preventing F'' to become zero, which contradicts $F''(b^*) = 0$. Therefore, $F''(x) < 0$ and consequently $F'(x) > 1$ on $[0, b^*)$.
In a similar way one can show $G''(x) < 0$ and $G'(x) > \frac{\lambda_1}{\lambda_1+\delta_1}$ on $[0, b^*]$.
3.) We already know that F and G have the form (9), where $G' > \frac{\lambda_1}{\lambda_1+\delta_1} > 1$ on $\mathbb{R}_+$ and $F' > 1$ on $[0, b^*)$. Therefore, our aim is to insert the function $x - b^* + F(b^*)$ into the differential equation (8) and show that the obtained expression $L(x)$ is non-positive on $[b^*,\infty)$, where

$$L(x) := \mu - (\lambda_2 + \delta_2)\big(x - b^* + F(b^*)\big) + \lambda_2 G(x)\ .$$

Note first that $L(b^*) = \mathcal{L}(V_{(\infty,b^*)})(b^*, 2) = 0$ by definition. Deriving the differential equation for F on $[0, b^*)$ we get

$$0 = \frac{\sigma^2}{2}F'''(b^*-) + \mu F''(b^*) - (\lambda_2 + \delta_2)F'(b^*) + \lambda_2 G'(b^*) = \frac{\sigma^2}{2}F'''(b^*-) + L'(b^*)\ .$$

We know that $F''(x) < 0$ on $[0, b^*)$ and $F''(b^*) = 0$ meaning $F'''(b^*-) \ge 0$ and consequently $L'(b^*) = -(\lambda_2 + \delta_2) + \lambda_2 G'(b^*) \le 0$. Further, because $L''(x) = \lambda_2 G''(x) < 0$ on $[b^*,\infty)$ (see **1.)** above), it follows $L'(x) < 0$ on (b^*,∞) implying $L(x) < 0$ on (b^*,∞). □

Now, we are ready to prove

Theorem 3.8 (Verification Theorem)
In the case of classical ruin, a barrier strategy with the barrier pair (∞, b^*) is an optimal strategy and its return function $V_{(\infty,b^*)}$ is the value function V defined in (2).

Proof: See Appendix. □

3.2 Dependence of the value function on the switching intensities

In this section, we investigate the dependence of the value function and of the optimal strategy on the switching intensities. We restrict our analysis to the switching intensity from the state 2 into the state 1, i.e. from the state with a positive preference rate into the state with a negative preference, i.e. to λ_2, as the results for λ_1 can be obtained in a similar way.
We consider two different switching intensities λ_2 and $\tilde\lambda_2$. W.l.o.g. we assume that $\tilde\lambda_2 > \lambda_2$, all other parameters being unchanged, and denote the optimal barriers corresponding to $\tilde\lambda_2$ and λ_2 by $\tilde b^*$ and b^* respectively. The objective is to show that with increasing switching intensity λ_2, the optimal barrier will increase as well, i.e. in the newly adopted notation $\tilde b^* > b^*$.

Notation 3.9
In the following, by adding a tilde to the already existing notation we indicate the quantities corresponding to $\tilde\lambda_2$.

Lemma 3.10
For the function $V = (G, F)$, given in (7)*, it holds $G(x) > F(x)$ and $G'(x) > F'(x)$ on $(0, \infty)$.*

Proof: See Appendix. □

Lemma 3.11
For every $x > 0$ it holds $F(x) < \tilde F(x)$ and $G(x) < \tilde G(x)$.

Proof: Consider the Markov chain $\{r_s\}$ with the two states δ_1 and δ_2 and intensities λ_1, λ_2. By changing the time, we can construct a Markov chain with the states δ_1 and δ_2 but with the intensities λ_1 and $\tilde\lambda_2$ respectively. For that purpose define

$$\varphi(x) := 1\!\!1_{[x=\delta_1]} + \frac{\tilde\lambda_2}{\lambda_2} 1\!\!1_{[x=\delta_2]}.$$

Because $\varphi(x)$ is positive on the set $\{\delta_1, \delta_2\}$, the integral

$$\int_0^t \frac{1}{\varphi(r_s)}\,\mathrm{d}s = \int_0^t 1\!\!1_{[r_s=\delta_1]} + \frac{\lambda_2}{\tilde\lambda_2} 1\!\!1_{[r_s=\delta_2]}\,\mathrm{d}s$$

is strictly increasing in t, attains 0 at $t = 0$ and converges to infinity as $t \to \infty$. Since we assumed $\tilde\lambda_2 > \lambda_2$, $\frac{1}{\varphi(r_s)}$ is smaller than one, implying $\frac{\mathrm{d}}{\mathrm{d}t}\Big(t - \int_0^t 1\!\!1_{[r_s=\delta_1]} + \frac{\lambda_2}{\tilde\lambda_2} 1\!\!1_{[r_s=\delta_2]}\,\mathrm{d}s\Big) > 0$.
Hence, there is a unique strictly increasing function $\xi(t) > t$ such that $\int_0^{\xi(t)} \frac{1}{\varphi(r_s)}\,\mathrm{d}s = t$. Note that the infinitesimal generator of the Markov chain $\{r_s\}$ is given by

$$\mathfrak{A}f(x) = \lambda_1\big(f(\delta_2) - f(\delta_1)\big) 1\!\!1_{[x=\delta_1]} + \lambda_2\big(f(\delta_1) - f(\delta_2)\big) 1\!\!1_{[x=\delta_2]}\ .$$

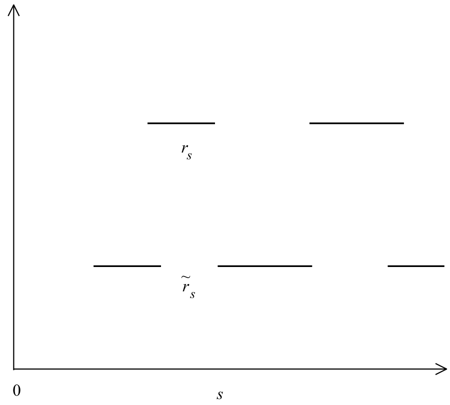

Figure 2: Occupation times in the state δ_1 for the processes $\{r_s\}$ and $\{\tilde{r}_s\}$.

Due to Volkonskii [29], the process $\tilde{r}_s := r_{\xi(s)}$ is a two-state Markov chain with the generator $\varphi(x)\mathfrak{A}f(x)$. It means $\tilde{r}_s$ is a Markov process with the states δ_1 and δ_2 and the corresponding intensities λ_1 and $\tilde{\lambda}_2$.
Now, what is the connection between $\{r_t\}$ and $\{\tilde{r}_t\}$? Denote by Λ_t and $\tilde{\Lambda}_t$ the time which the processes r_t and $\tilde{r}_t$ respectively have spent in the state δ_1 up to time t. In Figure 2 we see the times (represented as solid lines) spent by the processes $\{r_s\}$ and $\{\tilde{r}_s\}$ in the state δ_1. The solid lines for $\{r_s\}$ and $\{\tilde{r}_s\}$ have equal lengths. However, the intervals between the lines corresponding to the times spent in δ_2 have different lengths, due to $\tilde{\lambda}_2 > \lambda_2$.
Since $\xi(t) > t$ for every t and ω, it holds that $\Lambda_t \le \tilde{\Lambda}_t$, see Figure 2. Therefore, it holds

$$\int_0^t r_s \,\mathrm{d}s = \delta_2 t + (\delta_1 - \delta_2)\Lambda_t \ge \delta_2 t + (\delta_1 - \delta_2)\tilde{\Lambda}_t = \int_0^t \tilde{r}_s \,\mathrm{d}s \;.$$

Let now D be the barrier strategy corresponding to the barrier (b^*, ∞). Then, D is an admissible strategy for the pair $(W, \tilde{r}_s)$ because $\xi(t) > t$. Thus,

$$F(x) = \mathbb{E}_{x,2}\Big[\int_0^\tau \mathrm{e}^{-\int_0^t r_s}\,\mathrm{d}D_t\Big] < \mathbb{E}_{x,2}\Big[\int_0^\tau \mathrm{e}^{-\int_0^t \tilde{r}_s}\,\mathrm{d}D_t\Big] \le \tilde{F}(x) \;.$$

The same relationship holds true for G and $\tilde{G}$. □

Proposition 3.12
It holds $\tilde{b}^* > b^*$.

Proof: The proof uses on the one hand the relations given by the differential equations for $\tilde{F}, \tilde{G}$ and F, G and on the other hand Lemma 3.11. For the sake of clarity of presentation the technicalities are shifted to Appendix. □

We have seen that Assumption (4): $\delta_1 > -\frac{\lambda_1\delta_2}{\lambda_2+\delta_2}$ is crucial for the well-definiteness of the stated problem. Proposition 3.12 states that the bigger λ_2 the bigger would be the optimal barrier. We now examine what happens to the optimal barrier if and λ_2 approaches the critical value leading to $\delta_1 = -\frac{\lambda_1\delta_2}{\lambda_2^{\max}+\delta_2}$:

$$\lambda_2^{\max} := -\delta_2\Big(\frac{\lambda_1}{\delta_1} + 1\Big) \;,$$

with the interpretation $\lambda_2^{\max} := \infty$, if $\delta_1 = 0$.

Proposition 3.13
It holds $b^* \to \infty$ as $\lambda_2 \to \lambda_2^{\max}$.

Proof: We have seen in Proposition 3.12 that the optimal barrier $b^*(\lambda_2)$, as a function of λ_2, is strictly increasing and therefore has a unique limit. Assume $\lim\limits_{\lambda_2 \to \lambda_2^{\max}} b^*(\lambda_2) = \xi < \infty$. For every λ_2 the optimal barrier b^* leading to $F, G \in \mathcal{C}^2$ fulfils due to (11)

$$C_2(b^*)\psi'(b^*)\left(\frac{\psi''(b^*)}{\psi'(b^*)} - \frac{\phi''(b^*)}{\phi'(b^*)}\right) = -\frac{\phi''(b^*)}{\phi'(b^*)} .$$

Note that due to Lemma 3.4, $\frac{\phi'(x)}{\phi''(x)}$ uniformly converges to $-\frac{2\mu}{\sigma^2}$ and $\frac{\psi''(x)}{\psi'(x)}$ uniformly converges to a continuously differentiable function $\hat{\psi}^{\max}(x)$ on $[0, 2\xi]$ as $\lambda_2 \to \lambda_2^{\max}$. Also for $\lambda_2 = \lambda_2^{\max}$ one easily gets $B = \frac{\lambda_1}{\lambda_1+\delta_1}$. Thus, letting $\lambda_2 \to \lambda_2^{\max}$ in the above equality yields

$$\frac{\frac{2\mu}{\sigma^2}\frac{\lambda_1}{\lambda_1+\delta_1}}{\frac{2\mu}{\sigma^2}\frac{\lambda_1}{\lambda_1+\delta_1} + A\hat{\psi}^{\max}(\xi) + (\frac{\lambda_1}{\lambda_1+\delta_1} - A)q_5}\left(\hat{\psi}^{\max}(\xi) + \frac{2\mu}{\sigma^2}\right) = \frac{2\mu}{\sigma^2} ,$$

which is equivalent to $\hat{\psi}^{\max}(\xi) = q_5 < 0$. This is a contradiction as for any optimal barrier b^* it holds $\frac{\psi''(b^*)}{\psi'(b^*)} > 0$, see Lemma 3.4. Therefore, we can conclude that our claim holds true. □

Remark 3.14 (Dependence on λ_1.)
Considering the dependence of the value function and the optimal strategy on the switching intensity λ_1 and letting all other parameters unchanged leads to the following results.
Let $-\frac{\delta_1(\lambda_2+\delta_2)}{\delta_2} < \lambda_1 < \tilde{\lambda}_1$ and denote the corresponding value functions by $V = (G, F)$ and $\tilde{V} = (\tilde{G}, \tilde{F})$ respectively.

- *It holds $\tilde{G} < G$, $\tilde{F} < F$ on $(0, \infty)$ and $\tilde{b}^* < b^*$.*
- *The optimal barrier as a function of λ_1 converges to infinity as $\lambda_1 \to -\frac{\delta_1(\lambda_2+\delta_2)}{\delta_2}$.*

Thus, the smaller the intensity to switch from the state with a negative preference to the state with a positive preference the smaller will be the positive barrier in order to be able to collect dividends. It is a trade off between paying as much dividends as possible and avoiding ruin during the positive preference rate phase.
We omit the proof as it follows closely all steps presented above for λ_2. ■

Example 3.15
Choose $\delta_1 = -0.01$, $\delta_2 = 0.04$, $\lambda_1 = 0.5$, $\lambda_2 = 0.2$, $\mu = 0.1$, $\sigma = 0.3$. Then $\delta_1 > -\frac{\lambda_1\delta_2}{\lambda_2+\delta_2}$. The optimal strategy is given by the barrier $b^* = 1.729236$ in the phases with a positive interest rate and no dividend payments in the phases with a negative rate. The value function (G, F) along with the optimal barrier (dotted line) are illustrated on the lhs of Figure 3. Figure 1 in the beginning of this section shows a simulation of the ex-dividend process for this parameter set.
Now, we compare two value functions with different switching intensities $\tilde{\lambda}_2 = 0.5$ and $\lambda_2 = 0.2$ by keeping all other parameters unchanged. The switching intensities $\tilde{\lambda}_2$ and λ_2 lead to the optimal barriers $\tilde{b}^* = 2.200543$ and $b^* = 1.729236$ respectively. The lhs of Figure 3 illustrates the value functions (G, F), $(\tilde{G}, \tilde{F})$ with the corresponding barriers b^* and $\tilde{b}^*$ (dotted lines).

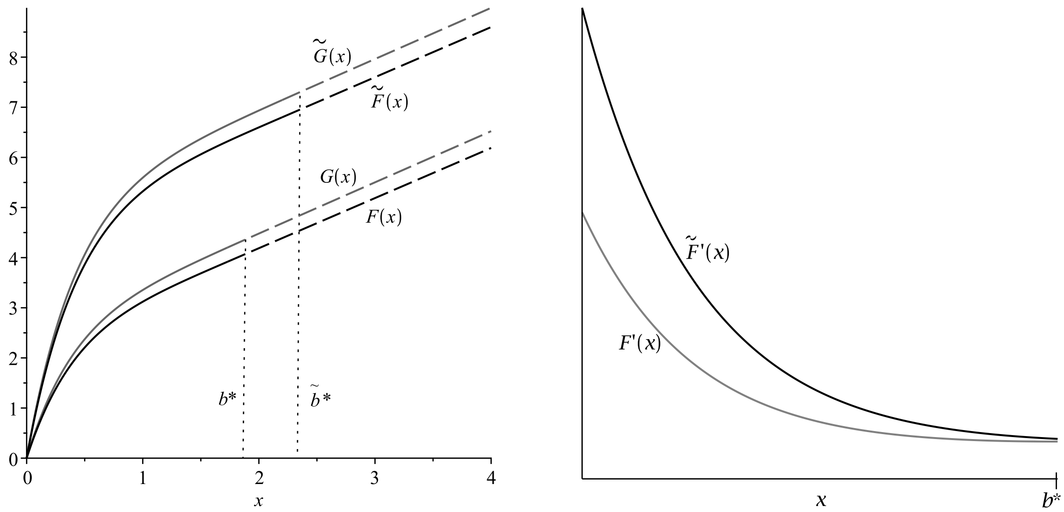

Figure 3: Lhs: The value functions and the barriers $\tilde{b}^*$ and b^* (dotted lines). Rhs: The derivatives $\tilde{F}'(x)$ and $F'(x)$ on $[0, b^*]$.

The rhs in Figure 3 shows that the derivative $\tilde{F}'$ lies above F' leading in this way to a bigger optimal barrier.
The dependence of the optimal barrier on the switching intensities λ_2 (lhs) and λ_1 (rhs) is illustrated in Figure 4, where the zeros x_ϕ and x_ψ are given as dashed lines above and below the curves $b^*(\lambda_2)$ (lhs), $b^*(\lambda_1)$ (rhs) respectively. As explained in Section 3.2, the optimal barrier b^* is strictly increasing in λ_2, decreasing in λ_1 and converges to infinity as λ_2 goes to $-\frac{\delta_2(\lambda_1+\delta_1)}{\delta_1}$ or λ_1 goes to $-\frac{\delta_1(\lambda_2+\delta_2)}{\delta_2}$. ■

Remark 3.16
Note that because the optimal barrier b^* from the pair (∞, b^*) fulfils $b^* \in (x_\psi, x_\phi)$, see Lemma 3.6, and $0 < x_\psi < x_\phi$, see Lemma 3.4, we can conclude that there are no parameter sets leading to $b^* = 0$. ■

4 Parisian Ruin

In this section, we introduce a random delay in declaring the event of ruin. Every time the surplus process becomes negative an independent random clock is activated. The ruin is said to have occurred if the running maximum of the surplus process stays negative during a given random period, which is assumed to be exponentially distributed with some parameter $\gamma > 0$. Thus, the expected delay is decreasing in γ, i.e. the bigger γ the shorter will be the expected ruin delay and vice versa. We do not allow for dividend payments if the surplus process is negative.
Formally: Let $T \sim \text{Exp}(\gamma)$ be independent of X and r and define the ruin time of the surplus

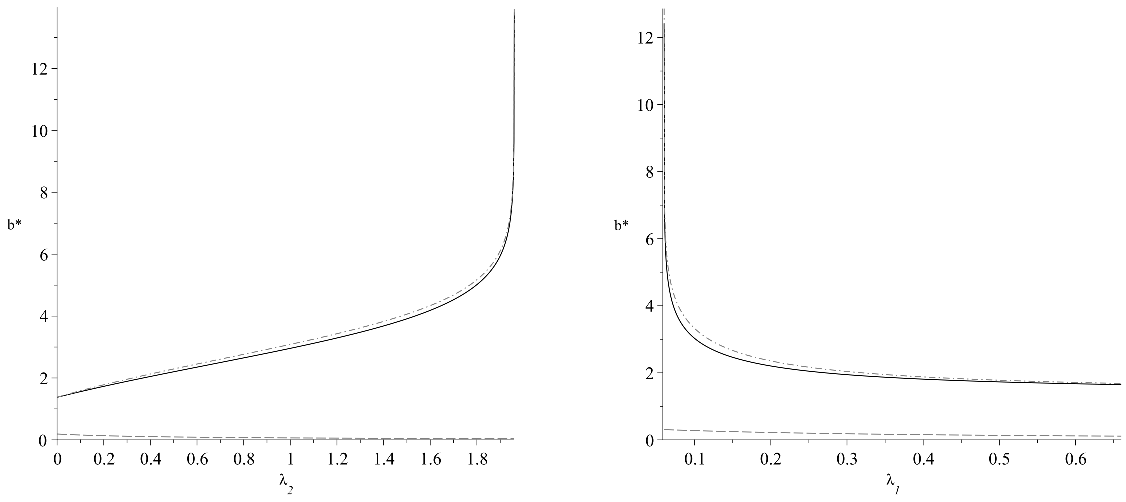

Figure 4: Dependence of the optimal barrier on the switching intensity λ_2 (lhs) and λ_1 (rhs).

process X^D under a dividend strategy D to be

$$\tau := \inf\{t+T : \sup_{t\le s\le t+T} X_s^D < 0\}\,,$$

The set of admissible strategies $\mathcal{D}_p$ contains all strategies D with $D_0 \ge 0$, D is cadlag, $\max\{X_{t-}^D, 0\} \ge D_t - D_{t-} \ge 0$ and D_t constant during $X_t^D < 0$ for all $t \le \tau$. The last condition reflects that no dividend payments are allowed during a phase with negative capital. We again target to maximise the expected discounted dividend payments until the time of ruin, given by (2). We assume that condition (4) is fulfilled. The proof of the well-definiteness of the problem follows closely the proof of Proposition 3.1. The HJB equation for the modified problem and $x \ge 0$ is then

$$\max\Big\{\frac{\sigma^2}{2}V''(x,i) + \mu V'(x,i) - (\delta_i+\lambda_i)V(x,i)(x) + \lambda_i V(x,j), 1 - V'(x,i)\Big\} = 0\,. \qquad (12)$$

Notation 4.1
In this section we define for a pair H of sufficiently smooth functions $H(x,1)$ and $H(x,2)$ and $i \ne j$:

$$\mathcal{L}^+ H(x,i) = \frac{\sigma^2}{2}H''(x,i) + \mu H'(x,i) - (\lambda_i+\delta_i)H(x,i) + \lambda_i H(x,j)\,, \quad x \ge 0\,,$$
$$\mathcal{L}^- H(x,i) = \frac{\sigma^2}{2}H''(x,i) + \mu H'(x,i) - (\lambda_i+\delta_i+\gamma)H(x,i) + \lambda_i H(x,j)\,, \quad x \le 0\,,$$

Remark 4.2
Note that because we do not allow any dividend payments if the surplus is negative, for $x < 0$, any return function V_D fulfils system of differential equations

$$\mathcal{L}^- V_D(x,1) = 0 \quad \text{and} \quad \mathcal{L}^- V_D(x,2) = 0\,, \qquad (13)$$

with boundary conditions $\lim_{x\to-\infty} V_D(x,i) = 0$ for $i \in \{1,2\}$. The general solutions are

$$V_D(x,1) = Ac_1 e^{u_1 x} + Bc_3 e^{u_3 x}, \qquad V_D(x,2)(x) = c_1 e^{u_1 x} + c_3 e^{u_3 x}\,, \qquad (14)$$

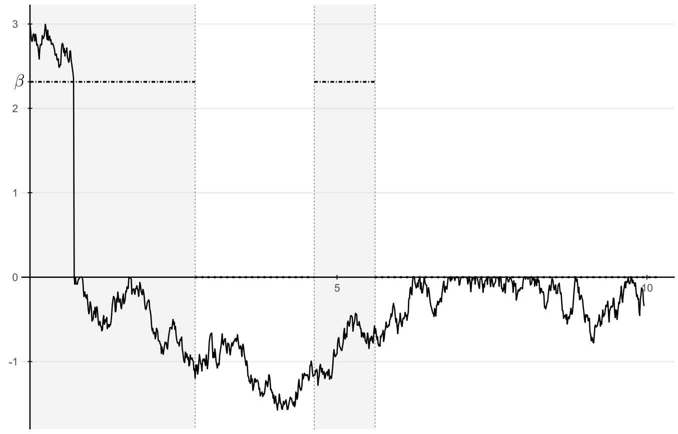

Figure 5: Ex-dividend process following a band strategy with parameter β (compare Example 4.13 below). The time intervals with $r_t = \delta_1$ are shown shaded in grey. The band parameter β is represented by the dash-dotted line.

where

$$u_1 := \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2(\Delta^+ + \gamma)}}{\sigma^2}, \qquad u_3 := \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2(\Delta^- + \gamma)}}{\sigma^2}.$$

The coefficients c_1 and c_3 will be uniquely determined by the requirement of $\mathcal{C}^1$-fit at zero . The proof follows closely the proof of Lemma 3.6 with dividends set to 0. ■

4.1 Barrier/band strategies and their return functions

In case of a Parisian ruin the exponential delay and hence the parameter γ will impact the optimal strategy. We conjecture that the optimal strategy will be of a **barrier** or **band** type and study the properties and the corresponding return functions. A band strategy is based on a sequence of intervals $\{(a_0, b_0], (a_1, b_1], ..., (a_n, b_n]\}$ and acts according to where the surplus is located. If the surplus is in a band, $(a_k, b_k]$, no dividends are paid. On the top of this region is the band $(b_k, a_{k+1}]$, where dividends are paid immediately such that the process is brought back to the frontier b_k between the both bands.

Definition 4.3
In the present paper, a strategy is called ***band strategy*** *with parameter $\beta > 0$ if no dividends are paid in the negative rate phases when the surplus lies in (β, ∞). Otherwise the entire positive surplus is paid out as dividends.*

Notation 4.4
• *We write $(G_\infty, F_b, g_\infty, f_b)$ for the return function, $V_{(\infty,b)}$, corresponding to a barrier (∞, b), where*

$$V_{(\infty,b)}(x,i) = \begin{cases} F_b(x) & : \text{if } x \in \mathbb{R}_+,\ i = 2 \\ G_\infty(x) & : \text{if } x \in \mathbb{R}_+,\ i = 1 \end{cases}, \qquad V_{(\infty,b)}(x,i) = \begin{cases} f_b(x) & : \text{if } x \in \mathbb{R}_-,\ i = 2 \\ g_\infty(x) & : \text{if } x \in \mathbb{R}_-,\ i = 1 \end{cases},$$

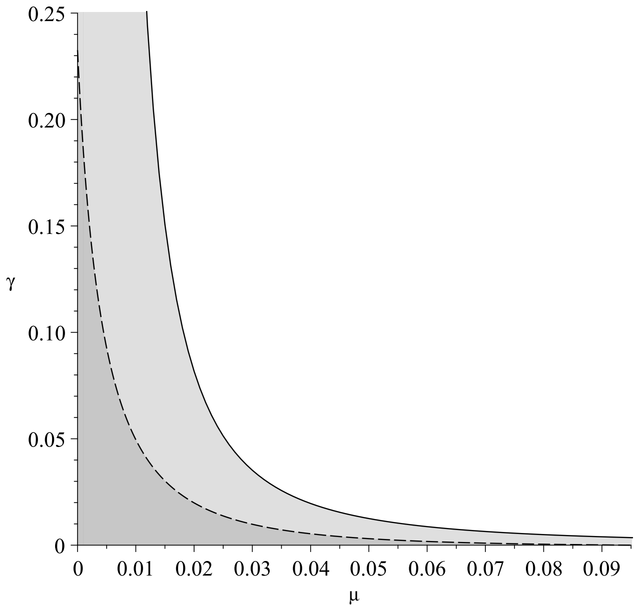

Figure 6: (γ, μ)-combinations for different optimal strategy types.

where $f_b(x) = c_1 e^{u_1 x} + c_3 e^{u_3 x}$ *and* $g_\infty(x) = Ac_1 e^{u_1 x} + Bc_3 e^{u_3 x}$ *as given in* (14).
• *The functions* ψ *and* ϕ *from Section 3 will be now replaced by the functions*

$$\psi_\gamma(x) := e^{q_1 x} - \frac{u_1 - q_1}{u_1 - q_2} e^{q_2 x}, \qquad \phi_\gamma(x) := e^{q_3 x} - \frac{u_3 - q_3}{u_3 - q_4} e^{q_4 x},$$

i.e. $F_b(x) = C_2(b)\psi_\gamma(x) + C_4(b)\phi_\gamma(x)$ *and* $G_\infty(x) = AC_2(b)\psi_\gamma(x) + BC_4(b)\phi_\gamma(x)$. *The alternating structure of* $\psi_\gamma(x)$ *and* $\phi_\gamma(x)$ *accounts for the new boundary condition at* 0*: instead of* $F_b(0) = G_\infty(0) = 0$ *we have a* $\mathcal{C}^1$*-fit of the pairs* F_b *and* f_b, G_∞ *and* g_∞.
• *Analogously, we write* $(G_\beta, F_0, g_\beta, f_0)$ *for the return function,* $V_{(\beta,0)}$, *corresponding to a band strategy* $(\beta, 0)$.

Remark 4.5
It holds that $u_1 > q_1 > 0$ *and* $u_3 > q_3 > 0$, *where* $\frac{\sigma^2}{2}u_1^2 + \mu u_1 = \Delta^+ + \gamma$ *and* $\frac{\sigma^2}{2}u_3^2 + \mu u_3 = \Delta^- + \gamma$. *The functions* ψ_γ *and* ϕ_γ *are strictly increasing solutions to the equations*

$$\frac{\sigma^2}{2}\psi_\gamma''(x) + \mu\psi_\gamma'(x) = \Delta^+\psi_\gamma(x), \qquad \frac{\sigma^2}{2}\phi_\gamma''(x) + \mu\phi_\gamma'(x) = \Delta^-\phi_\gamma(x).$$

with strictly increasing second derivatives. The (unique) zeros of ψ_γ'' *and* ϕ_γ'' *will be denoted in the following by* χ_ψ *and* χ_ϕ *respectively. Depending on the parameter set,* χ_ψ *and* χ_ϕ *can become negative. However, the relation* $\chi_\psi < \chi_\phi$ *holds true in all relevant cases, i.e.* $\gamma > \frac{(\Delta^-)^2\sigma^2}{2\mu^2}$ *(see the proof of Lemma 4.7).* ■

The following ***roadmap*** outlines the key steps in finding the optimal strategy for different values of γ.

• First, we assume that the optimal strategy is of a barrier or a band type in the sense of Definitions 3.2 and 4.3. We subdivide $\mathbb{R}_+$ in three intervals $(\bar{\gamma}(\mu), \infty)$, $(\underline{\gamma}(\mu), \bar{\gamma}(\mu)]$ and $(0, \underline{\gamma}(\mu)]$ with some functions $\bar{\gamma}(\mu) > \underline{\gamma}(\mu) > 0$ depending on the drift of the surplus process.

• We find the optimal barriers/bands for values of γ from each of the three intervals and

calculate the boundaries $\bar{\gamma}(\mu)$ (implicitly given by $\xi = 0$ with ξ in (16)) and $\underline{\gamma}(\mu)$ (implicitly given by $\eta = 0$ with η in (20)).
The three different areas are illustrated in Figure 6. The pairs (γ, μ) lying to the right of the solid curve, $\bar{\gamma}(\mu)$, lead to a barrier strategy with a positive barrier $b > 0$. The pairs (γ, μ) between the solid, $\bar{\gamma}(\mu)$, and the dashed, $\underline{\gamma}(\mu)$, curves correspond to a strategy with barrier $b = 0$. And finally, the area to the left of the dashed curve $\underline{\gamma}(\mu)$ produces a band strategy with a parameter $\beta > 0$.

• We show that the critical curve $\bar{\gamma}(\mu)$ exists for all parameter sets λ_1, δ_1, λ_2, δ_2 and σ, and $\underline{\gamma}(\mu)$ exists only under the additional condition that $|\delta_1|$ is not too large.

4.1.1 Short expected delay $\gamma \in (\bar{\gamma}(\mu), \infty)$

We know that the bigger γ the smaller will be the expected delay. Therefore, Parisian ruin with big γ values may be close to the case of a classical ruin considered in Section 3. Hence, it seems likely that the optimal strategy will be a barrier strategy with the barrier of the type (∞, b), $b > 0$ in the sense of Definition 3.2, i.e. no dividends are paid during the regime with a negative preference rate. In the following, we consider barrier strategies with a barrier pair (∞, b) and calculate the corresponding "optimality interval" $(\bar{\gamma}(\mu), \infty)$ in dependence of μ.

Remark 4.6
• *Since we consider barrier strategies from Section 3 we can use the results derived there. As it has been shown in Lemma 3.6, for a barrier strategy with a barrier (∞, b), for some $b > 0$, the corresponding return function $V_{(\infty,b)}(x, i) = (G_\infty, F_b)$ on $[0, \infty)$ has the form (9) with coefficients $C_2^s(b)$, $C_4^s(b)$, $C_5^s(b)$. According to the $\mathcal{C}^1$-requirement at zero, we get*

$$C_2^s(b) = \frac{-B\big(\frac{\phi_\gamma''(b)}{\phi_\gamma'(b)} - q_5\big) - \frac{\lambda_1}{\lambda_1+\delta_1}q_5}{-B\psi_\gamma'(b)\big(\frac{\phi_\gamma''(b)}{\phi_\gamma'(b)} - q_5\big) + A\psi_\gamma'(b)\big(\frac{\psi_\gamma''(b)}{\psi_\gamma'(b)} - q_5\big)}, \quad C_4^s(b) = \frac{1 - \psi_\gamma'(b)C_2(b)}{\phi_\gamma'(b)} \tag{15}$$

$$C_5^s(b) = \frac{-(B-A)\psi_\gamma'(b)c_2^s(b) + B - \frac{\lambda_1}{\lambda_1+\delta_1}}{q_5 e^{q_5 b}}\,, \; c_1^s(b) = \frac{q_1 - q_2}{u_1 - q_2}\cdot C_2^s(b)\,, \; c_3^s(b) = \frac{q_3 - q_4}{u_3 - q_4}\cdot C_4^s(b)\,.$$

• *The system of differential equations $\mathcal{L}^+(V_{(\infty,b)})(x, i) = 0$ for (G_∞, F_b) and the system $\mathcal{L}^-(V_{(\infty,b)})(x, i) = 0$ for (g_∞, f_b) indicate that a $\mathcal{C}^2$-fit at 0 is possible just if $F_b(0) = 0$, $G_\infty(0) = 0$, i.e. in case of an immediate ruin when the surplus reaches zero. Thus, due to $\gamma > 0$ the smooth fit at $x = 0$ will remain a $\mathcal{C}^1$-fit.* ■

In order to prove that the optimal barrier pair is given by (∞, b) for some parameters we need to specify the boundary $\bar{\gamma}$. Roughly speaking, we are searching for the values of γ allowing $F_b''(b) = 0$ for some $b > 0$. As we already know the structure of the coefficients $C_2^s(b)$ and $C_4^s(b)$, the following result is straight-forward.

Lemma 4.7
There is a unique function $\bar{\gamma}(\mu)$ on $(0, \infty)$, implicitly given by $\xi(\gamma, \mu) = 0$ where

$$\xi(\gamma, \mu) := \Big(A - \frac{\lambda_1}{\lambda_1 + \delta_1}\Big)\frac{\psi_\gamma'(0)}{\psi_\gamma''(0)} - \Big(B - \frac{\lambda_1}{\lambda_1 + \delta_1}\Big)\frac{\phi_\gamma'(0)}{\phi_\gamma''(0)} + \frac{B - A}{q_5}, \tag{16}$$

such that for all $\gamma > \bar{\gamma}(\mu)$ there exists a unique $b > 0$ leading to $F_b''(b) = 0$.

Proof: **1.)** If $\chi_\psi \geq 0$, which is equivalent to $\gamma \geq \frac{(\Delta^+)^2\sigma^2}{2\mu^2}$, then the existence of a $b > 0$ with $F_b''(b) = 0$ follows like in the proof of Lemma 3.6.
2.) Assume now $\chi_\psi < 0$ and $\chi_\phi > 0$, i.e. $\frac{(\Delta^-)^2\sigma^2}{2\mu^2} < \gamma < \frac{(\Delta^+)^2\sigma^2}{2\mu^2}$.
• From the proof of Lemma 3.6 we know already that in case $F_b''(b) = 0$ it should hold $b \in (0, \chi_\phi)$ and

$$C_2^s(b)\psi_\gamma'(b) = \frac{-\phi_\gamma''(b)/\phi_\gamma'(b)}{\psi_\gamma''(b)/\psi_\gamma'(b) - \phi_\gamma''(b)/\phi_\gamma'(b)} .$$

The proof of Lemma 3.6 yields us the existence and uniqueness of a $b > 0$ with $F_b''(b) = 0$ if $h(\gamma, \mu) < 0$, where

$$h(\gamma,\mu) := (A-B)C_2^s(0)\psi_\gamma'(0)\Big(q_5 - \frac{\psi_\gamma''(0)}{\psi_\gamma'(0)}\Big) + \Big(B - \frac{\lambda_1}{\lambda_1+\delta_1}\Big)q_5 , \tag{17}$$

see Lemma 3.6, (11).

• The condition $h(\gamma, \mu) < 0$ becomes $\xi(\gamma, \mu) > 0$, if we rearrange the terms like in the proof of Lemma 3.6. Thus, we are searching for the pairs (γ, μ) yielding $\xi(\gamma, \mu) > 0$.

• The function $\xi(\gamma, \mu)$ is strictly decreasing in γ, approaching $-\infty$ if $\gamma \nearrow \frac{(\Delta^+)^2\sigma^2}{2\mu^2}$ and ∞ if $\gamma \searrow \frac{(\Delta^-)^2\sigma^2}{2\mu^2}$. This follows because it holds that

$$\frac{\psi_\gamma''(0)}{\psi_\gamma'(0)} = q_1 + q_2 - q_1q_2/u_1 = \frac{-\mu + \Delta^+/u_1}{\sigma^2/2}, \quad \frac{\phi_\gamma''(0)}{\phi_\gamma'(0)} = q_3 + q_4 - q_3q_4/u_3 = \frac{-\mu + \Delta^-/u_3}{\sigma^2/2}, \tag{18}$$

and u_1, u_3 are increasing in γ.

• The behaviour of $\xi(\gamma, \mu)$ in the variable μ is not easy to obtain. Therefore, we estimate the derivative of ξ with respect to μ by the function ξ itself.
Note first that $\frac{\mathrm{d}}{\mathrm{d}\mu}q_5 = q_5/\sqrt{\mu^2 + 2\sigma^2(\lambda_1+\delta_1)}$ and $\frac{\mathrm{d}}{\mathrm{d}\mu}u_1 = -u_1/\sqrt{\mu^2 + 2\sigma^2(\Delta^+ + \gamma)}$. A similar result can be obtained for u_3, if one substitutes Δ^+ by Δ^-.
Second, it holds that $\Delta^- < \lambda_1+\delta_1 < \Delta^+$ and $-\mu+\frac{\Delta^+}{u_1} > 0$, $-\mu+\frac{\Delta^-}{u_3} < 0$ for $\gamma \in \big(\frac{(\Delta^-)^2\sigma^2}{2\mu^2}, \frac{(\Delta^+)^2\sigma^2}{2\mu^2}\big)$. This gives by using (18)

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\Big(\frac{\psi_\gamma''(0)}{\psi_\gamma'(0)}\Big) = \frac{-\sqrt{\mu^2 + 2\sigma^2(\Delta^+ + \gamma)} + \Delta^+/u_1}{\sqrt{\mu^2 + 2\sigma^2(\Delta^+ + \gamma)} \cdot \sigma^2/2} < \frac{1}{\sqrt{\mu^2 + 2\sigma^2(\lambda_1+\delta_1)}} \cdot \frac{\psi_\gamma''(0)}{\psi_\gamma'(0)} .$$

A similar result can be obtained for $\frac{\phi_\gamma''(0)}{\phi_\gamma'(0)}$.

• Assembling the above results yields then the following estimation

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\xi(\gamma,\mu) < -\frac{\xi(\gamma,\mu)}{\sqrt{\mu^2 + 2\sigma^2(\lambda_1+\delta_1)}} .$$

Thus, if $\xi(\hat\gamma, \hat\mu) \geq 0$ then it holds that $\frac{\mathrm{d}}{\mathrm{d}\mu}\xi(\hat\gamma, \hat\mu) < 0$. In other words, if $\xi(\hat\gamma, \hat\mu) = 0$ for some pair $(\hat\gamma, \hat\mu)$ then it holds that $\xi(\hat\gamma, \mu) < 0$ for all $\mu > \hat\mu$.
By the implicit function theorem, we can find a unique strictly decreasing $\bar\gamma(\mu) \in \big(\frac{(\Delta^-)^2\sigma^2}{2\mu^2}, \frac{(\Delta^+)^2\sigma^2}{2\mu^2}\big)$ leading to $\xi(\bar\gamma(\mu), \mu) = 0 = h(\bar\gamma(\mu), \mu)$, $h(\gamma, \mu) < 0$ for all $\gamma > \bar\gamma(\mu)$ and $h(\gamma, \mu) > 0$ for all $\frac{(\Delta^-)^2\sigma^2}{2\mu^2} < \gamma < \bar\gamma(\mu)$. In particular, it holds $\lim_{\mu\to 0}\bar\gamma(\mu) = \infty$ and $\lim_{\mu\to\infty}\bar\gamma(\mu) \to 0$. □

Corollary 4.8
If $\gamma > \bar{\gamma}(\mu)$, with $\bar{\gamma}(\mu)$ given in Lemma 4.7, then there exists a unique $b^* \in (\chi_\psi \vee 0, \chi_\phi)$ such that the return function $V_{(\infty,b^*)} = (G_\infty, F_{b^*})$, corresponding to the barrier strategy with a barrier pair (∞, b^*) fulfils: $F_{b^*}, G_\infty \in \mathcal{C}^2(\mathbb{R}_+)$ and solves the HJB equation (12).

Proof: The proof follows closely the proof of Lemma 3.7. □

4.1.2 Medium expected delay $\gamma \in (\underline{\gamma}(\mu), \bar{\gamma}(\mu)]$

In case the expected delay is relatively long, one would expect the optimal barriers to change. If γ lies below the critical boundary $\bar{\gamma}(\mu)$, a barrier $b > 0$ is not optimal anymore. It seems likely that the expected ruin delay is now so long that it becomes optimal to payout the entire positive surplus as dividends during the positive preference rate periods. However, the delay may not be long enough to induce dividend payments during the negative preference phases. The trade off between maximising dividends and avoiding ruin by accumulating more surplus at a rate μ is reflected in the boundary $\underline{\gamma}(\mu)$, to be determined in this section. We will see that the existence of the aforementioned boundary $\underline{\gamma}(\mu)$ depends on the negative preference rate δ_1. If $|\delta_1|$ is big enough it would be always optimal to wait during the negative preference phases in order to benefit from the accumulated negative rate at the switch time to the positive rate regime.
We start by considering the barrier strategy with the barrier pair $(\infty, 0)$, i.e. during the positive preference phases the entire positive surplus is paid as dividends, and no dividend payments occur during the negative preference intervals, and investigate when this strategy is optimal. Note that the return function $V_{(\infty,0)} = (G_\infty, F_0, g_\infty, f_0)$ fulfils, compare Lemma 3.6:

$$F_0(x) = x + c_1^m + c_3^m, \qquad G_\infty(x) = C_5^m e^{q_5 x} + \frac{\lambda_1}{\lambda_1+\delta_1}\Big(x + c_1^m + c_3^m + \frac{\mu}{\lambda_1+\delta_1}\Big),$$
$$f_0(x) = c_1^m e^{u_1 x} + c_3^m e^{u_3 x}, \qquad g_\infty(x) = Ac_1^m e^{u_1 x} + Bc_3^m e^{u_3 x}.$$

Using $\mathcal{C}^1$-fit at $x = 0$ yields

$$\begin{aligned} c_1^m u_1 &= \frac{\big(B - \frac{\lambda_1}{\lambda_1+\delta_1}\big)\big(\frac{1}{q_5} - \frac{1}{u_3}\big) + \frac{\lambda_1\mu}{(\lambda_1+\delta_1)^2}}{\big(B - \frac{\lambda_1}{\lambda_1+\delta_1}\big)\big(\frac{1}{q_5} - \frac{1}{u_3}\big) - \big(A - \frac{\lambda_1}{\lambda_1+\delta_1}\big)\big(\frac{1}{q_5} - \frac{1}{u_1}\big)}, \qquad c_3^m = \frac{1 - c_1^m u_1}{u_3}, \\ C_5^m &= \frac{(A-B)}{q_5} c_1^m u_1 + \frac{B - \frac{\lambda_1}{\lambda_1+\delta_1}}{q_5}. \end{aligned} \tag{19}$$

Remark 4.9
1.) *For all $\gamma > \bar{\gamma}(\mu)$, due to the $\mathcal{C}^2$-fit at the optimal barrier $b^* > 0$ it holds for $C_2(b)$ defined in (15), see the proof of Lemma 3.6: $C_2(b)\psi_\gamma'(b) = \frac{-\phi_\gamma''(b)/\phi_\gamma'(b)}{\psi_\gamma''(b)/\psi_\gamma'(b) - \phi_\gamma''(b)/\phi_\gamma'(b)}$.*
We know from Lemma 4.7 that for all $\gamma \le \bar{\gamma}(\mu)$ it holds $\xi(\gamma,\mu) \ge 0$, with ξ defined in (16). And it is a straightforward calculation to show that $\xi(\gamma,\mu) \ge 0$ is equivalent to $C_2(0)\psi_\gamma'(0) \ge \frac{-\phi_\gamma''(0)/\phi_\gamma'(0)}{\psi_\gamma''(0)/\psi_\gamma'(0) - \phi_\gamma''(0)/\phi_\gamma'(0)}$.
2.) *For $C_2^s(b)$ and $c_1^s(b)$ defined in (15) it holds, using $\psi_\gamma'(0) = u_1\frac{q_1-q_2}{u_1-q_2}$, (18) and $B - \frac{\lambda_1}{\lambda_1+\delta_1} =$*

$B\frac{\Delta^-}{u_3}$ and the corresponding relation for A:

$$c_1^s(0)u_1 = \psi_\gamma'(0)C_2^s(0) = \frac{-B\big(\frac{\phi_\gamma''(0)}{\phi_\gamma'(0)} - q_5\big) - \frac{\lambda_1}{\lambda_1+\delta_1}q_5}{-B\big(\frac{\phi_\gamma''(0)}{\phi_\gamma'(0)} - q_5\big) + A\big(\frac{\psi_\gamma''(0)}{\psi_\gamma'(0)} - q_5\big)} = c_1^m u_1 > 0\,.$$

3.) *For C_5^m it holds then that $C_5^m = C_5^s(0)$. Furthermore, for $\gamma = \bar\gamma(\mu)$ it holds $C_2^s(0)\psi_\gamma'(0) \ge 0$ implying*

$$C_5^m q_5 + \frac{\lambda_1}{\lambda_1+\delta_1} = C_2^s(0)\psi_\gamma'(0)(A-B) + B \le B < \frac{\lambda_2+\delta_2}{\lambda_2}\,.$$

This means that the cases of short and medium delay "fit together" at $\bar\gamma(\mu)$, which is the largest γ corresponding to a zero barrier. ■

In order to investigate when the barrier strategy with the barrier $(\infty, 0)$ is optimal, we need to insert the obtained the return function into the HJB equation (12).

Lemma 4.10
There exists a unique $\delta_1^* < 0$ such that for $\delta_1 \le \delta_1^*$ the pair (G_∞, F_0) fulfils $G_\infty, F_0 \in \mathcal{C}^2(\mathbb{R}_+)$ and solves the HJB equation (12) for all $\gamma < \bar\gamma(\mu)$. If $\delta_1 > \delta_1^*$ then there exist a $\mu_0 > 0$ and a unique strictly decreasing function $0 \le \underline{\gamma}(\mu) < \bar\gamma(\mu)$, $\mu \in [0,\mu_0]$ such that if $\gamma \in (\underline{\gamma}(\mu\wedge\mu_0), \bar\gamma(\mu)]$ the pair (G_∞, F_0) fulfils $G_\infty, F_0 \in \mathcal{C}^2(\mathbb{R}_+)$ and solves the HJB equation (12) , i.e.
1.) $G_\infty'(x) \ge 1$ for all $x \ge 0$,
2.) $\mu - (\lambda_2+\delta_2)F_0(x) + \lambda_2 G_\infty(x) \le 0$ for all $x \ge 0$.

Proof: 1.) • Note that $G_\infty''(x) = C_5^m q_5^2 e^{q_5 x}$. Thus, if $C_5^m q_5 \ge 0$ (i.e. $(A-B)c_1^m u_1 + B - \frac{\lambda_1}{\lambda_1+\delta_1} \ge 0$) then G_∞' is decreasing with $G_\infty'(x) \ge \lim\limits_{y\to\infty} G_\infty'(y) = \frac{\lambda_1}{\lambda_1+\delta_1} > 1$.

• If $C_5^m q_5 < 0$ (i.e. $(A-B)c_1^m u_1 + B - \frac{\lambda_1}{\lambda_1+\delta_1} < 0$) then G_∞' is increasing with $G_\infty'(x) > G_\infty'(0)$. We need to find the values of γ such that $G_\infty'(0) \ge 1$. Note that $G_\infty'(x) = C_5^m q_5 e^{q_5 x} + \frac{\lambda_1}{\lambda_1+\delta_1} \ge 1$ for all $x \ge 0$ holds true if $C_5^m q_5 + \frac{\lambda_1}{\lambda_1+\delta_1} \ge 1$. Hence, using the definition of C_5^m in (19), the crucial condition becomes $(A-B)c_1^m u_1 + B - 1 \ge 0$. Having in mind Definition (19), Remark 4.9 and noting that $q_5 - \frac{\psi_\gamma''(0)}{\psi_\gamma'(0)} = \frac{1}{\sigma^2/2}\Big(\frac{\lambda_1+\delta_1}{q_5} - \frac{\Delta^+}{u_1}\Big) < 0$ the condition for $G_\infty'(0) \ge 1$ becomes

$$0 \ge h(\gamma,\mu) - \big(B - \frac{\lambda_1}{\lambda_1+\delta_1}\big)q_5 + (B-1)\Big(q_5 - \frac{\psi_\gamma''(0)}{\psi_\gamma'(0)}\Big)\,,$$

where h is given in (17). Because $\frac{\lambda_1}{\lambda_1+\delta_1} > 1$ and $\psi_\gamma''(0) > 0$ as $\chi_\psi < 0$, we have

$$-\big(B - \frac{\lambda_1}{\lambda_1+\delta_1}\big)q_5 + (B-1)\Big(q_5 - \frac{\psi_\gamma''(0)}{\psi_\gamma'(0)}\Big) < -(B-1)\frac{\psi_\gamma''(0)}{\psi_\gamma'(0)} < 0\,,$$

meaning that for all $\gamma \ge \bar\gamma(\mu)$ it holds $h(\gamma,\mu) - \big(B - \frac{\lambda_1}{\lambda_1+\delta_1}\big)q_5 + (B-1)\Big(q_5 - \frac{\psi_\gamma''(0)}{\psi_\gamma'(0)}\Big) < 0$. Define now

$$\eta(\gamma,\mu) := \frac{(A-1)\big(B - \frac{\lambda_1}{\lambda_1+\delta_1}\big)\Big(\frac{1}{q_5} - \frac{1}{u_3}\Big) - (B-1)\big(A - \frac{\lambda_1}{\lambda_1+\delta_1}\big)\Big(\frac{1}{q_5} - \frac{1}{u_1}\Big)}{A-B} + \frac{\lambda_1\mu}{(\lambda_1+\delta_1)^2}, \quad (20)$$

where the dependence on γ and μ is hidden in u_1, u_3 and q_5. Since the denominator of $c_1^m u_1$ is negative, $(A-B)c_1^m u_1 + B - 1 \ge 0$ is equivalent to $\eta(\gamma,\mu) \ge 0$.

• There exists a unique value δ_1^*, such that $\eta(0,0) = 0$. For all $\delta > \delta_1^*$, there is a unique strictly decreasing function $\underline{\gamma}(\mu)$ defined on $[0,\mu_0]$, $\mu_0 > 0$, $\underline{\gamma}(\mu_0) = 0$ and $\underline{\gamma}(0) < \infty$ such that

$$\eta(\gamma,\mu) = \begin{cases} < 0 & : \gamma < \underline{\gamma}(\mu) \text{ and } \mu \in [0,\mu_0] \\ \ge 0 & : \text{otherwise.} \end{cases}$$

Since the proof of these facts is just a technical calculation of derivatives, we postpone it to the appendix.
We can conclude that $G'_\infty(x) \ge 1$ holds true for all $x \ge 0$ if $\bar{\gamma}(\mu) \ge \gamma \ge \underline{\gamma}(\mu \wedge \mu_0)$.

2.) It remains to show $\mu - (\lambda_2+\delta_2)F_0(x) + \lambda_2 G_\infty(x) \le 0$ for all $x \ge 0$.
Due to $\mathcal{C}^1$-fit at zero, one has $\mu - (\lambda_2+\delta_2)F_0(0) + \lambda_2 G_\infty(0) = -\frac{\sigma^2}{2} f_0''(0) + \gamma f_0(0)$. Further,

$$\begin{aligned} -\frac{\sigma^2}{2} f_0''(0) + \gamma f_0(0) &= c_1^m u_1 \big(-\frac{\sigma^2}{2} u_1 + \frac{\gamma}{u_1} + \frac{\sigma^2}{2} u_3 - \frac{\gamma}{u_3} \big) - \frac{\sigma^2}{2} u_3 + \frac{\gamma}{u_3} \\ &= c_1^m u_1 \big(\mu - \frac{\Delta^+}{u_1} - \mu + \frac{\Delta^-}{u_3} \big) + \mu - \frac{\Delta^-}{u_3} \\ &= -\frac{\sigma^2}{2} C_2^s(0) \psi_\gamma'(0) \left(\frac{\psi_\gamma''(0)}{\psi_\gamma'(0)} - \frac{\phi_\gamma''(0)}{\phi_\gamma'(0)} \right) - \frac{\sigma^2}{2} \frac{\phi_\gamma''(0)}{\phi_\gamma'(0)} \,. \end{aligned} \tag{21}$$

Because $\frac{\psi_\gamma''(0)}{\psi_\gamma'(0)} - \frac{\phi_\gamma''(0)}{\phi_\gamma'(0)} > 0$, see Lemma 3.4, we can conclude using Remark 4.9 1.) that for all $\gamma < \bar{\gamma}(\mu)$ the expression in (21) is negative. This means in particular, $\mu - (\lambda_2+\delta_2)F_0(0) + \lambda_2 G_\infty(0) < 0$ for all $\gamma < \bar{\gamma}(\mu)$.

If $C_5^m \ge 0$ then $G''_\infty(x) = \lambda_2 C_5^m q_5^2 e^{q_5 x} \ge 0$, i.e. $\frac{d^2}{dx^2}\Big(\mu - (\lambda_2+\delta_2)F_0(x) + \lambda_2 G_\infty(x)\Big) \ge 0$. Assumption (4) implies for the first derivative at ∞

$$\lim_{x\to\infty} \Big(-(\lambda_2+\delta_2) + \lambda_2 G'_\infty(x) \Big) = -(\lambda_2+\delta_2) + \lambda_2 \frac{\lambda_1}{\lambda_1+\delta_1} = \frac{\lambda_2+\delta_2}{\lambda_1+\delta_1} \Big(-\delta_1 - \frac{\lambda_1\delta_2}{\lambda_2+\delta_2} \Big) < 0\,,$$

meaning that $-(\lambda_2+\delta_2) + \lambda_2 G'_\infty(x) < 0$ for all $x \ge 0$. Thus, $\mu - (\lambda_2+\delta_2)F_0(x) + \lambda_2 G_\infty(x)$ is decreasing in x, negative at zero, and consequently negative for all $x \ge 0$.

If $C_5^m < 0$ then $G''_\infty(x) < 0$, meaning that the first derivative of $\mu - (\lambda_2+\delta_2)F_0(x) + \lambda_2 G_\infty(x)$ is decreasing in x.
Due to Remark 4.9, it holds $G'_\infty(0) < \frac{\lambda_2+\delta_2}{\lambda_2}$, leading to $-(\lambda_2+\delta_2) + \lambda_2 G'_\infty(0) < 0$ and consequently to $-(\lambda_2+\delta_2) + \lambda_2 G'_\infty(x) < 0$. Therefore, the biggest and still negative (see (21)) value of $\mu - (\lambda_2+\delta_2)F_0(x) + \lambda_2 G_\infty(x)$ is attained at $x = 0$. □

4.1.3 Long expected delay $\gamma \in (0, \underline{\gamma}(\mu)]$

In this section, we assume $\delta_1 > \delta_1^*$ and let the parameter γ be very small, i.e. the expected delay is almost infinite. Here, we consider band strategies with bands $\{(-\infty,0], (\beta,\infty)\}$ for some $\beta > 0$. It means, the entire positive surplus is paid as dividends in both phases. However during the negative preference regime no dividends are paid if the surplus lies in (β,∞). The idea behind such a strategy is the following. Since the expected delay is long one would expect, similar to the case $\gamma \in (\underline{\gamma}(\mu), \bar{\gamma}(\mu)]$, that the entire positive surplus is paid as dividends during the positive preference regime. During the negative preference periods, since we assume that the surplus rate fulfils $\mu < \mu_0$, i.e. is relatively low, it may be more optimal to pay the entire surplus as dividends if the initial value is so small that the process ruins easily. On the other hand, if the initial surplus is big the process may stay above the barrier until entering the positive preference regime and be entirely paid as dividends. Here, the accumulated negative preference increases the expected discounted dividends. Of course, the band parameter β will depend on γ and μ. The return function $V_{(\beta,0)} = (G_\beta, F_0, g_\beta, f_0)$ corresponding to a band strategy with a parameter β is:

$$F_0(x) = x + c_1^l + c_3^l, \qquad G_\beta(x) = \begin{cases} C_5^l e^{q_5 x} + \frac{\lambda_1}{\lambda_1+\delta_1}\Big(x + c_1^l + c_3^l + \frac{\mu}{\lambda_1+\delta_1}\Big) & : x > \beta \\ x + Ac_1^l + Bc_3^l & : x \le \beta \end{cases},$$

$$f_0(x) = c_1^l e^{u_1 x} + c_3^l e^{u_3 x}, \qquad g_\beta(x) = Ac_1^l e^{u_1 x} + Bc_3^l e^{u_3 x}$$

compare Lemma 3.6. Using $\mathcal{C}^1$-fit at $x = 0$ and at $x = \beta$ yields

$$c_1^l = \frac{B-1}{(B-A)u_1} > 0\,, \qquad c_3^l = \frac{1-A}{(B-A)u_3} > 0\,, \qquad C_5^l = \frac{\delta_1}{(\lambda_1+\delta_1)q_5 e^{q_5\beta}} > 0$$

and the barrier β is given in dependence of the function $\eta(\gamma,\mu)$ defined in (20):

$$\beta = \frac{1}{\delta_1}\Big(\lambda_1 f_0(0) - (\lambda_1+\delta_1)g_\beta(0) + \frac{\delta_1}{q_5} + \frac{\lambda_1\mu}{\lambda_1+\delta_1}\Big) = \frac{(\lambda_1+\delta_1)^2}{\delta_1}\eta(\gamma,\mu)\,. \tag{22}$$

Note that $\beta = 0$ if $\gamma = \underline{\gamma}(\mu)$ and $\beta > 0$ for $\gamma < \underline{\gamma}(\mu)$ because $\delta_1 < 0$.

Lemma 4.11
The pair (G_β, F_0) solves the HJB equation (12) if
1.) $G'_\beta(x) \ge 1$ for all $x \ge \beta$.
2.) $\mu - (\lambda_1+\delta_1)G_\beta(x) + \lambda_1 F_0(x) \le 0$ for $x \in [0,\beta]$.
3.) $\mu - (\lambda_2+\delta_2)F_0(x) + \lambda_2 G_\beta(x) \le 0$ for $x \ge 0$.
These conditions are fulfilled if $\delta_1 > \tilde{\delta}_1$ and $\gamma \in (0, \underline{\gamma}(\mu\wedge\mu_0)]$. In this case, $(G_\beta, F_0) \in \mathcal{C}^2(\mathbb{R}_+\backslash\beta)$.

Proof: 1.) Because $C_5^l > 0$ and $G'_\beta(\beta) = 1$, we can conclude $G'_\beta(x) \ge 1$ for $x \ge \beta$.
2.) Since $\delta_1 < 0$, the biggest value of $\mu - (\lambda_1+\delta_1)G_\beta(x) + \lambda_1 F_0(x)$ on $[0,\beta]$ is attained at β. The function G_β fulfils $G''_\beta(\beta) = q_5^2 C_5^l e^{q_5\beta} > 0$ and for $x \le \beta$

$$0 > -\frac{\sigma^2}{2}G''_\beta(\beta) = \mu - (\lambda_1+\delta_1)G_\beta(\beta) + \lambda_1 F_0(\beta) \ge \mu - (\lambda_1+\delta_1)G_\beta(x) + \lambda_1 F_0(x)\,.$$

3.) Since $\delta_2 > 0$, the biggest value of $\mu - (\lambda_2+\delta_2)F_0(x) + \lambda_2 G_\beta(x)$ on $[0,\beta]$ is attained at 0. Therefore, using the $\mathcal{C}^1$-fit at zero:

$$\begin{aligned} \mu - (\lambda_2+\delta_2)F_0(0) + \lambda_2 G_\beta(0) &= -c_1^l u_1\Big(\frac{\sigma^2 u_1}{2} - \frac{\gamma}{u_1}\Big) - c_3^l u_3\Big(\frac{\sigma^2 u_3}{2} - \frac{\gamma}{u_3}\Big) \\ &= -\frac{B-1}{(B-A)}\Big(-\mu + \frac{\Delta^+}{u_1}\Big) + \frac{A-1}{(B-A)}\Big(-\mu + \frac{\Delta^-}{u_3}\Big) \end{aligned}$$

Because $\frac{B-1}{B-A}, \frac{1-A}{B-A} > 0$ and u_1, u_3 are increasing in γ, we can conclude that the above expression is increasing in γ, attaining its maximum at $\gamma = \underline{\gamma}(\mu)$. Note that for $\gamma = \underline{\gamma}(\mu)$ it holds $c_1^m u_1 = \frac{B-1}{B-A} = c_1^l u_1$ by definition of $\underline{\gamma}(\mu)$. Lemma 4.10, (21) yields the desired result. □

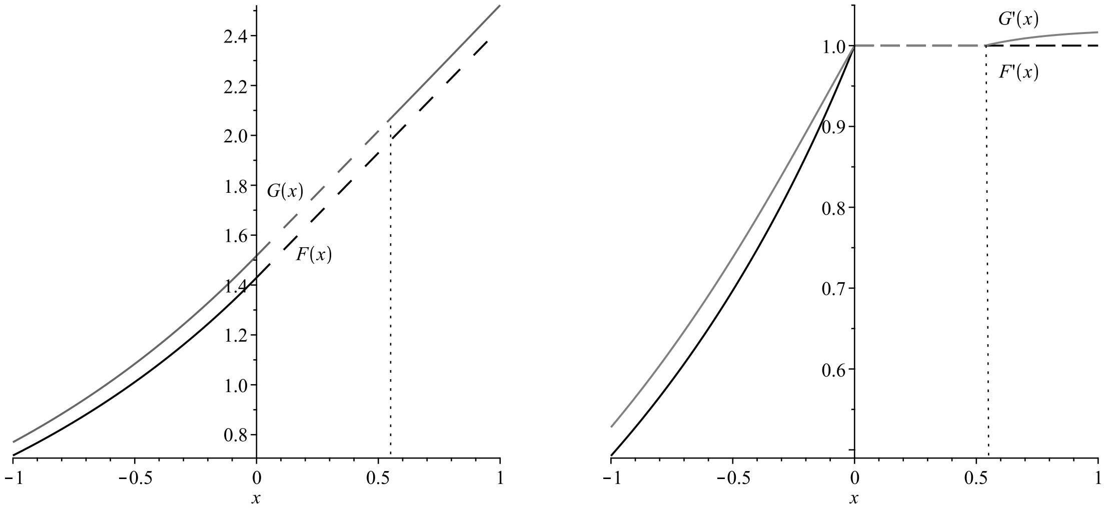

Figure 7: Lhs: The value functions G and F and the barrier (dotted line). Rhs: The derivatives $G'(x)$ and $F'(x)$ and the barrier (dotted line).

4.1.4 Verification Theorem and Numerical Example

We summarise our findings with the following

Theorem 4.12 (Verification Theorem)
In the case of Parisian ruin, the optimal strategy depends on the parameter γ of the exponential clock:
1.) If $\gamma \in (\bar{\gamma}(\mu), \infty)$, a barrier strategy with the barrier pair (∞, b^*) is an optimal strategy and its return function $V_{(\infty,b^*)}$ is the value function V in (2).
2.) If $\delta_1 \le \delta_1^*$ and $\gamma \in (0, \bar{\gamma}(\mu))$ or $\delta_1 > \delta_1^*$ and $\gamma \in (\underline{\gamma}(\mu \wedge \mu_0), \bar{\gamma}(\mu))$, a barrier strategy with the barrier pair $(\infty, 0)$ is an optimal strategy and its return function $V_{(\infty,0)}$ is the value function V in (2).
3.) If $\delta_1 > \delta_1^*$ and $\gamma \in (0, \underline{\gamma}(\mu \wedge \mu_0))$, a band strategy with the parameters $(\beta, 0)$ is an optimal strategy and its return function $V_{(\beta,0)}$ is the value function V in (2).

Proof: See Appendix. □

Example 4.13
Consider the parameters from Example 3.15: $\delta_1 = -0.01$, $\delta_2 = 0.04$, $\lambda_1 = 0.5$, $\lambda_2 = 0.2$, $\sigma = 0.3$. The separating curve $\underline{\gamma}(\mu)$ is given in Figure 6, dashed line. The pairs (γ, μ) lying above the curve indicate that it is not optimal to pay dividends if the interest rate is negative. The pairs below the curve imply the existence of a band $[0, \beta]$ such that during the negative preference phases it is optimal to pay out the whole positive surplus as dividends if the surplus lies in $[0, \beta]$ and to wait until the phase with a positive rate if the surplus is outside $[0, \beta]$. The separating curve $\underline{\gamma}(\mu)$ is strictly decreasing in μ, fulfils $\underline{\gamma}(0.232581) = 0$ and $\underline{\gamma}(0) = 0.09494$.
Thus, choosing $\mu = 0.02$ and $\gamma = 0.01$ leads to $\beta = 2.3145$. The functions G, F, f, g, their derivatives and the parameter β (dotted line) are illustrated in Figure 7. A simulation of the ex-dividend process is displayed in Figure 5. ■

5 Conclusion

In this paper, we have considered optimal dividend strategies in a regime switching model with two states, one of which allows a negative preference rate or technical discounting.
We have shown that a barrier strategy is optimal in case ruin is defined as the first time the ex-dividend surplus is negative. In the positive regime, b^* marks the critical value below which dividend payments would be to risky and might cause premature ruin. Since the value of dividends is perceived as increasing with time in the negative regime, the insurer waits until the end of the current phase to pay out a lump sum. We have shown that this barrier is always strictly positive. This is not surprising because paying out the full surplus would lead to immediate ruin, and, by our assumption $\mu > 0$, the insurer expects the future business to be profitable.
In addition, we were able to prove a monotone relation between the switching intensities and the dividend barriers using differential equations connected to the problem. As it turns out, the more time (on average) is spent in the negative regime, the larger the barrier. One explanation for this may be that due to long and frequent "collecting"-phases the value of future dividends is perceived as higher than the value of immediate payments.
The second part of our analysis has dealt with the case in which the observation period ends when an excursion of the ex-dividend process to the negative half plane outlasts an independent, $\exp(\gamma)$-distributed time (i.e. Parisian ruin). In line with the economic intuition, one of the key assumptions is that no dividend payments are allowed if the surplus is negative. We have shown that optimal dividend strategies are of one of three different types, depending on the average surplus and the expected time limit (expressed through the pair (μ, γ)). We have proven existence of a curve $(\mu, \overline{\gamma}(\mu))$ which is asymptotic to the μ- and the γ-axis, such that a barrier strategy with a strictly positive barrier is optimal for all pairs (μ, γ) above the curve. This is the case which is closest to the case of regular ruin, which is not surprising: If γ is large, this means that the "reprieve" for recovery after the surplus becomes negative is very short. Moreover, the larger the general drift of the process, the shorter the duration of a negative excursion will be. This also explains why the critical $\overline{\gamma}(\mu)$ is decreasing with growing drift μ. This curve exists for all parameter sets.
A remarkable result of our analysis of Parisian ruin is that a barrier strategy with barrier $b^* = 0$ is actually possible. In fact, all points lying on the critical curve correspond to a zero barrier. This means that the insurer retains and collects all payments during the negative preference regime and then pays out all earnings at the first regime switch as a lump sum.
Now the optimal behaviour for pairs (μ, γ) which lie below the curve, i.e. there is a relatively long time to recover, depends on the preference rate. If the weight of the negative preference is rather large, meaning that $\delta_1 \leq \delta_1^*$, the zero barrier remains optimal for all pairs (μ, γ) below the critical curve. A remarkable and somewhat surprising result is that if the negative preference rate only has a mild impact (i.e. $\delta_1 > \delta_1^*$), the optimal strategy can take a new shape. We have shown that in this case, there exists another curve $(\mu, \underline{\gamma}(\mu))$ with $\underline{\gamma}(\mu) < \overline{\gamma}(\mu)$, such that in between the curves the zero barrier is optimal. Below the curve $(\mu, \underline{\gamma}(\mu))$, while the zero barrier remains optimal in the positive regime, dividends are paid out according to a band strategy in the negative regime. Under this strategy, if the process starts above the band parameter β in the negative regime, the insurer waits until either β is reached or the first switch occurs. From this time on, dividends are paid out at a zero barrier in both regimes. Likely, this type of strategies arises because of two influential factors. The first factor is that

with a low value of γ, the ruin event occurs very late, such that the insurer will aim to pay out as much dividends as possible. A natural extension to this strategy would be to pay out dividends in the positive preference regime, even if the surplus is negative. However, this behaviour is not allowed. Therefore, the second factor is the extent of the negative preference in the corresponding regime. If the positive impact of waiting to pay out dividends is not too large, the insurer resorts to payments in the negative regime.
As the techniques we have used are predominantly based on the relation of the corresponding differential equations, an interesting and promising topic for future research would be to extend these results to more general models.
An interesting and important extension to the model would be to include jumps in the surplus, for example, by considering a compound Poisson process. In this case, the HJB-equation becomes a system of two integro-differential equations. For instance, Lu and Li, [22, Section 6], characterise the return function of a constant barrier dividend strategy for a classical risk model in a regime switching environment by such a system of equations. The case of exponentially distributed claim sizes (in a classical risk model) may be a good starting point for further investigations of the optimisation problem incorporating a negative interest rate phase. However, due to the expected complexity of the HJB equation in the compound Poisson setting, the results and proof techniques used in this paper (which heavily rely on the differential equations) cannot be easily transferred. In fact, the assumption of exponential claims, would produce a system of differential equations of 3rd order. As stated in the introduction, an extension of these methods to the case of deterministic Parisian delay is not trivial. Here, a consideration of Erlangian time horizons (which can be used to approximate deterministic times) as in [19] might be a starting point to re-establish the relation to the deterministic case. Still, it should be noted that under this assumption the memoryless property is lost. This means, the related differential equations change substantially in this case, as they additionally depend on the time parameter.

6 Appendix

Proof of Lemma 3.4:
1.) We know that $\psi(x) = \mathrm{e}^{q_1 x} - \mathrm{e}^{q_2 x}$ and $\phi(x) = \mathrm{e}^{q_3 x} - \mathrm{e}^{q_4 x}$.
It is clear that $\psi'(x), \phi'(x), \psi'''(x), \phi'''(x) > 0$ for all $x > 0$ and $\psi''(0), \phi''(0) < 0$. Thus, if ψ'' has a zero, $\hat{x}$, this zero is unique. The existence of $\hat{x}$ follows from $\lim\limits_{x\to\infty} \psi''(x) = \infty$.
In the same way, we get the existence of a unique zero of ϕ''. It is straightforward to explicitly calculate x_ψ and x_ϕ and to show $0 < x_\psi < x_\phi$. In step 2.) below, we give an alternative proof of $0 < x_\psi < x_\phi$ based on the corresponding differential equations. The advantage of this method is that it does not require any technical calculations.
2.) Now, divide the derived differential equations for ψ' and ϕ' by ψ' and ϕ' respectively:

$$\frac{\sigma^2}{2}\frac{\psi'''(x)}{\psi'(x)} + \mu\frac{\psi''(x)}{\psi'(x)} = \Delta^+ , \quad \frac{\sigma^2}{2}\frac{\phi'''(x)}{\phi'(x)} + \mu\frac{\phi''(x)}{\phi'(x)} = \Delta^- .$$

Let $\tilde{\psi}(x) := \frac{\psi''(x)}{\psi'(x)}$ and $\tilde{\phi}(x) := \frac{\phi''(x)}{\phi'(x)}$. Then, the above differential equations become

$$\frac{\sigma^2}{2}\big(\tilde{\psi}'(x) + \tilde{\psi}(x)^2\big) + \mu\tilde{\psi}(x) = \Delta^+ , \quad \frac{\sigma^2}{2}\big(\tilde{\phi}'(x) + \tilde{\phi}(x)^2\big) + \mu\tilde{\phi}(x) = \Delta^-$$

with the boundary condition $\tilde{\psi}(0) = -\frac{2\mu}{\sigma^2} = \tilde{\phi}(0)$. Comparison theorem, see Walter [30, p. 138], yields then $\tilde{\psi}(x) > \tilde{\phi}(x)$ for all $x > 0$. In particular, we can conclude that the zero of ψ'', denoted by x_ψ, is positive and strictly smaller than the zero of ϕ'', denoted by x_ϕ.
3.) It is easy to see that $q_2 < \tilde{\psi}(x) < q_1$ and $q_4 < \tilde{\phi}(x) < q_3$ for all $x \geq 0$. Since the differential equations for $\tilde{\phi}$ and $\tilde{\psi}$ can be written as

$$\tilde{\psi}'(x) + \big(\tilde{\psi}(x) - q_1\big)\big(\tilde{\psi}(x) - q_2\big) = 0\ , \quad \tilde{\phi}'(x) + \big(\tilde{\phi}(x) - q_3\big)\big(\tilde{\phi}(x) - q_4\big) = 0$$

we can conclude that $\tilde{\psi}', \tilde{\phi}' > 0$.
4.) A simple calculation shows that Δ^+ is increasing in λ_2. Comparison theorem Walter [30, p. 138] yields then that $\psi'_{\lambda_2} < \psi'_{\tilde{\lambda}_2}$ when ψ_{λ_2} and $\psi_{\tilde{\lambda}_2}$ correspond to the parameters λ_2 and $\tilde{\lambda}_2$ respectively and $\lambda_2 < \tilde{\lambda}_2$.
Note that $\big(\tilde{\psi}_{\lambda_2}(x)\big)_{0\leq\lambda_2<-\frac{\delta_2(\lambda_1+\delta_1)}{\delta_1}}$ is a monotonically increasing sequence of continuous functions that converges pointwise on every compact set to the continuous function $\tilde{\psi}_{-\frac{\delta_2(\lambda_1+\delta_1)}{\delta_1}}(x)$. Dini's theorem yields the uniform convergence.
In a similar way, one can show that $\big(\tilde{\phi}_{\lambda_2}(x)\big)_{0\leq\lambda_2<-\frac{\delta_2(\lambda_1+\delta_1)}{\delta_1}}$ converges to $-\frac{2\mu}{\sigma^2}$ uniformly on compacts. □

Proof of Lemma 3.6:
1.) It is straight forward to check that the functions F_b and G_∞ defined in (9) solve the system of differential equations (8) on $[0, b]$. The coefficients given in (10) result as the unique solutions to the equation system $F_b'(b) = 1$, $G_\infty'(b-) = C_5(b)q_5 e^{q_5 b} + \frac{\lambda_1}{\lambda_1+\delta_1}$ and $G_\infty''(b-) = C_5(b)q_5^2 e^{q_5 b}$.
2.) We show now the existence of a unique b^* such that the corresponding return function $V_{(\infty,b^*)} = (G, F)$ fulfils $G, F \in \mathcal{C}^2(\mathbb{R}_+)$. Solving the equations $F'(b-) = 1$, $F''(b-) = 0$, $G'(b-) = G'(b+)$ and $G''(b-) = G''(b+)$ yields

$$C_2(b)\psi'(b) = \frac{-\frac{\phi''(b)}{\phi'(b)}}{\frac{\psi''(b)}{\psi'(b)} - \frac{\phi''(b)}{\phi'(b)}} \quad \text{and} \quad (A-B)C_2(b)\psi'(b)\Big(q_5 - \frac{\psi''(b)}{\psi'(b)}\Big) + \Big(B - \frac{\lambda_1}{\lambda_1+\delta_1}\Big)q_5 = 0\ . \tag{23}$$

Note that due to Lemma (3.4) it holds $\frac{\psi''(b)}{\psi'(b)} - \frac{\phi''(b)}{\phi'(b)} > 0$ for all $b > 0$. Therefore, defining

$$\alpha(b) := -q_5\Big(A - \frac{\lambda_1}{\lambda_1+\delta_1}\Big)\frac{\phi''(b)}{\phi'(b)} + q_5\Big(B - \frac{\lambda_1}{\lambda_1+\delta_1}\Big)\frac{\psi''(b)}{\psi'(b)} + (A-B)\frac{\phi''(b)}{\phi'(b)}\frac{\psi''(b)}{\psi'(b)}$$

we have $\alpha(b) = 0$ is equivalent to (23). Since $\psi', \phi' > 0$, $x_\psi < x_\phi$ and it holds $\alpha(x_\psi) > 0$, $\alpha(x_\phi) < 0$, we can consider $\frac{\psi'(b)\phi'(b)}{\psi''(b)\phi''(b)}\alpha(b)$ on $\mathbb{R}_+\backslash\{x_\psi, x_\phi\}$. Due to Lemma 3.4 the functions $\frac{\psi'(b)}{\psi''(b)}$ and $\frac{\phi'(b)}{\phi''(b)}$ are strictly decreasing, so that

$$\tilde{\alpha}(b) := \frac{\psi'(b)\phi'(b)}{\psi''(b)\phi''(b)}\alpha(b) = -q_5\Big(A - \frac{\lambda_1}{\lambda_1+\delta_1}\Big)\frac{\psi'(b)}{\psi''(b)} + q_5\Big(B - \frac{\lambda_1}{\lambda_1+\delta_1}\Big)\frac{\phi'(b)}{\phi''(b)} + (A-B)$$

is strictly increasing in b on $(0, x_\psi)$, (x_ψ, x_ϕ) and (x_ϕ, ∞). Because for all $b > x_\phi$ it holds $\psi'', \phi'' > 0$ one has $\tilde{\alpha}(b) < 0$ on (x_ϕ, ∞), i.e. no zeros. For $b = 0$ we have, see the proof of Lemma 3.4 2.): $\frac{\psi'(0)}{\psi''(0)} = \frac{\phi'(0)}{\phi''(0)} = -\frac{\sigma^2}{2\mu}$ and consequently using $A - B < 0$ and $q_5 < 0$

$$\tilde{\alpha}(0) = q_5\Big(A - \frac{\lambda_1}{\lambda_1+\delta_1}\Big)\frac{\sigma^2}{2\mu} - q_5\Big(B - \frac{\lambda_1}{\lambda_1+\delta_1}\Big)\frac{\sigma^2}{2\mu} + (A-B) = \frac{(A-B)(\lambda_1+\delta_1)}{\mu q_5} > 0\ .$$

Therefore, no zeros can exist in the interval $(0, x_\psi)$. On the other hand, because $\lim\limits_{b\searrow x_\psi} \tilde{\alpha}(b) = -\infty$ and $\lim\limits_{b\nearrow x_\phi} \tilde{\alpha}(b) = \infty$, we can conclude that there is a unique $b^* \in (x_\psi, x_\phi)$ fulfilling (23). It is a direct consequence of $b^* \in (x_\psi, x_\phi)$ that $1 > \psi'(b^*)C_2(b^*) > 0$ with $C_2(b)$ given in (23). The coefficients $C_5(b^*)$ and $C_4(b^*)$ given in (10) yield $\phi'(b^*)C_2(b^*) = 1 - \psi'(b^*)C_2(b^*) > 0$ and

$$C_5(b^*) = \frac{(A-B)\psi'(b^*)C_2(b^*) + B - \frac{\lambda_1}{\lambda_1+\delta_1}}{q_5 \mathrm{e}^{q_5 b^*}} = \frac{(A-B)\psi''(b^*)C_2(b^*) + B - \frac{\lambda_1}{\lambda_1+\delta_1}}{q_5^2 \mathrm{e}^{q_5 b^*}} < 0\,.$$

3.) It remains to show that for a $b > 0$ the function $V_{(\infty,b)} = (G_\infty, F_b)$ as given in (9) is the return function corresponding to a barrier strategy with a barrier pair (∞, b). This is done by applying a suitable version of Itô's formula to the function $f(x,y,\alpha) = \mathrm{e}^{-x} V_{(\infty,b)}(y,\alpha)$ and the three dimensional process consisting of the discounting, the surplus and the switching mechanism. This scheme applies to all b. To avoid repetition, we refer to the proof of Theorem 3.8 below, in which we show in detail how to obtain the result for b^* and $V_{(\infty,b^*)} = (G,F)$. □

Proof of Theorem 3.8 (Verification Theorem):
Let D denote an arbitrary admissible dividend strategy and X^D the ex-dividend process under D and by τ the ruin time of X^D. Let further b^* be such $\tilde{V} = (G,F)$ with $G = G_\infty, F = F_{b^*}$ given in (9) solves the HJB equation. Let D^c be the continuous part of D and ΔD the pure jump part. For the sake of clarity of presentation we let $\alpha_t = 1\!\!1_{[r_t=\delta_2]} + 1$, i.e. $\alpha_t = 1$ if $r_t = \delta_1$ and $\alpha_t = 2$ if $r_t = \delta_2$. $\tilde{V}$ is twice continuously differentiable. We apply Itô's formula to the intervals $[Y_{n-1}, Y_n]$ with $Y_n = \sum_{i=1}^n T_i$ (compare Protter [25], pp. 214,216) and get:

$$\begin{aligned} \mathrm{e}^{-\int_0^{t\wedge\tau} r_\nu \mathrm{d}\nu}\, \tilde{V}(X^D_{t\wedge\tau}, \alpha_{t\wedge\tau}) &= \tilde{V}(X^D_0, \alpha_0) + \int_0^{t\wedge\tau} \mathrm{e}^{-\int_0^s r_\nu \mathrm{d}\nu}\, \sigma \tilde{V}'(X^D_s, \alpha_s)\, \mathrm{d}W_s + M^{\tilde{V}}_{t\wedge\tau} \\ &+ \int_0^{t\wedge\tau} \mathrm{e}^{-\int_0^s r_\nu \mathrm{d}\nu}\, \mathcal{L}\tilde{V}(X^D_s, \alpha_s)\, \mathrm{d}s - \int_0^{t\wedge\tau} \mathrm{e}^{-\int_0^s r_\nu \mathrm{d}\nu}\, \tilde{V}'(X^D_s, \alpha_s)\, \mathrm{d}D^c_s \\ &+ \sum_{0\le s\le t\wedge\tau} \mathrm{e}^{-\int_0^s r_\nu \mathrm{d}\nu} [\tilde{V}(X^D_{s-} - \Delta D_s, \alpha_s) - \tilde{V}(X^D_{s-}, \alpha_s)] 1\!\!1_{[\Delta D_s > 0]}\,, \end{aligned} \tag{24}$$

where $M^{\tilde{V}}$ is the local martingale associated to the regime switching mechanism. That is, writing for a Poisson random measure N with intensity $\mathrm{d}t \times \lambda(\mathrm{d}y)$

$$\mathrm{d}\alpha_t = \int_{\mathbb{R}} h(\alpha(t-), z)\, N(\mathrm{d}t, \mathrm{d}y)\,, \qquad h(a,z) = \begin{cases} 1\!\!1_{[0,\lambda_1)}(z)\,, & \text{if } a = 1\,, \\ -1\!\!1_{[\lambda_1, \lambda_1+\lambda_2)}(z)\,, & \text{if } a = 2\,, \end{cases}$$

and compensated Poisson random measure $\tilde{N}(\mathrm{d}t, \mathrm{d}y) := N(\mathrm{d}t, \mathrm{d}y) - \mathrm{d}t \times \lambda(\mathrm{d}y)$, the process M^f is given by

$$M^f_t = \int_0^t \int_{\mathbb{R}} [f(X^D_{s-}, \alpha_{s-} + h(\alpha_{s-}, z)) - f(X^D_{s-}, \alpha_{s-})]\, \tilde{N}(\mathrm{d}s, \mathrm{d}z)\,.$$

Since $\tilde{V}$ solves the HJB-equation, either it holds $\mathcal{L}\tilde{V}(X^D_s, \alpha_s) \le 0$ and $\tilde{V}'(X^D_s, \alpha_s) = 1$, or $\mathcal{L}(\tilde{V})(X^D_s, \alpha_s) = 0$ and $\tilde{V}'(X^D_s, \alpha_s) \ge 1$. Because the derivative of $\tilde{V}$ is at least equal to one, additionally we have $\tilde V(X^D_{s-}-\Delta D_s,\alpha_s)-\tilde V(X^D_{s-},\alpha_s)\le -\Delta D_s$. Thus,

$$\begin{aligned}&\int_0^{t\wedge\tau} \mathrm{e}^{-\int_0^s r_\nu \mathrm{d}\nu}\, \mathcal{L}\tilde V(X^D_s,\alpha_s)\,\mathrm{d}s - \int_0^{t\wedge\tau} \mathrm{e}^{-\int_0^s r_\nu \mathrm{d}\nu}\, \tilde V'(X^D_s,\alpha_s)\,\mathrm{d}D^c_s \\ &\qquad + \sum_{0\le s\le t\wedge\tau} \mathrm{e}^{-\int_0^s r_\nu \mathrm{d}\nu}[\tilde V(X^D_{s-}-\Delta D_s,\alpha_s)-\tilde V(X^D_{s-},\alpha_s)]\mathbb{1}_{\Delta D_s>0} \\ &\le 0 - \int_0^{t\wedge\tau} \mathrm{e}^{-\int_0^s r_\nu \mathrm{d}\nu}\,\mathrm{d}D^c_s - \sum_{0\le s\le t\wedge\tau} \mathrm{e}^{-\int_0^s r_\nu \mathrm{d}\nu}\Delta D_s \mathbb{1}_{\Delta D_s>0} = -\int_0^{t\wedge\tau} \mathrm{e}^{-\int_0^s r_\nu \mathrm{d}\nu}\,\mathrm{d}D_s\,.\end{aligned}$$

Now rearranging the terms of (24), we conclude

$$\begin{aligned}&\tilde V(X^D_0,\alpha_0) + \int_0^{t\wedge\tau} \mathrm{e}^{-\int_0^s r_\nu \mathrm{d}\nu}\,\sigma\tilde V'(X^D_s,\alpha_s)\,\mathrm{d}W_s + M^{\tilde V}_{t\wedge\tau} \\ &\qquad\ge \mathrm{e}^{-\int_0^{t\wedge\tau} r_\nu \mathrm{d}\nu}\,\tilde V(X^D_{t\wedge\tau},\alpha_{t\wedge\tau}) + \int_0^{t\wedge\tau} \mathrm{e}^{-\int_0^s r_\nu \mathrm{d}\nu}\,\mathrm{d}D_s \ge 0\,,\end{aligned}$$

so the local martingale on the left hand side is bounded from below and is therefore a supermartingale. Taking expectations we find

$$\begin{aligned}\tilde V(x,i) &\ge \mathbb{E}_{x,i}\Big(\tilde V(X^D_0,\alpha_0) + \int_0^{t\wedge\tau} \mathrm{e}^{-\int_0^s r_\nu \mathrm{d}\nu}\,\sigma\tilde V'(X^D_s,\alpha_s)\,\mathrm{d}W_s + M^{\tilde V}_t\Big) \\ &\ge \mathbb{E}_{x,i}\Big(\mathrm{e}^{-\int_0^{t\wedge\tau} r_\nu \mathrm{d}\nu}\,\tilde V(X^D_{t\wedge\tau},\alpha_{t\wedge\tau})\Big) + \mathbb{E}_{x,i}\Big(\int_0^{t\wedge\tau} \mathrm{e}^{-\int_0^s r_\nu \mathrm{d}\nu}\,\mathrm{d}D_s\Big)\,.\end{aligned}$$

Now if we let $t\to\infty$ the second term on the right hand side converges to the return of the dividend strategy D by monotone convergence. If the first term goes to zero, we can conclude $\tilde V(x,i)\ge V(x,i)$. Since $X^D_t\le X_t\le \frac{\mathrm{e}^{aX_t}}{a}$ for a defined as in Proposition 3.1, and since additionally, $\tilde V$ is linearly bounded, we consider

$$\begin{aligned}&\mathbb{E}_{x,i}\Big(\mathrm{e}^{-\int_0^{t\wedge\tau} r_\nu \mathrm{d}\nu}\,\tilde V(X^D_{t\wedge\tau},\alpha_{t\wedge\tau})\Big) = 0 + \mathbb{E}_{x,i}\Big(\mathbb{1}_{\{\tau>t\}}\mathrm{e}^{-\int_0^t r_\nu \mathrm{d}\nu}\,\tilde V(X^D_t,\alpha_t)\Big) \\ &\le c_1\mathbb{E}_{x,i}\Big(\mathbb{1}_{\{\tau>t\}}\mathrm{e}^{-\int_0^t r_\nu \mathrm{d}\nu}\,X^D_t\Big) + c_2\mathbb{E}_{x,i}\Big(\mathbb{1}_{\{\tau>t\}}\mathrm{e}^{-\int_0^t r_\nu \mathrm{d}\nu}\Big) \\ &\le \frac{c_1}{a}\mathbb{E}_{x,i}\Big(\mathbb{1}_{\{\tau>t\}}\mathrm{e}^{aX_t-\int_0^t r_\nu \mathrm{d}\nu}\Big) + c_2\mathbb{E}_{x,i}\Big(\mathrm{e}^{-\int_0^t r_\nu \mathrm{d}\nu}\Big) \\ &\le \frac{c_1\mathrm{e}^{ax}}{a}\,\frac{(\delta_i-\Delta^-)\mathrm{e}^{-\left(\Delta^+-\frac{\Delta^-}{2}\right)t}-(\delta_i-\Delta^+)\mathrm{e}^{-\frac{\Delta^-}{2}t}}{\Delta^+-\Delta^-} + c_2\frac{(\delta_i-\Delta^-)\mathrm{e}^{-\Delta^+t}-(\delta_i-\Delta^+)\mathrm{e}^{-\Delta^-t}}{\Delta^+-\Delta^-}\,.\end{aligned}$$

for positive constants c_1, c_2 which are determined by the function $\tilde V$. The expressions on the right hand side were obtained in Lemma 2.1 and the proof of Proposition 3.1. Both converge to zero for $i\in\{1,2\}$ as $t\to\infty$. Since the left hand side is non-negative for all $t\ge 0$, we conclude that the expectation converges to zero.

Now we show that the function $\tilde V$ is the value of the strategy $D^*=(\infty,b^*)$. We shortly write X^* for the ex-dividend process X^{D^*}. In the same way as before we obtain

$$\begin{aligned}&\tilde V(X^*_0,\alpha_0) + \int_0^{t\wedge\tau} \mathrm{e}^{-\int_0^s r_\nu \mathrm{d}\nu}\,\sigma\tilde V'(X^*_s,\alpha_s)\,\mathrm{d}W_s + M^{\tilde V}_{t\wedge\tau} \\ &\qquad = \mathrm{e}^{-\int_0^{t\wedge\tau} r_\nu \mathrm{d}\nu}\,\tilde V(X^*_{t\wedge\tau},\alpha_{t\wedge\tau}) + \int_0^{t\wedge\tau} \mathrm{e}^{-\int_0^s r_\nu \mathrm{d}\nu}\,\mathrm{d}D^*_s\,, \qquad (25)\end{aligned}$$

because the jumps of the dividend process under strategy D^* occur exactly at the times when the preference regime switches from negative to positive. Again using that $\tilde{V}$ is linearly bounded, we have:

$$\mathbb{E}_{x,i}\Big(\sup_{t\le T} \mathrm{e}^{-\int_0^{t\wedge\tau} r_\nu \mathrm{d}\nu}\tilde{V}(X^*_{t\wedge\tau},\alpha_{t\wedge\tau})\Big) = \mathbb{E}_{x,i}\Big(\sup_{t\le T} \mathrm{e}^{-\int_0^{t} r_\nu \mathrm{d}\nu}\tilde{V}(X^*_{t\wedge\tau},\alpha_{t\wedge\tau})\mathbb{1}_{\{t\le\tau\}}\Big)$$
$$\le c_1\mathrm{e}^{-\delta_1 T}\mathbb{E}_{x,i}\Big(\sup_{t\le T} X_t\Big) + c_2\mathrm{e}^{-\delta_1 T} < \infty\,,$$

which is finite for all $T>0$. Moreover, it follows from the proof of Proposition 3.1, that

$$\mathbb{E}_{x,i}\Big(\sup_{t\le T}\int_0^{t\wedge\tau} \mathrm{e}^{-\int_0^s r_\nu \mathrm{d}\nu}\,\mathrm{d}D^*_s\Big) \le \mathbb{E}_{x,i}\Big(\int_0^{\tau} \mathrm{e}^{-\int_0^s r_\nu \mathrm{d}\nu}\,\mathrm{d}D^*_s\Big) < \infty\,.$$

So, this time, the local martingale in (25) is a martingale. Therefore,

$$\tilde{V}(x,i) = \mathbb{E}_{x,i}\Big(\mathrm{e}^{-\int_0^{t\wedge\tau} r_\nu \mathrm{d}\nu}\,\tilde{V}(X^*_{t\wedge\tau},\alpha_{t\wedge\tau})\Big) + \mathbb{E}_{x,i}\Big(\int_0^{t\wedge\tau} \mathrm{e}^{-\int_0^s r_\nu \mathrm{d}\nu}\,\mathrm{d}D^*_s\Big)$$

for all $t\ge 0$. Letting $t\to\infty$ and using the same arguments as above, we conclude $\tilde{V}(x,i) = \mathbb{E}_{x,i}\Big(\int_0^\tau \mathrm{e}^{-\int_0^s r_\nu \mathrm{d}\nu}\,\mathrm{d}D^*_s\Big)$. □

Proof of Lemma 3.10:

• Assume first that it holds $G'(0)-F'(0)<0$. From the differential equations for F and G:

$$\frac{\sigma^2}{2}F''(x) + \mu F'(x) - (\lambda_2+\delta_2)F(x) + \lambda_2 G(x) = 0,$$
$$\frac{\sigma^2}{2}G''(x) + \mu G'(x) - (\lambda_1+\delta_1)G(x) + \lambda_1 F(x) = 0,$$

we get using $G(0)=F(0)=0$ that $G''(0)-F''(0)>0$. Let now $\hat{x} := \inf\{x>0 : G'(x)-F'(x)>0\}$. Because $G(0)=F(0)=0$ we immediately obtain that $G(x)<F(x)$ on $(0,\hat{x}]$. Subtracting the differential equations for G and F at $\hat{x}$, one has

$$\frac{\sigma^2}{2}\big(G''(\hat{x})-F''(\hat{x})\big) = -\delta_2F(\hat{x}) + \delta_1 G(\hat{x}) + (\lambda_2+\lambda_1)\big(G(\hat{x})-F(\hat{x})\big) < 0$$

which means that $G'(\hat{x})-F'(\hat{x})$ cannot become zero, contradicting our assumption.
Assuming $G'(0)=F'(0)$ would yield $G''(0)=F''(0)$ and $G'''(0)-F'''(0)=\delta_1F'(0)-\delta_2F'(0)<0$, meaning that in an ε-environment of 0 it holds $G''(x)<F''(x)$ and $G'(x)<F(x)$. Hence, we can use the contradiction argument from above.

• It remains to consider the case when $F'(0)<G'(0)$ but there is an $0<\bar{x}<b^*$ with $F'(\bar{x})>G'(\bar{x})$ and $F''(\bar{x})>G''(\bar{x})$. From

$$0 > \frac{\sigma^2}{2}\big(G''(\bar{x})-F''(\bar{x})\big) = -\mu\big(G'(\bar{x})-F'(\bar{x})\big) - \delta_2F(\bar{x}) + \delta_1G(\bar{x}) + (\lambda_2+\lambda_1)\big(G(\bar{x})-F(\bar{x})\big)\,.$$

we can conclude that $-\delta_2F(\bar{x})+\delta_1G(\bar{x})+(\lambda_2+\lambda_1)\big(G(\bar{x})-F(\bar{x})\big)<0$. As long as $F'(x)>G'(x)$ the function $\big(G(x)-F(x)\big)$ is decreasing, meaning that $-\delta_2F(x)+\delta_1G(x)+(\lambda_2+\lambda_1)\big(G(x)-$

$F(x)\big) < 0$. Hence, $F'(x) = G'(x)$ would imply $G''(x) - F''(x) < 0$, i.e. $G'(x) < F'(x)$ for all $x \in [\bar{x}, b^*]$ contradicting $F'(b^*) = 1 < G'(b^*)$ stated in Lemmata 3.5 and 3.6. □

Proof of Proposition 3.12:
We prove the claim by contradiction and assume first $\tilde{b}^* \le b^*$.
1.) Then, it follows from the properties of the value function $\tilde{F}'(\tilde{b}^*) - F'(\tilde{b}^*) \le 0$ and $\tilde{F}''(\tilde{b}^*) - F''(\tilde{b}^*) \ge 0$. On the other hand, because $\tilde{F}(x) > F(x)$ and $\tilde{G}(x) > G(x)$ for $x > 0$ (Lemma 3.11), we conclude that $\tilde{F}'(0) - F'(0) > 0$ and $\tilde{G}'(0) - G'(0) > 0$, which implies $\tilde{F}''(0) - F''(0) < 0$ and $\tilde{G}''(0) - G''(0) < 0$ due to differential equations (8) for F and G.

2.) Let $\omega := \inf\{x > 0 :\ \tilde{F}'(x) - F'(x) \le 0\}$ and $v := \sup\{x > 0 :\ \tilde{G}'(x) - G'(x) \le 0\}$.
We know already that

$$\tilde{G}'(x) - G'(x) = (A - B)\big(\psi'_{\tilde{\lambda}_2}(x) - \psi'_{\lambda_2}(x)\big) + B\big(\tilde{F}'(x) - F'(x)\big) \, .$$

Since $\psi'_{\tilde{\lambda}_2}(x) - \psi'_{\lambda_2}(x) > 0$ (Lemma 3.4) and $B > \frac{\lambda_1}{\lambda_1+\delta_1}$ we can conclude that $v < \omega$, and for all x with $\tilde{F}'(x) - F'(x) \le 0$ it holds

$$\tilde{G}'(x) - G'(x) < \frac{\lambda_1}{\lambda_1 + \delta_1}\big(\tilde{F}'(x) - F'(x)\big) \, . \tag{26}$$

3.) Deriving and subtracting the differential equations for $\tilde{G}$ and G on $[0, b^*]$, yields

$$\frac{\sigma^2}{2}\{\tilde{G}'''(x) - G'''(x)\} + \mu\{\tilde{G}''(x) - G''(x)\} = (\lambda_1 + \delta_1)\{\tilde{G}'(x) - G'(x)\} - \lambda_1\{\tilde{F}'(x) - F'(x)\} \, . \tag{27}$$

For $x = v$ it holds $\tilde{G}''(v) - G''(v) \le 0$ and $(\lambda_1 + \delta_1)\{\tilde{G}'(v) - G'(v)\} - \lambda_1\{\tilde{F}'(v) - F'(v)\} < 0$, meaning that $\tilde{G}'' - G''$ stays negative. Therefore, we can conclude using (27) and (26) that $(\lambda_1 + \delta_1)\{\tilde{G}'(x) - G'(x)\} - \lambda_1\{\tilde{F}'(x) - F'(x)\} < 0$ for all $b^* \ge x \ge v$, meaning that $\tilde{G}''(x) - G''(x) < 0$ and consequently $\tilde{G}'(x) - G'(x) < 0$ for all $b^* \ge x \ge v$. However, due to Lemma 3.5 there is an $R \in \mathbb{R}$ such that $\tilde{G}'(b^*) - G'(b^*) = Rq_5 e^{q_5 b^*}$ and $\tilde{G}''(b^*) - G''(b^*) = Rq_5^2 e^{q_5 b^*}$ leading to a contradiction $Rq_5 e^{q_5 b^*} < 0$ and $Rq_5^2 e^{q_5 b^*} < 0$, as $q_5 < 0$. □

Proof of Lemma 4.10:
Here, we prove that the function η defined in (20) is decreasing in γ and μ and there is a unique function $\underline{\gamma}(\mu)$ such that $\eta(\underline{\gamma}(\mu), \mu) \equiv 0$.

1.) Recall that $\eta(\gamma, \mu) = \frac{(A-1)\left(B - \frac{\lambda_1}{\lambda_1+\delta_1}\right)\left(\frac{1}{q_5} - \frac{1}{u_3}\right) - (B-1)\left(A - \frac{\lambda_1}{\lambda_1+\delta_1}\right)\left(\frac{1}{q_5} - \frac{1}{u_1}\right)}{A-B} + \frac{\lambda_1\mu}{(\lambda_1+\delta_1)^2}$. Since

$$A - \frac{\lambda_1}{\lambda_1 + \delta_1} = \frac{A\Delta^+}{\lambda_1 + \delta_1}, \qquad B - \frac{\lambda_1}{\lambda_1 + \delta_1} = \frac{B\Delta^-}{\lambda_1 + \delta_1}, \qquad \frac{1}{q_5} = \frac{\sigma^2}{2(\lambda_1 + \delta_1)} q_5 + \frac{\mu}{\lambda_1 + \delta_1}$$

we can rearrange the terms in η and get

$$\eta(\gamma, \mu) = -\frac{B-1}{B-A} \cdot \frac{A}{\lambda_1 + \delta_1}\frac{\Delta^+}{u_1} - \frac{1-A}{B-A} \cdot \frac{B}{\lambda_1 + \delta_1}\frac{\Delta^-}{u_3} + \frac{\delta_1}{(\lambda_1 + \delta_1)q_5} + \frac{\lambda_1\mu}{(\lambda_1 + \delta_1)^2} \, .$$

Note that the dependence on γ and μ is hidden in u_1, u_3 and q_5. Taking derivatives with respect to γ and μ yields because $\Delta^+ > \Delta^-$, $u_1 > u_3$ and $\frac{A(B-1)}{(B-A)} < 0$:

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}\eta(\gamma,\mu) = \frac{\frac{A(B-1)}{(B-A)}\frac{\Delta^+}{u_1(\lambda_1+\delta_1)}}{u_1\sqrt{\mu^2+2\delta\sigma^2(\Delta^+ + \gamma)}} + \frac{\frac{B(1-A)}{(B-A)}\frac{\Delta^-}{u_3(\lambda_1+\delta_1)}}{u_3\sqrt{\mu^2+2\delta\sigma^2(\Delta^- + \gamma)}} > \frac{-\eta(\gamma,\mu)+\frac{\sigma^2}{2}\frac{\delta_1 q_5}{(\lambda_1+\delta_1)^2}}{u_3\sqrt{\mu^2+2\delta\sigma^2(\Delta^- + \gamma)}} .$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\mu}\eta(\gamma,\mu) &= \frac{\frac{A(B-1)}{(B-A)}\frac{\Delta^+}{u_1(\lambda_1+\delta_1)}}{\sqrt{\mu^2+2\delta\sigma^2(\Delta^+ + \gamma)}} + \frac{\frac{B(1-A)}{(B-A)}\frac{\Delta^-}{u_3(\lambda_1+\delta_1)}}{\sqrt{\mu^2+2\delta\sigma^2(\Delta^- + \gamma)}} + \frac{\frac{\sigma^2}{2}\frac{\delta_1 q_5}{(\lambda_1+\delta_1)^2}}{\sqrt{\mu^2+2\sigma^2(\lambda_1+\delta_1)}} \\ &> \frac{-\eta(\gamma,\mu)+\frac{\sigma^2}{2}\frac{\delta_1 q_5}{(\lambda_1+\delta_1)^2}}{\sqrt{\mu^2+2\delta\sigma^2(\Delta^- + \gamma)}} + \frac{\frac{\sigma^2}{2}\frac{\delta_1 q_5}{(\lambda_1+\delta_1)^2}}{\sqrt{\mu^2+2\sigma^2(\lambda_1+\delta_1)}} .\end{aligned}$$

Thus, we can immediately conclude that $\eta(\gamma,\mu)$ is strictly increasing in γ and in μ if $\eta(\gamma,\mu) < \frac{\sigma^2}{2}\frac{\delta_1 q_5}{(\lambda_1+\delta_1)^2}$. In particular, if $\eta(\tilde\gamma,\tilde\mu) = 0$ for some $(\tilde\gamma,\tilde\mu)$ then $\frac{\mathrm{d}}{\mathrm{d}\gamma}\eta(\tilde\gamma,\tilde\mu) > 0$ and $\frac{\mathrm{d}}{\mathrm{d}\mu}\eta(\tilde\gamma,\tilde\mu) > 0$ meaning that $\eta(\gamma,\mu) > 0$ for all $(\gamma,\mu) \in (\tilde\gamma,\infty)\times(\tilde\mu,\infty)$.

2.) It remains to check when the function η can attain negative values.
Let $\gamma = 0$ and $\mu = 0$. Then, $u_1 = q_1 = \sqrt{2\Delta^+/\sigma^2}$, $u_3 = q_3 = \sqrt{2\Delta^-/\sigma^2}$, $q_5 = -\sqrt{2(\lambda_1+\delta_1)/\sigma^2}$,

$$\begin{aligned}\eta(0,0) &= \frac{(1-A)\Big(B-\frac{\lambda_1}{\lambda_1+\delta_1}\Big)\Big(\frac{1}{q_5}-\frac{1}{q_3}\Big) + (B-1)\Big(A-\frac{\lambda_1}{\lambda_1+\delta_1}\Big)\Big(\frac{1}{q_5}-\frac{1}{q_1}\Big)}{B-A} \\ &= \frac{(B-1)A\Delta^-\Big(\frac{1}{q_5}-\frac{1}{q_3}\Big)}{(\lambda_1+\delta_1)(B-A)}\cdot\left\{\frac{(1-A)B}{(B-1)A} + \frac{\Delta^+}{\Delta^-}\cdot\frac{\frac{1}{q_1}-\frac{1}{q_5}}{\frac{1}{q_3}-\frac{1}{q_5}}\right\} .\end{aligned}$$

As $B = \frac{\lambda_1}{\lambda_1+\delta_1}$ if $\delta_1 = -\frac{\lambda_1\delta_2}{\lambda_1+\delta_1}$ and $A < 0$, it is easy to see that

$$\eta(0,0) = \begin{cases} \frac{(1-A)(B-1)}{B-A}\big(\frac{1}{q_1}-\frac{1}{q_3}\big) < 0 & : \text{ if } \delta_1 = 0, \\ \frac{B-1}{B-A}\cdot\big(A-\frac{\lambda_1}{\lambda_1+\delta_1}\big)\big(\frac{1}{q_5}-\frac{1}{q_1}\big) > 0 & : \text{ if } \delta_1 = -\frac{\lambda_1\delta_2}{\lambda_1+\delta_1}. \end{cases}$$

By the intermediate value theorem, there is at least one δ_1 leading to $\eta(0,0) = 0$. In order to show the uniqueness, we consider the expression in front of the curly brackets, which is positive as $A, q_5 < 0$, $B > 1$. Using the definitions of A and B, see Notation 3.3, and considering δ_1 as a variable we can define

$$\frac{(1-A)B}{(B-1)A} = -\frac{\Delta^+ - \delta_1}{\Delta^- - \delta_1}, \qquad \frac{\Delta^+}{\Delta^-}\frac{\frac{1}{q_1}-\frac{1}{q_5}}{\frac{1}{q_3}-\frac{1}{q_5}} = \frac{\Delta^+ + \sqrt{(\lambda_1+\delta_1)\Delta^+}}{\Delta^- + \sqrt{(\lambda_1+\delta_1)\Delta^-}} .$$

$$u_1(\delta_1) := -\frac{\Delta^+ - \delta_1}{\Delta^+ + \sqrt{(\lambda_1+\delta_1)\Delta^+}} \qquad u_2(\delta_1) := \frac{\Delta^- - \delta_1}{\Delta^- + \sqrt{(\lambda_1+\delta_1)\Delta^-}}$$

It is straightforward to get that the function $\frac{x-\delta_1}{x+\sqrt{(\lambda_1+\delta_1)x}}$ fulfils

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x-\delta_1}{x+\sqrt{(\lambda_1+\delta_1)x}}\right) &= \frac{1}{(x+\sqrt{(\lambda_1+\delta_1)x})^2}\left\{x+\sqrt{(\lambda_1+\delta_1)x} - (x-\delta_1)\big(1+\frac{\sqrt{\lambda_1+\delta_1}}{2\sqrt{x}}\big)\right\} \\ &= \frac{\sqrt{\lambda_1+\delta_1}}{(x+\sqrt{(\lambda_1+\delta_1)x})^2}\left\{\frac{1}{2}\sqrt{x} + \frac{\delta_1}{\sqrt{\lambda_1+\delta_1}} + \frac{\delta_1}{2\sqrt{x}}\right\} .\end{aligned}$$

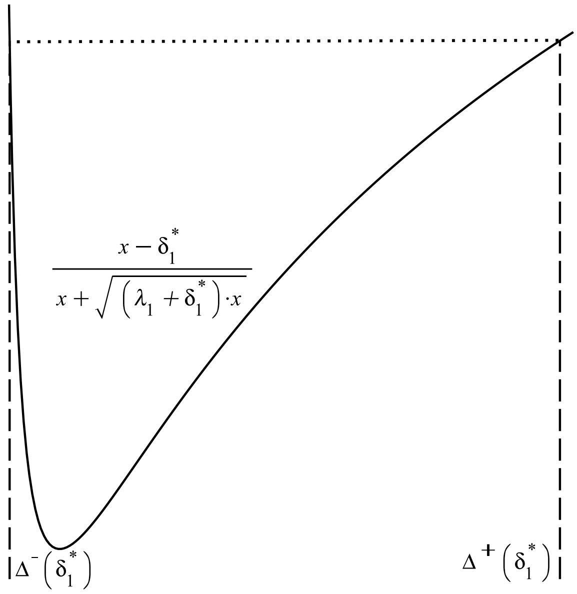

Figure 8: The function $(x-\delta_1^*)/\big(x+\sqrt{(\lambda_1+\delta_1^*)x}\big)$.

Let now $f(x,\delta_1) := \frac{1}{2}\sqrt{x} + \frac{\delta_1}{\sqrt{\lambda_1+\delta_1}} + \frac{\delta}{2\sqrt{x}}$. Then, it is clear that f is strictly increasing in x, fulfilling $\lim\limits_{x\to 0} f(x,\delta_1) = -\infty$ and $\lim\limits_{x\to\infty} f(x,\delta_1) = \infty$.
Also, it is clear that f is strictly increasing in δ_1. Thus, there is a unique, strictly decreasing function $v(\delta_1)$ such that $f(v(\delta_1),\delta_1) \equiv 0$.

Let $\delta_1^* := \inf\{a>0:\ u_1(a)-u_2(a)<0\}$. Note that $v(\delta_1^*) \ge \Delta^+(\delta_1^*)$ or $v(\delta_1^*) \le \Delta^+(\delta_1^*)$ is impossible, as in this way it would hold $-u_1(\delta_1) \ne u_2(\delta_1^*)$, see Figure 8. If $v(\delta_1^*) \in \big(\Delta^-(\delta_1^*), \Delta^+(\delta_1^*)\big)$ then $f(\Delta^+) > f(\Delta^-)$ for all $\delta_1 > \delta_1^*$ as Δ^+ and Δ^- are increasing in δ_1 and $v(\delta_1)$ is decreasing. To explain it on the example of Figure 8: $u_2(\delta_1)$ will move down the decreasing arm and change at some point to the increasing arm of the function $(x-\delta_1^*)/\big(x+\sqrt{(\lambda_1+\delta_1^*)x}\big)$, whereas $-u_1(\delta_1)$ will move up on the increasing arm for increasing values of δ_1. The value $v(\delta_1)$ will move to the left, so that it will hold either $v(\delta_1) \in \big(\Delta^-(\delta_1), \Delta^+(\delta_1)\big)$ or $v(\delta_1) \le \Delta^-(\delta_1)$ for $\delta_1 > \delta_1^*$.
And we can conclude that δ_1^* is the unique zero of $\eta(0,0)$.
Further, we know that $\eta(\gamma,0)$ is strictly increasing in γ at least until $\eta(\gamma,0)$ attains zero and

$$\lim_{\gamma\to\infty} \eta(\gamma,0) = \frac{\delta_1}{(\lambda_1+\delta_1)q_5} > 0 \quad \text{and} \quad \lim_{\mu\to\infty} \eta(0,\mu) = \lim_{\mu\to\infty} \frac{-\mu\delta_1}{(\lambda_1+\delta_1)^2} = \infty\ .$$

By the intermediate value theorem, there are unique γ_0 and μ_0 such that $\eta(\gamma_0,0) = 0 = \eta(0,\mu_0)$. In particular, due to 1.) we can conclude that $\eta(\gamma,\mu_0) > 0$ and $\eta(\gamma_0,\mu) > 0$ for $\gamma > 0$, $\mu > 0$.

3.) By the implicit function theorem there is a unique decreasing function $\underline{\gamma}(\mu) \le \gamma_0$, given as a dashed line in Figure 6, and defined on the interval $[0,\mu_0]$, $\mu_0 < \infty$, such that $\eta(\underline{\gamma}(\mu),\mu) \equiv 0$. All pairs (γ,μ) below or on the dashed curve in Figure 6 yield $\eta(\gamma,\mu) \le 0$ and consequently $\beta \ge 0$. □

Proof of Theorem 4.12:
Let $\tilde{V}$ denote the respective solution to the HJB equation for the cases of short, medium and long expected delay given in Corollary 4.8, Lemma 4.10 and Lemma 4.11 extended to the negative half plane. We note that in all cases the functions are piecewise twice continuously

differentiable. The second derivative has at most two discontinuities, which lie at $x = 0$ and $x = \beta > 0$. Now we apply Itô's formula in the formulation of Protter [25], pp. 214,216, to the intervals $[Y_{n-1}, Y_n]$ with $Y_n = \sum_{i=1}^n T_i$. This is possible due to Lemma 45.9 of Rogers and Williams [27]. Similar to the proof of Theorem 3.7, we therefore obtain for every admissible dividend strategy D:

$$
\begin{aligned}
\mathrm{e}^{-\int_0^{t\wedge\tau} r_\nu \mathrm{d}\nu}\, \tilde V(X^D_{t\wedge\tau}, \alpha_{t\wedge\tau}) &= \tilde V(X^D_0, \alpha_0) + \int_0^{t\wedge\tau} \mathrm{e}^{-\int_0^s r_\nu \mathrm{d}\nu}\, \sigma \tilde V'(X^D_s, \alpha_s)\, \mathrm{d}W_s + M^{\tilde V}_{t\wedge\tau} \\
&\quad + \int_0^{t\wedge\tau} \mathrm{e}^{-\int_0^s r_\nu \mathrm{d}\nu}\, \mathcal{L}\tilde V(X^D_s, \alpha_s)\, \mathrm{d}s - \int_0^{t\wedge\tau} \mathrm{e}^{-\int_0^s r_\nu \mathrm{d}\nu}\, \tilde V'(X^D_s, \alpha_s)\, \mathrm{d}D^c_s \\
&\quad + \sum_{0\le s\le t\wedge\tau} \mathrm{e}^{-\int_0^s r_\nu \mathrm{d}\nu} [\tilde V(X^D_{s-} - \Delta D_s, \alpha_s) - \tilde V(X^D_{s-}, \alpha_s)] \mathbb{1}_{[\Delta D_s > 0]}\,, \qquad (28)
\end{aligned}
$$

where $\mathcal{L}\tilde V(x, i) = (\mathcal{L}^- \tilde V(x, i))\mathbb{1}_{(-\infty,0)} + (\mathcal{L}^+ \tilde V(x, i))\mathbb{1}_{[0,\infty)}$. Since $\tilde V$ solves the HJB-equation, either it holds $\mathcal{L}\tilde V(X^D_s, \alpha_s) \le 0$ and $\tilde V'(X^D_s, \alpha_s) = 1$, or $\mathcal{L}\tilde V(X^D_s, \alpha_s) = 0$ and $\tilde V'(X^D_s, \alpha_s) \ge 1$ whenever $X^D_s \ge 0$. By (13), $\mathcal{L}\tilde V(X^D_s, \alpha_s) = 0$ if $X^D_s < 0$. Moreover, since dividend payments are forbidden if the surplus is negative, D_s is constant if $X^D_s < 0$. Combining these findings, we arrive at

$$
\begin{aligned}
\tilde V(X^D_0, \alpha_0) &+ \int_0^{t\wedge\tau} \mathrm{e}^{-\int_0^s r_\nu \mathrm{d}\nu}\, \sigma \tilde V'(X^D_s, \alpha_s)\, \mathrm{d}W_s + M^{\tilde V}_{t\wedge\tau} \\
&\ge \mathrm{e}^{-\int_0^{t\wedge\tau} r_\nu \mathrm{d}\nu}\, \tilde V(X^D_{t\wedge\tau}, \alpha_{t\wedge\tau}) + \int_0^{t\wedge\tau} \mathrm{e}^{-\int_0^s r_\nu \mathrm{d}\nu}\, \mathrm{d}D_s \ge 0\,.
\end{aligned}
$$

Now the assertion follows by the same arguments as used in the proof of Theorem 3.7. □

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