Measuring the Suboptimality of Dividend Controls in a Brownian Risk Model

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Abstract

We consider an insurance company modelling its surplus process by a Brownian motion with drift. Our target is to maximise the expected exponential utility of discounted dividend payments, given that the dividend rates are bounded by some constant.
The utility function destroys the linearity and the time homogeneity of the considered problem. The value function depends not only on the surplus, but also on time. Numerical considerations suggest that the optimal strategy, if it exists, is of a barrier type with a non-linear barrier. In the related article [14], it has been observed that standard numerical methods break down in certain parameter cases and no close form solution has been found.
For these reasons, we offer a new method allowing to estimate the distance of an arbitrary smooth enough function to the value function. Applying this method, we investigate the goodness of the most obvious suboptimal strategies – payout on the maximal rate, and constant barrier strategies – by measuring the distance of its performance function to the value function.

Key words: suboptimal control, Hamilton–Jacobi–Bellman equation, dividend payouts, Brownian risk model, exponential utility function.
2010 Mathematical Subject Classification: Primary 93E20
Secondary 91B30, 60H30

1 Introduction

Dividend payments of companies is one of the most important factors for analytic investors when they have to decide whether they invest into the firm. Furthermore, dividends serve as a sort of litmus paper, indicating the financial health of the considered company. Indeed, the reputation, and consequently commercial success of a company with a long record of dividend payments would be negatively impacted in the case the

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company will drop the payments. On the other hand, new companies can additionally strengthen their position by declaring dividends. For the sake of fairness, it should be mentioned that there are also some serious arguments against dividend payouts, for example for tax reasons it might be advantageous to withhold dividend payments. A discussion of the pros and contras of dividends distribution is beyond the scope of the present manuscript. We refer to surveys on the topic by Avanzi [6] or Albrecher and Thonhauser [4].

Due to its importance, the value of expected discounted dividends has been for a long time, and still remains, one of the most popular risk measures in the actuarial literature. Modelling the entire surplus of an insurance company by a Brownian motion, a compound Poisson process or a general Levy process with an infinite or finite time horizon – lots of papers have been written on maximising expected discounted dividends. Gerber [13], Bühlmann [10], Azcue and Muler [7], Albrecher and Thonhauser [3] are just some of the results obtained since de Finetti's path-breaking paper [11].
Shreve, Lehoczky and Gaver [20] considered the problem for a general diffusion process, where the drift and the volatility fulfil some special conditions. Modelling the surplus process via a Brownian motion with drift, was considered by Asmussen and Taksar [5], who could find the optimal strategy to be a constant barrier.

All the papers mentioned above deal with linear dividend payments, in the sense that the lump sum payments or dividend rates are not skewed by a utility function. Hubalek and Schachermayer [15] apply various utility functions to the dividend rates before accumulation. Their result differs a lot from the classical result described in Asmussen and Taksar [5].

An interesting question is to consider the expected "present utility" of the discounted dividend payments. It means the utility function will be applied on the value of the accumulated discounted dividend payments up to ruin. In this way, one considers as a risk measure the utility of the present value of dividends. The dividend payments are not attributed to a specific owner (the shareholders), they serve as the only cash-flow stream used to evaluate the company's financial health. Therefore, the present utility of the accumulated payments accounts for the company's risk aversion by exercising a dividend payment strategy. The fact that the considerations are stopped at ruin indicates that the negative surplus is considered as a high risk. A higher utility of the present value of future dividends payments makes the company more attractive for potential investors. An early ruin will of course lead to a smaller utility of the present value of dividends. Thus, the event of ruin is a technical feature and does not mean that the company actually goes out of business.
Some big companies, for instance Munich Re (see [17]), do not reduce their dividends also in crisis times for strategic and reputational reasons. Recently, researchers have started to investigate the problem of non-decreasing dividend payments, some examples are Albrecher et al. [2], [1]. In this case, even with a linear utility function, the problem becomes two dimensional. Adding a non-linear utility function to this setting would further complicate the solution to the problem.

Modelling the surplus by a Brownian motion with drift, Grandits et. al applied in

[14] an exponential utility function to the value of unrestricted discounted dividends. In other words, they considered the expected utility of the present value of dividends and not the expected discounted utility of the dividend rates. In that paper, the existence of the optimal strategy could not be shown. We will investigate the related problem where the dividend payments are restricted to a finite rate. Note that using a non-linear utility function increases the dimension of the problem. Therefore, tackling the problem via the Hamilton–Jacobi–Bellman (HJB) approach in order to find an explicit solution seems to be an unsolvable task. Of course, one can prove the value function to be the unique viscosity solution to the corresponding Hamilton–Jacobi–Bellman equation and then try to solve the problem numerically. However, on this path one faces two problems that are not easy to tackle. First, the proof that the value function is a (unique) viscosity solution to the corresponding HJB equation can be very complex, time- and space-consuming. In particular, if one chooses a non-linear and non-exponential utility function the value function will depend on 3 variables: the time t, the surplus x and the accumulated dividend payments prior to the starting time t. Using an exponential utility allows to get rid of the third variable. This is also one of the reasons why an exponential utility is considered in the present paper. Having just two variables to consider allows to represent the proposed method in a more clear way, avoiding unnecessary details.
Second, if the maximal allowed dividend rate is quite big the standard numerical methods like finite differences and finite elements break down. We discuss some numerical problems in Section 5.

In this paper, we offer a new approach. Instead of proving the value function to be the unique viscosity solution to the corresponding Hamilton–Jacobi–Bellman equation, we investigate the "goodness" of suboptimal strategies. In this way, one avoids both problems described above. There is no need to prove that the value function is a classical or a viscosity solution to the HJB equation, and no need to solve the HJB equation numerically. One simply chooses an arbitrary control with an easy-to-calculate return function and compares its performance, or rather an approximation of its performance, against the unknown value function.
The method is based on sharp occupation bounds which we find by a method developed for sharp density bounds in Baños and Krühner [8]. This enables us to make an educated guess and to check if our pick is indeed almost as good as the optimal strategy.
This approach drastically differs from procedures usually used for calculation of the value function in two ways. First, unlike most numerical schemes there is no convergence to the value function, i.e. one only gets a bound for the performance of one given strategy but no straightforward procedure to get better strategies. Second, our criterion has almost no influence from the dimension of the problem and is consequently directly applicable in higher dimensions.

The paper is organised as follows. In the next section, we motivate the problem and derive some basic properties of the value function. In Section 3, we consider the case of the maximal constant dividend rate strategy, the properties of the corresponding return function and the goodness of this strategy (a bound for the distance of the return function to the unknown value function). Section 4 investigates the goodness of a constant barrier strategy. In Section 5, we consider examples illustrating the classical and the new approach. Finally, in the appendix we gather technical proofs and establish occupation bounds.

2 Motivation

We consider an insurance company whose surplus is modelled as a Brownian motion with drift

$$X_t = X_0 + \mu t + \sigma W_t \quad t \geq 0$$

where $\mu, \sigma, X_0 \in \mathbb{R}$. We will use the Markov property of X. To be exact we mean that $(\Omega, \mathfrak{A})$ is a measurable space, $\mathbb{P}_{(t,x)}$, $x \in \mathbb{R}$, $t \geq 0$ is a family of measures, $X, W : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ are continuous sample paths processes and under $\mathbb{P}_{(t,x)}$ we have that $(W_{s+t})_{s\geq 0}$ is a standard Brownian motion $W_u = 0$, $u \in [0,t]$, $X_s = x + \mu \max\{t-s, 0\} + W_s$, $s \geq 0$ and $(\mathcal{F}_t)_{t\geq 0}$ is the right-continuous filtration generated by X. In particular, we have $\mathbb{P}_{(t,x)}(X_t = x) = 1$. Note that the process X is defined for all time points $s \geq 0$ but we have $\mathbb{P}_{(t,x)}(X_s = x) = 1$ for $0 \leq s \leq t$, which basically means that X is constant equal to its starting value x before its starting time t. We denote by $\mathbb{E}_{(t,x)}$ the expectation corresponding to $\mathbb{P}_{(t,x)}$, also we use the notation $\mathbb{E}_x := \mathbb{E}_{(0,x)}$.

Further, we assume that the company has to pay out dividends, characterised by a dividend rate. Denoting the dividend rate process by C, we can write the ex-dividend surplus process as

$$X_t^C = x + \mu t + \sigma W_t - \int_0^t C_s \,\mathrm{d}s \,.$$

In the present manuscript we only allow dividend rate processes C which are progressively measurable and satisfy $0 \leq C_s \leq \xi$ for some maximal rate $\xi > 0$ at any time $s \geq 0$. We call these strategies *admissible*. Let $U(x) = \frac{1}{\gamma} - \frac{1}{\gamma} e^{-\gamma x}$, $\gamma > 0$, be the underlying utility function and $\tau^C := \inf\{s \geq t : X_s^C < 0\}$ the ruin time corresponding to the strategy C under the measure $\mathbb{P}_{(t,x)}$. Our objective is to maximise the expected exponential utility of the discounted dividend payments until ruin. Since we can start our observation in every time point $t \in \mathbb{R}_+$, the target functional is given by

$$V^C(t,x) := \mathbb{E}_{(t,x)}\Big[U\Big(\int_t^{\tau^C} e^{-\delta s} C_s \,\mathrm{d}s\Big)\Big] \,,$$

Here, $\delta > 0$ denotes the preference rate of the insurer, helping to transfer the dividend payments to the starting time t.

Further, we assume that the dividend payout up to t equals 0, for a rigorous simplification confer [14] or simply note that with already paid dividends $\bar{C}$ up to time t we have

$$\mathbb{E}_{(t,x)}\Big[U\Big(\bar{C} + \int_t^{\tau^C} e^{-\delta s} C_s \,\mathrm{d}s\Big)\Big] = U(\bar{C}) + e^{-\gamma \bar{C}} V^C(t,x).$$

The corresponding value function V is defined by

$$V(t,x) := \sup_C \mathbb{E}_{(t,x)}\Big[U\Big(\int_t^{\tau^C} e^{-\delta s} C_s \,\mathrm{d}s\Big)\Big]$$

where the supremum is taken over all admissible strategies C. Note that $V(t,0)=0$, because ruin will happen immediately due to the oscillation of Brownian motion, i.e. $\tau^C = \min\{s\ge t : X_s^C = 0\}$ for any strategy C under $\mathbb{P}_{(t,x)}$. The Hamilton–Jacobi–Bellman (HJB) equation corresponding to the problem can be found similar as in [14], for general explanations confer for instance [19]:

$$V_t + \mu V_x + \frac{\sigma^2}{2} V_{xx} + \sup_{0\le y\le \xi}\Big[y\Big(-V_x + e^{-\delta t}(1-\gamma V)\Big)\Big] = 0\;. \tag{1}$$

We like to stress at this point that we neither show that the value function solves the HJB in some sense, nor that a good enough solution is the value function. In fact, our approach of evaluating the goodness of a given strategy compared to the unknown optimal strategy does not assume any knowledge about the optimal strategy or its existence.

Assuming that the HJB equation has a classical solution (i.e. smooth enough), one would expect that an optimal strategy C^* is the maximiser in the HJB equation at any given point of time which would depend on the state of the optimal strategy, i.e.

$$C^*(s,X_s^*) = \begin{cases} 0 & \text{if } -V_x(s,X_s^*) + e^{-\delta s}(1-\gamma V(s,X_s^*)) < 0,\\ \in[0,\xi] & \text{if } -V_x(s,X_s^*) + e^{-\delta s}(1-\gamma V(s,X_s^*)) = 0,\\ \xi & \text{if } -V_x(s,X_s^*) + e^{-\delta s}(1-\gamma V(s,X_s^*)) > 0. \end{cases}$$

$\mathbb{P}_{(t,x)}$-a.s. for any $s\ge t$.

Remark 2.1
For every dividend strategy C it holds:

$$V^C(t,x) = \mathbb{E}_{(t,x)}\Big[U\Big(\int_t^{\tau^C} C_s e^{-\delta s}\,\mathrm{d}s\Big)\Big] \le U\Big(\xi\int_t^\infty e^{-\delta s}\,\mathrm{d}s\Big) = U\Big(\frac{\xi}{\delta}e^{-\delta t}\Big)$$

We conclude

$$\lim_{x\to\infty} V(t,x) \le U\Big(\frac{\xi}{\delta}e^{-\delta t}\Big)\;,$$

and V is a bounded function. Consider now a constant strategy $C_t\equiv\xi$, i.e. we always pay on the rate ξ. The ex-dividend process becomes a Brownian motion with drift $\mu-\xi$ and volatility σ. Define further for $n\ge 1$

$$\eta_n := \frac{\xi-\mu-\sqrt{(\xi-\mu)^2+2\delta\sigma^2 n}}{\sigma^2} < 0\;, \tag{2}$$

and let $\tau^\xi := \inf\{s\ge t:\ x+(\mu-\xi)s+\sigma W_s \le 0\}$, i.e. τ^ξ is the ruin time under the strategy ξ. Here and in the following we define

$$\Delta := \xi\gamma/\delta. \tag{3}$$

With help of change of measure technique, see for example [19, p. 216], we can calculate the return function V^ξ of the constant strategy $C_t \equiv \xi$ by using the power series representation of the exponential function:

$$\begin{aligned}
V^\xi(t,x) &= \mathbb{E}_x\Big[U\Big(\xi\int_t^{\tau^\xi} e^{-\delta s}\,\mathrm{d}s\Big)\Big] = \frac{1}{\gamma} - \frac{1}{\gamma}\mathbb{E}_x\Big[e^{-\Delta\left(e^{-\delta t}-e^{-\delta(t+\tau^\xi)}\right)}\Big] \\
&= \frac{1}{\gamma} - \frac{1}{\gamma}e^{-\Delta e^{-\delta t}}\mathbb{E}_x\Big[e^{\Delta e^{-\delta(t+\tau^\xi)}}\Big] = \frac{1}{\gamma} - \frac{e^{-\Delta e^{-\delta t}}}{\gamma}\sum_{n=0}^\infty \frac{e^{-\delta t n}\Delta^n}{n!}\mathbb{E}_x[e^{-\delta\tau^\xi n}] \\
&= \frac{1}{\gamma} - \frac{e^{-\Delta e^{-\delta t}}}{\gamma} - \frac{e^{-\Delta e^{-\delta t}}}{\gamma}\sum_{n=1}^\infty \frac{e^{-\delta t n}\Delta^n}{n!}e^{\eta_n x}\;.
\end{aligned}\tag{4}$$

It is obvious, that in the above power series $\lim\limits_{x\to\infty}$ and summation can be interchanged yielding $\lim\limits_{x\to\infty} V^\xi(t,x) = U\Big(\frac{\xi}{\delta}e^{-\delta t}\Big)$. In particular, we can now conclude

$$\lim_{x\to\infty} V(t,x) = \frac{1}{\gamma} - \frac{1}{\gamma}\exp\big(-\Delta e^{-\delta t}\big) = U\Big(\frac{\xi}{\delta}e^{-\delta t}\Big)\;.$$

uniformly in $t \in [0,\infty)$. ■

Next, we show that for some special values of the maximal rate ξ with a positive probability the ex-dividend surplus process remains positive up to infinity.

Remark 2.2
Let C be an admissible strategy, where X^C is the process under the strategy C. Let further X^ξ be the process under the constant strategy ξ, i.e. X^ξ is a Brownian motion with drift $(\mu-\xi)$ and volatility σ. Then it is clear that

$$X_s^\xi \le X_s^C\;.$$

If $\mu > \xi$ then it holds, see for example [9, p. 223], $\mathbb{P}_{(t,x)}[\tau^C=\infty] \ge \mathbb{P}_{(t,x)}[\tau^\xi=\infty] > 0$. ■

Finally, we gather one structural property of the value function which, however, is not used later.

Proposition 2.3
The value function is Lipschitz-continuous, strictly increasing in x and decreasing in t.

Proof: • Let $h>0$, $\varepsilon>0$ be arbitrary but fixed. Let further C be an ε-optimal strategy for $(t,x)\in\mathbb{R}_+^2$, i.e. $V(t,x)\le V^C(t,x)+\varepsilon$. Define the strategy $\tilde C$ for $(t,x+h)$ in the following way:

$$\tilde C_s = \begin{cases} C_s & : t\le s<\tau^C, \\ \xi & : \text{otherwise.}\end{cases}$$

Then, $\tilde{C}$ is an admissible strategy and does actually the same as the strategy C until the process $X^{\tilde{C}}$ reaches the level h. Afterwards it pays at maximal rate until ruin which is strictly later $\tau^C < \tau^{\tilde{C}}$. Note that $U(x+y) = U(x) + e^{-\gamma x}U(y)$ and, hence, we have

$$\begin{aligned} V(t,x+h) - V(t,x) &\ge V^{\tilde{C}}(t,x+h) - V^C(t,x) - \varepsilon \\ &= \mathbb{E}_{(t,x+h)}\Big[U\Big(\int_t^{\tau^{\tilde{C}}} \tilde{C}_s e^{-\delta s}\,\mathrm{d}s\Big)\Big] - \mathbb{E}_{(t,x)}\Big[U\Big(\int_t^{\tau^C} C_s e^{-\delta s}\,\mathrm{d}s\Big)\Big] - \varepsilon \\ &= \mathbb{E}_{(t,x+h)}\Big[e^{-\gamma\int_t^{\tau^C}\tilde{C}_s e^{-\delta s}\,\mathrm{d}s}U\Big(\int_{\tau^C}^{\tau^{\tilde{C}}} \xi e^{-\delta s}\,\mathrm{d}s\Big)\Big] - \varepsilon \\ &\ge \mathbb{E}_{(t,x+h)}\Big[e^{-\gamma\int_t^{\infty}\xi e^{-\delta s}\,\mathrm{d}s}U\Big(\int_{\tau^C}^{\tau^{\tilde{C}}} \xi e^{-\delta s}\,\mathrm{d}s\Big)\Big] - \varepsilon \\ &\ge K_h - \varepsilon\,, \end{aligned}$$

where $K_h > 0$ and can be chosen independent of the strategy C. Thus we find that $V(t,x+h) - V(t,x) \ge K_h$.

• Consider further $(t,0)$ with $t \in \mathbb{R}_+$. Let $h,\varepsilon > 0$ and C be an arbitrary admissible strategy. Let τ^0 be the ruin time for the strategy which is constant zero. Define

$$\varrho_n := \frac{\sqrt{\mu^2 + 2\sigma^2\delta n}}{\sigma^2}\,, \quad \theta_n := \frac{-\mu}{\sigma^2} + \varrho_n \quad \text{and} \quad \zeta_n := \frac{-\mu}{\sigma^2} - \varrho_n \tag{5}$$

for any $n \in \mathbb{N}$. Using $\mathbb{E}_h[e^{-\delta\tau^0}] = e^{\zeta_1 h}$, confer for instance [9, p. 295]. It follows with $X_s^0 \ge X_s^C$ and using the convexity of the exponential function, $U(x) = \frac{1-e^{-\gamma x}}{\gamma} \le x$:

$$\begin{aligned} V^C(t,h) = \mathbb{E}_{(t,h)}\Big[U\Big(\int_t^{\tau^C} e^{-\delta s}C_s\,\mathrm{d}s\Big)\Big] &\le \mathbb{E}_h\Big[U\Big(\xi\int_t^{t+\tau^0} e^{-\delta s}\,\mathrm{d}s\Big)\Big] \\ &= \mathbb{E}_h\Big[U\Big(\frac{\xi}{\delta}e^{-\delta t}(1-e^{-\delta\tau^0})\Big)\Big] \le \frac{\xi}{\delta}e^{-\delta t}\big(1-e^{\zeta_1 h}\big) \le -\frac{\xi}{\delta}\zeta_1 h\,. \end{aligned} \tag{6}$$

– Let $h \ge 0$ and τ^0 be the ruin time for the strategy which is constant zero. Let $(t,x) \in \mathbb{R}_+^2$ be arbitrary, C be an admissible strategy which is ε-optimal for the starting point $(t,x+h)$, i.e. $V(t,x+h) - V^C(t,x+h) \le \varepsilon$. Define further $\tilde{\tau} := \inf\{s \ge t:\ X_s^C = h\}$. Then $X_s^C \ge 0$ for $s \in [t,\tilde{\tau}]$ under $\mathbb{P}_{(t,x)}$ because $X_s^C \ge h$ for $s \in [t,\tilde{\tau}]$ under $\mathbb{P}_{(t,x+h)}$. Then, the strategy C, up to $\tilde{\tau}$ is an admissible strategy for (t,x) fulfilling

$$V^C(t,x) = \mathbb{E}_{(t,x)}\Big[U\Big(\int_t^{\tilde{\tau}} e^{-\delta s}C_s\,\mathrm{d}s\Big)\Big] = \mathbb{E}_{(t,x+h)}\Big[U\Big(\int_t^{\tilde{\tau}} e^{-\delta s}C_s\,\mathrm{d}s\Big)\Big]\,.$$

Note that $X_{\tilde{\tau}}^C = h$ and, hence, we have

$$\begin{aligned} \tau^C - \tilde{\tau} &= \inf\{u \ge 0 : X_{u+\tilde{\tau}}^C = 0\} = \inf\{u \ge 0 : h + (X_{u+\tilde{\tau}} - X_{\tilde{\tau}}) - \int_{\tilde{\tau}}^u C_r\,\mathrm{d}r = 0\} \\ &\le \inf\{u \ge 0 : h + (X_{u+\tilde{\tau}} - X_{\tilde{\tau}}) = 0\} =: \beta^0 \end{aligned}$$

where $\mathbb{P}^{\beta_0}_{(t,x+h)} = \mathbb{P}^{\tau^0}_{t,h}$. Here, $\mathbb{P}^{\tau^0}_{t,h}$ denotes the law of τ^0 under $\mathbb{P}_{t,h}$; analogously $\mathbb{P}^{\beta_0}_{(t,x+h)}$ is the law of β_0 under $\mathbb{P}_{t,x+h}$. Since, U fulfils $U(a+b) \le U(a) + U(b)$ for any $a, b \ge 0$, we have

$$\begin{aligned} V(t,x+h) &\le V^C(t,x+h) + \varepsilon = \mathbb{E}_{(t,x+h)}\left[U\Big(\int_t^{\tau^C} e^{-\delta s} C_s \,\mathrm{d}s\Big)\right] + \varepsilon \\ &= \mathbb{E}_{(t,x+h)}\left[U\Big(\int_t^{\tilde\tau} e^{-\delta s} C_s \,\mathrm{d}s + \int_{\tilde\tau}^{\tau^C} e^{-\delta s} C_s \,\mathrm{d}s\Big)\right] + \varepsilon \\ &\le \mathbb{E}_{(t,x+h)}\left[U\Big(\int_t^{\tilde\tau} e^{-\delta s} C_s \,\mathrm{d}s\Big)\right] + \mathbb{E}_{(t,x+h)}\left[U\Big(\int_{\tilde\tau}^{\tau^C} e^{-\delta s} C_s \,\mathrm{d}s\Big)\right] + \varepsilon \\ &\le V^C(t,x) + \mathbb{E}_{(t,x+h)}\left[U\Big(\frac{\xi}{\delta}\big(e^{-\delta\tilde\tau} - e^{-\delta\tau^C}\big)\Big)\right] + \varepsilon \\ &\le V(t,x) + \mathbb{E}_{(t,x+h)}\left[U\Big(\frac{\xi}{\delta}\big(1 - e^{-\delta(\tau^C-\tilde\tau)}\big)\Big)\right] + \varepsilon \\ &\le V(t,x) + \mathbb{E}_{h}\left[U\Big(\frac{\xi}{\delta}\big(1 - e^{-\delta\tau^0}\big)\Big)\right] + \varepsilon\,. \end{aligned}$$

Because ε was arbitrary and due to (6) we find

$$0 \le V(t,x+h) - V(t,x) \le -\frac{\xi}{\delta}\zeta_1 h\,.$$

Consequently, V is Lipschitz-continuous in the space variable x with Lipschitz-constant at most $-\frac{\xi}{\delta}\zeta_1$.

• Next, we consider the properties of the value function concerning the time variable. Because $\delta > 0$, it is clear that V is strictly decreasing in t. First we show that the value function is strictly decreasing in time. To this end let $(t,x) \in \mathbb{R}^2_+$, $h > 0$ and C be an admissible strategy which is constant zero before time $t+h$ and τ its ruin time. Since C is measurable with respect to the σ-algebra $\sigma(X_s : s \ge t)$ we find a measurable function $c : \mathbb{R}_+ \times C(\mathbb{R}_+, \mathbb{R}) \to \mathbb{R}$ such that $C_s(\omega) = c(s-t, (X_{t+u})_{u\ge 0})$. Defining $\tilde C_s := c(s-(t+h), (X_{t+h+u})_{u\ge 0}$ we see that the law of $(X_s, C_s)_{s\ge t}$ under $\mathbb{P}_{(t,x)}$ equals the law of $(X_{s+h}, \tilde C_{s+h})_{s\ge t}$ under $\mathbb{P}_{(t+h,x)}$.

The stopping time $\tilde\tau := \inf\{s \ge t+h : X^{\tilde C}_s = 0\}$ is the corresponding ruin time.

$$\begin{aligned} V^C(t,x) &= \mathbb{E}_{(t,x)}\left[U\Big(\int_t^{\tau} C_s e^{-\delta s} \,\mathrm{d}s\Big)\right] \\ &= \mathbb{E}_{(t+h,x)}\left[U\Big(\int_{t+h}^{\tilde\tau} \tilde C_s e^{-\delta(s-h)} \,\mathrm{d}s\Big)\right] \end{aligned}$$

Taking the supremum over all strategies yields

$$V(t,x) = \sup_{\tilde C} \mathbb{E}_{(t+h,x)}\left[U\left(e^{\delta h}\int_{t+h}^{\tilde\tau^{\tilde C}} \tilde C_s e^{-\delta s} \,\mathrm{d}s\right)\right] > V(t+h,x).$$

Let further $(t, x) \in \mathbb{R}_+^2$, $h > 0$ and C be an admissible strategy. Then, the strategy $\tilde{C}$ with $\tilde{C}_s := C_{s-h} \mathbb{I}_{\{s \geq h\}}$ is admissible. Since, U is concave we have

$$\begin{aligned} V(t+h, x) \geq V^{\tilde{C}}(t+h, x) &= \mathbb{E}_{(t+h,x)}\Big[U\Big(\int_{t+h}^{\tau^C+h} e^{-\delta s} C_{s-h} \,\mathrm{d}s\Big)\Big] \\ &= \mathbb{E}_{(t,x)}\Big[U\Big(e^{-\delta h}\int_t^{\tau^C} e^{-\delta s} C_s \,\mathrm{d}s\Big)\Big] \geq e^{-\delta h} V^C(t, x) \,. \end{aligned}$$

Building the supremum over all admissible strategies on the right side of the above inequality and using Remark 2.1, yields

$$0 \geq V(t+h, x) - V(t, x) \geq V(t, x)(e^{-\delta h} - 1) \geq -U\Big(\frac{\xi}{\delta}\Big)\delta h$$

and, consequently, V is Lipschitz-continuous as a function of t with constant $\delta U(\xi/\delta)$. $\square$

2.1 Heuristics

Heuristically, our approach to compare a given feedback strategy C with the unknown optimal strategy C^* works as follows:

1. We start with the performance function V^C corresponding to some feedback strategy $C_t = c(t, X_t^C)$. V^C satisfies (if smooth enough)

$$V_t^C + \mu V_x^C + \frac{\sigma^2}{2} V_{xx}^C + c\big\{-V_x^C + e^{-\delta t}(1 - \gamma V^C)\big\} = 0, \quad V^C(t, 0) = 0 \,.$$

2. However, sometimes V^C is not smooth enough or not known explicitly. In this case one would use a replacement H (simply any $\mathcal{C}^{1,2}$-function from $\mathbb{R}_+ \times \mathbb{R}$ to $\mathbb{R}$ with $H(t, 0) = 0$) and define the mismatch

$$\Psi := H_t + \mu H_x + \frac{\sigma^2}{2} H_{xx} + c\big\{-H_x + e^{-\delta t}(1 - \gamma H)\big\} \,.$$

where Ψ as close to zero as possible is desirable.

3. We consider an other strategy C^* and the corresponding controlled process $X^* = X^{C^*}$ as well as their performance $V^* := V^{C^*}$. Its ruin time is denoted by τ and

we obtain from Itô's formula using $X_\tau^* = 0$ and $H(t,0) \equiv 0$:

$$
\begin{aligned}
0 &= e^{-\gamma \int_t^\tau e^{-\delta u} C_u^* \,\mathrm{d}u} \cdot H(\tau, X_\tau^*) \\
&= H(t, X_t^*) + \int_t^\tau e^{-\gamma \int_t^s e^{-\delta u} C_u^* \,\mathrm{d}u} \cdot H_x \,\mathrm{d}W_s \\
&\quad + \int_t^\tau e^{-\gamma \int_t^s e^{-\delta u} C_u^* \,\mathrm{d}u} \cdot \left(H_t + \mu H_x + \frac{\sigma^2}{2} H_{xx} + C_s^* \big\{ - H_x - \gamma e^{-\delta s} H \big\} \right) \mathrm{d}s \\
&= H(t, X_t^*) + \int_t^\tau e^{-\gamma \int_t^s e^{-\delta u} C_u^* \,\mathrm{d}u} \cdot H_x \,\mathrm{d}W_s \\
&\quad + \int_t^\tau e^{-\gamma \int_t^s e^{-\delta u} C_u^* \,\mathrm{d}u} \cdot \left(H_t + \mu H_x + \frac{\sigma^2}{2} H_{xx} + c(-H_x + e^{-\delta s}(1 - \gamma H)) \right) \mathrm{d}s \\
&\quad + \int_t^\tau e^{-\gamma \int_t^s e^{-\delta u} C_u^* \,\mathrm{d}u} \cdot (C_s^* - c) \big\{ - H_x + e^{-\delta s}(1 - \gamma H) \big\} \,\mathrm{d}s \\
&\quad - \int_t^\tau e^{-\gamma \int_t^s e^{-\delta u} C_u^* \,\mathrm{d}u} \cdot e^{-\delta s} C_s^* \,\mathrm{d}s \\
&= H(t, X_t^*) + \int_t^\tau e^{-\gamma \int_t^s e^{-\delta u} C_u^* \,\mathrm{d}u} \cdot H_x \,\mathrm{d}W_s + \int_t^\tau e^{-\gamma \int_t^s e^{-\delta u} C_u^* \,\mathrm{d}u} \cdot \Psi \,\mathrm{d}s \\
&\quad + \int_t^\tau e^{-\gamma \int_t^s e^{-\delta u} C_u^* \,\mathrm{d}u} (C_s^* - c) \big\{ - H_x + e^{-\delta s}(1 - \gamma H) \big\} \,\mathrm{d}s \\
&\quad - U\Big(\int_t^\tau e^{-\delta u} C_u^* \,\mathrm{d}u \Big) .
\end{aligned}
$$

4. Taking $\mathbb{P}_{(t,x)}$-expectation (assuming the local martingale from the $\mathrm{d}W_s$-integral is a martingale) bringing the expectation of the U-term on the other side yields

$$
\begin{aligned}
V^*(t,x) &= H(t,x) + \mathbb{E}_{(t,x)} \left[\int_t^\tau e^{-\gamma \int_t^s e^{-\delta u} C_u^* \,\mathrm{d}u} \cdot \Psi \,\mathrm{d}s \right] \\
&\quad + \mathbb{E}_{(t,x)} \left[\int_t^\tau e^{-\gamma \int_t^s e^{-\delta u} C_u^* \,\mathrm{d}u} \cdot (C_s^* - c) \big\{ - H_x + e^{-\delta s}(1 - \gamma H) \big\} \,\mathrm{d}s \right] .
\end{aligned}
$$

5. Up to here, this is all standard. The performance function V^* is expressed in terms of a new function H plus two error terms which could have a negative sign. Several other stochastic control problems can be brought to a similar equation. The first error term corresponds to the usage of a function other than the performance function of our initial feedback control C. The second error term corresponds to the suboptimality of the feedback control C compared to the control C^* measured relatively by the function H.

6. Now, we need to control the error terms despite the appearance of the unknown optimal control. The first error term is simply bounded by

$$
\left| \mathbb{E}_{(t,x)} \left[\int_t^\tau e^{-\gamma \int_t^s e^{-\delta u} C_u^* \,\mathrm{d}u} \Psi \,\mathrm{d}s \right] \right| \le \mathbb{E}_{(t,x)} \left[\int_t^\tau |\Psi(s, X_s^*)| \,\mathrm{d}s \right] ,
$$

where one has to deal with the unknown process X^* but its control has disappeared. This is the point where occupation bounds as in the appendix yield explicit upper bounds.

7. The appearance of C^* in the second error term can be suppressed via maximising the integrand over all possible values of C^*. Since $C = C^*$ is a possible value, this maximum is positive and we obtain:

$$\mathbb{E}_{(t,x)}\left[\int_t^\tau \exp(-\gamma\int_t^s e^{-\delta u}C_u^*\,\mathrm{d}u)(C_s^* - c)(-H_x + e^{-\delta s}(1-\gamma H))\,\mathrm{d}s\right]$$
$$\leq \mathbb{E}_{(t,x)}\left[\int_t^\tau \sup_{y\in[0,\xi]}\Big((y-c)(-H_x + e^{-\delta s}(1-\gamma H))\Big)\,\mathrm{d}s\right].$$

8. Putting these together we obtain:

$$V^*(t,x) \leq H(t,x) + \mathbb{E}_{(t,x)}\left[\int_t^\tau |\Psi(s,X_s^*)|\,\mathrm{d}s\right]$$
$$+\mathbb{E}_{(t,x)}\left[\int_t^\tau \sup_{y\in[0,\xi]}\Big((y-c)(-H_x(s,X_s^*) + e^{-\delta s}(1-\gamma H(s,X_s^*)))\Big)\,\mathrm{d}s\right].$$

9. If we have a common upper bound $\Upsilon_{t,x} \geq 0$ for $\mathbb{E}_{(t,x)}\left[\int_t^\tau |\Psi(s,X_s^*)|\,\mathrm{d}s\right]$, then we may take the supremum over all strategies on the left hand side and obtain

$$V(t,x) \leq H(t,x) + \Upsilon_{t,x}$$
$$+\mathbb{E}_{(t,x)}\left[\int_t^\tau \sup_{y\in[0,\xi]}\Big((y-c)(-H_x(s,X_s^*) + e^{-\delta s}(1-\gamma H(s,X_s^*)))\Big)\,\mathrm{d}s\right].$$

   We will employ bounds for the expected occupation to obtain such a common upper bound which are summarised in the appendix. Note, that after choosing the optimal y in dependence on (s, X_x^*) allows to employ those common upper bounds also to the second summand.

Remark 2.4
We like to note that if $H = V^C$ (i.e. $\Psi = 0$) and if in the maximisation in Point 8 above the maximiser is attained in C, then both error terms vanish and we find

$$V^C \leq V \leq H = V^C$$

which yields that all are the same. That means, if a feedback control C is found such that its performance function V^C satisfies the HJB equation

$$\sup_{y\in[0,\xi]}\left(V_t^C + \mu V_x^C + \frac{\sigma^2}{2}V_{xx}^C + y(-V_x^C + e^{-\delta t}(1-\gamma V^C))\right) = 0, \quad V^C(t,0) = 0,$$

then we verified heuristically $V^C = V$. ■

3 Payout on the Maximal Rate

3.1 Could it be optimal to pay on the maximal rate up to ruin?

At first, we investigate the constant strategy ξ, i.e. the dividends will be paid out at the maximal rate ξ until ruin. In this section we find exact conditions under which this strategy is optimal. We already know from (4) that the corresponding return function is given by

$$V^\xi(t,x) = \frac{1}{\gamma} - \frac{1}{\gamma}e^{-\Delta e^{-\delta t}} - e^{-\Delta e^{-\delta t}}\sum_{n=1}^\infty \frac{\Delta^n}{\gamma n!}e^{-\delta t n}e^{\eta_n x}\;.$$

It is obvious that V^ξ is increasing and concave in x and decreasing in t. For further considerations we will need the following remark.

Remark 3.1
Consider η_n, defined in (2), as a function of ξ.

1. *Since*

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\eta_n = \frac{-\eta_n}{\sqrt{(\xi-\mu)^2+2\delta\sigma^2 n}},$$

it is easy to see that $\eta_n(\xi)$ and $\frac{\eta_{n+1}(\xi)n}{\eta_n(\xi)(n+1)}$ are increasing in ξ. Also, we have

$$\lim_{\xi\to\infty}\frac{\eta_{n+1}(\xi)n}{\eta_n(\xi)(n+1)} = 1\;.$$

We conclude that $\frac{\eta_{n+1}}{(n+1)} > \frac{\eta_n}{n}$, as $\eta_n, \eta_{n+1} < 0$.

2. *Further, we put to record*

$$\lim_{\xi\to\infty}\xi\eta_n(\xi) = -\delta n\;.$$

3. *Also, we have*

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\Big(\delta n + \xi\eta_n(\xi)\Big) = \eta_n\Big(1 - \frac{\xi}{\sqrt{(\xi-\mu)^2+2\delta\sigma^2 n}}\Big)\begin{cases} <0 & \xi < \frac{\mu^2+2\delta\sigma^2 n}{2\mu} \\ \ge 0 & \xi \ge \frac{\mu^2+2\delta\sigma^2 n}{2\mu}\end{cases}\;.$$

Thus, at $\xi=0$ the function $\xi\mapsto \delta n+\xi\eta_n(\xi)$ attains the value $\delta n>0$, at its minimum point $\xi^* = \frac{\mu^2+2\delta\sigma^2 n}{2\mu}$ we have

$$\delta n + \xi^*\eta_n(\xi^*) = \delta n - \frac{\mu^2+2\delta\sigma^2 n}{2\sigma^2} = -\frac{\mu^2}{2\sigma^2} < 0$$

and, finally, for $\xi\to\infty$ it holds, due to Item 2 above, that $\lim\limits_{\xi\to 0}\delta n + \xi\eta_n(\xi) = 0$. Thus, for every $n\in\mathbb{N}$ the function $\xi\mapsto 1+\frac{\eta_n(\xi)\xi}{\delta n}$ has a unique zero at $\frac{\delta n\sigma^2}{2\mu}$. ■

Further, it is easy to check that in V^ξ summation and differentiation can be interchanged. Derivation with respect to x yields

$$V_x^\xi(t,x) = -e^{-\Delta e^{-\delta t}} \sum_{n=1}^{\infty} \frac{\Delta^n}{\gamma n!} e^{-\delta t n} \eta_n e^{\eta_n x} \; .$$

In order to answer the optimality question, we have to look at the function $-V_x^\xi + e^{-\delta t}\big(1-\gamma V^\xi\big)$, appearing in the crucial condition in HJB equation (1). If this expression is positive for all $(t,x) \in \mathbb{R}_+^2$ the function V^ξ becomes a candidate for the value function. For simplicity, we multiply the expression $-V_x^\xi + e^{-\delta t}\big(1-\gamma V^\xi\big)$ by $e^{\delta t} e^{\Delta e^{-\delta t}}$ and define

$$\begin{aligned}
\psi(t,x) :&= \frac{e^{\Delta t}}{t}\Big\{ -V_x^\xi\Big(\frac{\ln(t)}{-\delta}, x\Big) + t\Big(1-\gamma V^\xi\Big(\frac{\ln(t)}{-\delta}, x\Big)\Big)\Big\} \\
&= \sum_{n=0}^{\infty} t^n \frac{\Delta^n}{n!}\Big\{\frac{\eta_{n+1}\xi}{\delta(n+1)} e^{\eta_{n+1}x} + e^{\eta_n x}\Big\} \; . \qquad (7)
\end{aligned}$$

If $\psi \geq 0$ on $[0,1] \times \mathbb{R}_+$, then V^ξ does solve the HJB equation and as we will see, it is the value function in that case.

Proposition 3.2
V^ξ is the value function if and only if $\xi \leq \frac{\delta\sigma^2}{2\mu}$. In that case V^ξ is a classical solution to the HJB equation (1) *and an optimal strategy is constant ξ.*

Proof: Since $\frac{\sigma^2}{2}\eta_n^2 = (\xi-\mu)\eta_n + \delta n$ for all $n \geq 1$, it is easy to check, using the power series representation of V^ξ, that V^ξ solves the differential equation

$$V_t^\xi + \mu V_x^\xi + \frac{\sigma^2}{2} V_{xx}^\xi + \xi\Big(-V_x^\xi + e^{-\delta t}(1-\gamma V^\xi)\Big) = 0 \; .$$

We first assume that $\xi \leq \frac{\delta\sigma^2}{2\mu}$ and show that V^ξ is the value function. Note that $\xi \leq \frac{\delta\sigma^2}{2\mu}$ is equivalent to $\frac{\eta_1\xi}{\delta} + 1 \geq 0$. We have $\xi \leq n\frac{\delta\sigma^2}{2\mu}$ for any $n \geq 1$ and Remark 3.1 **1.** yields for all $n \geq 2$

$$\eta_n \frac{\xi}{\delta n} + 1 > \eta_1 \frac{\xi}{\delta} + 1 \geq 0.$$

This gives immediately for all $(t,x) \in (0,1] \times \mathbb{R}_+$:

$$\psi(t,x) \geq \sum_{n=0}^{\infty} t^n \frac{\Delta^n}{n!}\Big\{\frac{\eta_{n+1}\xi}{\delta(n+1)} + 1\Big\} e^{\eta_n x} \geq 0 \; .$$

which is equivalent to

$$-V_x^\xi(t,x) + e^{-\delta t}\big(1-\gamma V^\xi(t,x)\big) \geq 0$$

for all $(t,x) \in \mathbb{R}_+^2$. This means that V^ξ solves HJB equation (1) if $\xi \leq \frac{\delta\sigma^2}{2\mu}$.
Let now C be an arbitrary admissible strategy, τ its ruin time and $\hat{X}_u := X_u^C$. Applying Ito's formula yields $\mathbb{P}_{(t,x)}$-a.s.

$$e^{-\gamma \int_t^{\tau\wedge s} e^{-\delta u} C_u \,\mathrm{d}u} V^\xi(\tau\wedge s, \hat{X}_{\tau\wedge s}) = V^\xi(t,x) + \sigma \int_t^{\tau\wedge s} e^{-\gamma \int_t^y e^{-\delta u} C_u \,\mathrm{d}u} V_x^\xi \,\mathrm{d}W_y$$
$$+ \int_t^{\tau\wedge s} e^{-\gamma \int_t^y e^{-\delta u} C_u \,\mathrm{d}u} \Big\{ V_t^\xi + (\mu - C_y) V_x^\xi + \frac{\sigma^2}{2} V_{xx}^\xi - \gamma C_y e^{-\delta y} V^\xi \Big\} \,\mathrm{d}y \,.$$

Since V_x^ξ is bounded, the stochastic integral above is a martingale with expectation zero. For the second integral one obtains using the differential equation for V^ξ:

$$\int_t^{\tau\wedge s} e^{-\gamma \int_t^y e^{-\delta u} C_u \,\mathrm{d}u} \Big\{ V_t^\xi + (\mu - C_y) V_x^\xi + \frac{\sigma^2}{2} V_{xx}^\xi - \gamma C_y e^{-\delta y} V^\xi \Big\} \,\mathrm{d}y$$
$$= \int_t^{\tau\wedge s} e^{-\gamma \int_t^y e^{-\delta u} C_u \,\mathrm{d}u} \Big\{ \big(C_y - \xi\big)\Big[- V_x^\xi + e^{-\delta y}\big(1 - \gamma V^\xi\big)\Big] - C_y e^{-\delta y} \Big\} \,\mathrm{d}y.$$

Building the expectations on the both sides and letting $s \to \infty$, we obtain by interchanging limit and expectation (due to the bounded convergence theorem):

$$0 = V^\xi(t,x)$$
$$+ \mathbb{E}_{(t,x)}\Big[\int_t^\tau e^{-\gamma \int_t^y e^{-\delta u} C_u \,\mathrm{d}u} \big(C_y - \xi\big) \Big\{ - V_x^\xi + e^{-\delta y}\big(1 - \gamma V^\xi\big) \Big\} \,\mathrm{d}y \Big] \tag{8}$$
$$- \mathbb{E}_{(t,x)}\Big[\int_t^\tau e^{-\gamma \int_t^y e^{-\delta u} C_u \,\mathrm{d}u} \cdot C_y e^{-\delta y} \,\mathrm{d}y \Big] \,. \tag{9}$$

Since $C_u \le \xi$ and $-V_x^\xi\big(y, \hat{X}_y\big) + e^{-\delta y}\big(1 - \gamma V^\xi\big(y, \hat{X}_y\big)\big) \ge 0$, the expectation in (8) is non-positive.
For (9) one has

$$\mathbb{E}_{(t,x)}\Big[\int_t^\tau e^{-\gamma \int_t^y e^{-\delta u} C_u \,\mathrm{d}u} \cdot C_y e^{-\delta y} \,\mathrm{d}y \Big] = -\mathbb{E}_{(t,x)}\Big[\int_t^\tau \mathrm{d} \frac{e^{-\gamma \int_t^y e^{-\delta u} C_u \,\mathrm{d}u}}{\gamma} \Big]$$
$$= \mathbb{E}_{(t,x)}\Big[U\Big(\int_t^\tau e^{-\delta u} C_u \,\mathrm{d}u \Big) \Big] = V^C(t,x) \,,$$

giving $V^C(t,x) \le V^\xi(t,x)$ for all admissible strategies C. Therefore, V^ξ is the value function.

Let now $\xi > \frac{\delta\sigma^2}{2\mu}$ and assume for contradiction that V^ξ is the value function. Then we have $\psi(0,0) = 1 + \eta_1 \xi/\delta < 0$. It means in particular, that the function ψ is negative also for some $(t,x) \in (0,1] \times \mathbb{R}_+$. Consequently, V^ξ does not solve the HJB equation (1). However, V^ξ is smooth enough and has a bounded x-derivative. Thus, classical verification results (see, for instance, [19] Section 2.5.1) yield that V^ξ solves the HJB equation. A contradiction. $\square$

In the following, we assume $\xi > \frac{\delta\sigma^2}{2\mu}$.

3.2 The goodness of the strategy ξ.

We now provide an estimate on the goodness of the constant payout strategy which relies only on the performance of the chosen strategy ξ and on deterministic constants. Recall from (2) and (5) that

$$\begin{aligned}\eta_n &= \frac{(\xi-\mu)-\sqrt{(\xi-\mu)^2+2n\delta\sigma^2}}{\sigma^2},\\ \theta_n &= \frac{-\mu+\sqrt{\mu^2+2n\delta\sigma^2}}{\sigma^2}, \quad \zeta_n = \frac{-\mu-\sqrt{\mu^2+2n\delta\sigma^2}}{\sigma^2}.\end{aligned}$$

We first present the main inequality of this section and then discuss in the following remark finiteness of the sum.

Proposition 3.3
Let $t, x \ge 0$. Then we have

$$\begin{aligned}V(t,x) &\le V^\xi(t,x)\\ &\quad + \xi e^{-\delta t}\sum_{n=0}^\infty e^{-\delta t n}\frac{\Delta^n}{n!}\int_0^\infty \left(\frac{-\eta_{n+1}\xi}{\delta(n+1)}e^{\eta_{n+1}y}-e^{\eta_n y}\right)^+ f_{n+1}(x,y)\,\mathrm{d}y,\end{aligned}$$

where

$$f_n(x,y) := \frac{2\left(e^{\theta_n(x\wedge y)}-e^{\zeta_n(x\wedge y)}\right)e^{\eta_n(x-y)^+}}{\sigma^2\left((\theta_n-\eta_n)e^{y\theta_n}-(\zeta_n-\eta_n)e^{y\zeta_n}\right)}, \quad y \ge 0.$$

Proof: We know that the return function $V^\xi \in \mathcal{C}^{1,2}$. Let C be an arbitrary admissible strategy. Then, using Ito's formula for $s > t$ under $\mathbb{P}_{(t,x)}$:

$$\begin{aligned}&e^{-\gamma\int_t^{s\wedge\tau^C} e^{-\delta u}C_u\,\mathrm{d}u}\cdot V^\xi(s\wedge\tau^C, X^C_{s\wedge\tau^C})\\ &= V^\xi(t,x)+\int_t^{s\wedge\tau^C} e^{-\gamma\int_t^r e^{-\delta u}C_u\,\mathrm{d}u}\cdot\left\{V^\xi_t+(\mu-C_r)V^\xi_x+\frac{\sigma^2}{2}V^\xi_{xx}-\gamma e^{-\delta r}C_rV^\xi\right\}\mathrm{d}r\\ &\quad+\sigma\int_t^{s\wedge\tau^C} e^{-\gamma\int_t^r e^{-\delta u}C_u\,\mathrm{d}u}\cdot V^\xi_x\,\mathrm{d}W_r\,.\end{aligned}$$

Using the differential equation for V^ξ, one obtains like in the last proof, using the definition of ψ from (7):

$$\begin{aligned}&e^{-\gamma\int_t^{s\wedge\tau^C} e^{-\delta u}C_u\,\mathrm{d}u}\cdot V^\xi(s\wedge\tau^C, X^C_{s\wedge\tau^C})\\ &\quad= V^\xi(t,x)+\int_t^{s\wedge\tau^C} e^{-\gamma\int_t^r e^{-\delta u}C_u\,\mathrm{d}u}\cdot(C_r-\xi)\cdot e^{-\delta r}e^{-\Delta e^{-\delta r}}\psi(e^{-\delta r}, X^C_r)\,\mathrm{d}r\\ &\qquad-\int_t^{s\wedge\tau^C} e^{-\gamma\int_t^r e^{-\delta u}C_u\,\mathrm{d}u}\cdot C_re^{-\delta r}\,\mathrm{d}r+\sigma\int_t^{s\wedge\tau^C} e^{-\gamma\int_t^r e^{-\delta u}C_u\,\mathrm{d}u}\cdot V^\xi_x\,\mathrm{d}W_r\,.\end{aligned}$$

Building the $\mathbb{P}_{(t,x)}$-expectations, letting $s \to \infty$ and rearranging the terms, one has

$$
\begin{aligned}
V^C(t,x) &= V^\xi(t,x) \\
&\quad + \mathbb{E}_{(t,x)}\Big[\int_t^{\tau^C} e^{-\gamma \int_t^r e^{-\delta u} C_u \,\mathrm{d}u} \cdot (C_r - \xi) \cdot e^{-\delta r} e^{-\Delta e^{-\delta r}} \psi(e^{-\delta r}, X_r^C) \,\mathrm{d}r \Big] .
\end{aligned}
$$

Our goal is to find a C-independent estimate for the expectation on the rhs. above, in order to gain a bound for the difference $V(t,x) - V^\xi(t,x)$. Since $e^{-\gamma \int_t^r e^{-\delta u} C_u \,\mathrm{d}u} \le 1$, $e^{-\Delta e^{-\delta r}} \le 1$ and $-(C_r - \xi) \le \xi$ we have

$$
\begin{aligned}
&\mathbb{E}_{(t,x)}\Big[\int_t^{\tau^C} e^{-\gamma \int_t^r e^{-\delta u} C_u \,\mathrm{d}u} \cdot (C_r - \xi) \cdot e^{-\delta r} e^{-\Delta e^{-\delta r}} \psi(e^{-\delta r}, X_r^C) \,\mathrm{d}r \Big] \\
&\le -\xi \mathbb{E}_{(t,x)}\Big[\int_t^{\tau^C} e^{-\gamma \int_t^r e^{-\delta u} C_u \,\mathrm{d}u} \cdot e^{-\delta r} e^{-\Delta e^{-\delta r}} \psi(e^{-\delta r}, X_r^C) \mathbb{1}_{\{\psi(e^{-\delta r}, X_r^C) < 0\}} \,\mathrm{d}r \Big] \\
&\le -\xi \mathbb{E}_{(t,x)}\Big[\int_t^{\tau^C} e^{-\delta r} \psi(e^{-\delta r}, X_r^C) \mathbb{1}_{\{\psi(e^{-\delta r}, X_r^C) < 0\}} \,\mathrm{d}r \Big] .
\end{aligned}
$$

Now, inserting the power series representation of ψ from (7), one gets

$$
\begin{aligned}
&- \xi \mathbb{E}_{(t,x)}\Big[\int_t^{\tau^C} e^{-\delta r} \psi(e^{-\delta r}, X_r^C) \mathbb{1}_{\{\psi(e^{-\delta r}, X_r^C) < 0\}} \,\mathrm{d}r \Big] \\
&\le \xi \sum_{n=0}^\infty e^{-\delta t(n+1)} \frac{\Delta^n}{n!} \mathbb{E}_{(t,x)}\Big[\int_t^{\tau^C} e^{-\delta(r-t)(n+1)} \left(\frac{-\eta_{n+1}\xi}{\delta(n+1)} e^{\eta_{n+1} X_r^C} - e^{\eta_n X_r^C} \right)^+ \mathrm{d}r \Big] \\
&\le \xi e^{-\delta t} \sum_{n=0}^\infty e^{-\delta t n} \frac{\Delta^n}{n!} \int_0^\infty \left(\frac{-\eta_{n+1}\xi}{\delta(n+1)} e^{y\eta_{n+1}} - e^{\eta_n y} \right)^+ f_{n+1}(x,y) \,\mathrm{d}y
\end{aligned}
$$

where the last inequality follows from Theorem A.1. □

Remark 3.4
One could wonder if the infinite sum appearing on the right hand side of Proposition 3.3 is finite. In order to see its finiteness we try to find an upper bound of the form A^n *for the integral. To this end we split the integral in two parts, from* 0 *to* x *and the remaining part. Since* $\theta_n \to \infty$ *while* $\eta_n, \zeta_n \to -\infty$ *for* $n \to \infty$*. We have for* $0 \le y \le x$ *that*

$$
\begin{aligned}
f_n(x,y) &:= \frac{2\left(1 - e^{(\zeta_n - \theta_n)y}\right) e^{\eta_n(x-y)}}{\sigma^2\left((\theta_n - \eta_n) - (\zeta_n - \eta_n) e^{(\zeta_n - \theta_n)y}\right)} \\
&\le K_1 \frac{1}{\theta_n - \eta_n} \\
&\le K_1
\end{aligned}
$$

for some suitable constant $K_1 > 0$ (not depending on n and y). For $0 \le x \le y$ we find

$$\begin{aligned} f_n(x,y) &:= \frac{2\left(e^{\theta_n(x-y)} - e^{\zeta_n x - \theta_n y}\right)}{\sigma^2\left((\theta_n - \eta_n) - (\zeta_n - \eta_n)e^{y(\zeta_n - \theta_n)}\right)} \\ &\le K_2 e^{\theta_n(x-y)} \end{aligned}$$

for some suitable constant $K_1 > 0$ (not depending on n and y). The bracket appearing inside the integral before f_{n+1} is bounded by some constant $K_3 > 0$. We find that

$$\int_0^\infty \left(\frac{-\eta_{n+1}\xi}{\delta(n+1)} e^{\eta_{n+1}y} - e^{\eta_n y}\right)^+ f_{n+1}(x,y)\,\mathrm{d}y \le xK_1K_3 + \frac{K_2K_3}{\theta_n} \le K_4$$

for some suitable constant $K_4 > 0$. Hence, the sum is bounded by

$$\exp(\Delta e^{-\delta t})K_4.$$

■

4 The Goodness of Constant Barrier Strategies

Shreve et al. [20] and Asmussen and Taksar [5] considered the problem of dividend maximisation for a surplus described by a Brownian motion with drift. The optimal strategy there turned out to be a barrier strategy with a constant barrier.
Let $q \in \mathbb{R}_+$ and C be given by $C_s = \xi 1\!\!1_{\{X_s^C > q\}}$, i.e. C is a barrier strategy with a constant barrier q and ruin time $\tau^C = \inf\{s \ge 0 :\ X_s^C = 0\}$. The corresponding return function fulfils due to the Markov-property of X^C

$$V^C(t,x) = \frac{1}{\gamma} - \frac{1}{\gamma}\mathbb{E}_x\left[e^{-\gamma \int_t^{t+\tau^C} e^{-\delta s} C_s\,\mathrm{d}s}\right].$$

Note that for every $a > 0$ we have

$$\mathbb{E}_x\left[e^{a\int_t^{t+\tau^C} e^{-\delta s}C_s\,\mathrm{d}s}\right] \le e^{a\int_t^\infty e^{-\delta s}\xi\,\mathrm{d}s} = e^{\frac{a\xi}{\delta}e^{-\delta t}} < \infty\,.$$

It means, the moment generating function of $\int_t^{t+\tau^C} e^{-\delta s} C_s\,\mathrm{d}s$ is infinitely often differentiable and all moments of $\int_t^{t+\tau^C} e^{-\delta s}C_s\,\mathrm{d}s$ exist.
Aiming at finding the performance function of a barrier strategy with a constant barrier q, we use the classical ansatz of calculating the performance “above the barrier”, “below the barrier” and putting these two solutions together via the smooth fit at the barrier (in our case a $\mathcal{C}^{(1,1)}$-fit). We define

$$\begin{aligned} M_n(q) &:= \mathbb{E}_q\Big[\Big(\Delta - \gamma\int_0^{\tau^C} e^{-\delta s}C_s\,\mathrm{d}s\Big)^n\Big] > 0, \\ \tau^{q,\xi} &:= \inf\{s \ge 0 :\ X_s^\xi = q\}, \\ \tau^{q,0} &:= \inf\{s \ge 0 :\ X_s^0 \notin (0,q)\}. \end{aligned}$$

Since, a barrier strategy depends on the surplus, but not on the time, we pretend to start at time 0 accounting for a different starting time $t > 0$ by shifting the corresponding stopping times by t. Starting at $x > q$, one will pay at the maximal rate ξ up to $\tau^{q,\xi}$ and then follow the barrier strategy with the starting value q. Starting at $x < q$, one would not pay dividends until $\tau^{q,0}$, i.e. until the surplus hits the level q or ruins. If the level q will be hit before ruin, then one would follow the barrier strategy starting at q. This means in particular, that after hitting the level q the strategy will be exactly the same does not matter whether starting at $x > q$ or at $x < q$. We will use this fact in order to enforce smooth fit ($\mathcal{C}^{(1,1)}$-fit) at the barrier. A $\mathcal{C}^{(1,2)}$-fit can usually be achieved just by a barrier strategy which turns out to be the optimal strategy and whose performance function is the value function, see for instance [5] and [19] for details, further explanations are given in Section 5.1. Figure 1 illustrates the $\mathcal{C}^{(1,1)}$-fit of the return function corresponding to the 5-barrier. The gray and black areas correspond to the "above the barrier" and "below the barrier" solutions. The right picture shows that the second derivative with respect to x of the performance function is not continuous at the barrier.
For $F(t,x) := V^C(t,x)$, $x > q$, and for $G(t,x) := V^C(t,x)$, $x < q$, it holds:

$$\begin{aligned}
F(t,x) &= \frac{1}{\gamma} - \frac{1}{\gamma}\mathbb{E}_x\Big[e^{-\gamma\xi\int_t^{t+\tau^{q,\xi}} e^{-\delta s}\,\mathrm{d}s - \gamma\int_{t+\tau^{q,\xi}}^{t+\tau^C} e^{-\delta s}C_s\,\mathrm{d}s}\Big]\\
&= \frac{1}{\gamma} - \frac{1}{\gamma}\mathbb{E}_x\Big[\exp\big(e^{-\delta t}(-\Delta(1-e^{-\delta\tau^{q,\xi}}) - \gamma\int_{\tau^{q,\xi}}^{\tau^C} e^{-\delta s}C_s\,\mathrm{d}s)\big)\Big]\\
&= \frac{1}{\gamma} - \frac{1}{\gamma}e^{-\Delta e^{-\delta t}}\mathbb{E}_x\Big[\exp\big(e^{-\delta t}e^{-\delta\tau^{q,\xi}}(\Delta - \gamma\int_0^{\tau^C-\tau^{q,\xi}} e^{-\delta s}C_{s+\tau^{q,\xi}}\,\mathrm{d}s)\big)\Big]\\
&= \frac{1}{\gamma} - \frac{1}{\gamma}e^{-\Delta e^{-\delta t}} - \frac{1}{\gamma}e^{-\Delta e^{-\delta t}}\sum_{n=1}^\infty \frac{e^{-\delta tn}}{n!}\mathbb{E}_x\big[e^{-\delta n\tau^{q,\xi}}\big]\mathbb{E}_q\Big[\Big(\Delta - \gamma\int_0^{\tau^C} e^{-\delta s}C_s\,\mathrm{d}s\Big)^n\Big]\\
&= \frac{1}{\gamma} - \frac{1}{\gamma}e^{-\Delta e^{-\delta t}} - \frac{1}{\gamma}e^{-\Delta e^{-\delta t}}\sum_{n=1}^\infty \frac{e^{-\delta tn}}{n!}e^{\eta_n(x-q)}M_n(q)\\
&= -\frac{1}{\gamma}\sum_{n=1}^\infty \frac{e^{-\delta tn}}{n!}\sum_{k=0}^n\binom{n}{k}(-\Delta)^{n-k}M_k(q)e^{\eta_k(x-q)}\;. \qquad (10)
\end{aligned}$$

$$\begin{aligned}
G(t,x) &= \mathbb{E}_x\big[F\big(t+\tau^{q,0},q\big); X^0_{\tau^{q,0}} = q\big]\\
&= -\frac{1}{\gamma}\sum_{n=1}^\infty \frac{e^{-\delta tn}}{n!}\cdot\frac{e^{\theta_n x} - e^{\zeta_n x}}{e^{\theta_n q} - e^{\zeta_n q}}\sum_{k=0}^n\binom{n}{k}(-\Delta)^{n-k}M_k(q)\;. \qquad (11)
\end{aligned}$$

where, for the fourth equality, we developed the first exponential function in the expectation into its power series and used the Markov property to see that the $\mathbb{P}_{0,x}$-law given $\mathcal{F}_{\tau^{q,\xi}}$ of $\tau^C - \tau^{q,\xi}$ equals the $\mathbb{P}_{0,q}$-law of τ^C. Also, for the last equality for G we inserted the formula for F and used the identities given in Borodin and Salminen [9, p. 309, formula 3.0.5 (b)]. The notation used in G means $\mathbb{E}_x[Y_t; A] = \mathbb{E}_x[Y_t 1\!\!1_A]$ for some process Y.

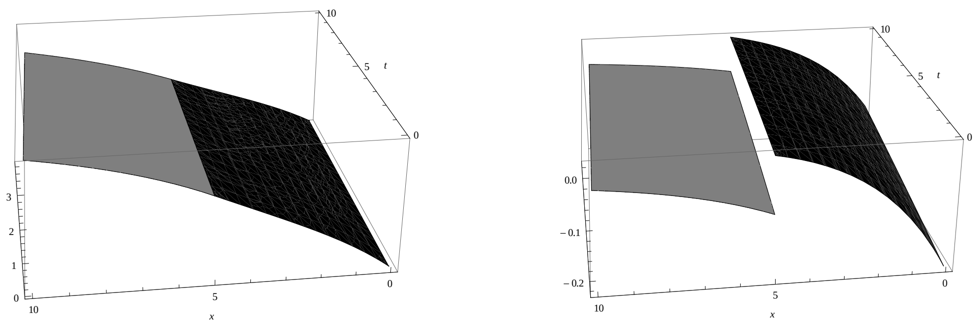

Figure 1: The return function corresponding to a 5-barrier strategy and its second derivative with respect to x.

In order to analyse the performance function of a barrier strategy we will develop the performance function into integer powers of $(e^{-\delta t})$ with x-dependent coefficients and truncate at some N. This will result in an approximation for the performance function which is much easier to handle but this incurs an additional truncation error. Inspecting Equations (10), (11) motivates the approximations

$$F^N(t,x) = \sum_{n=1}^{N} e^{-\delta tn} \sum_{k=0}^{n} A_{n,k} e^{\eta_k (x-q)}, \tag{12}$$

$$G^N(t,x) := \sum_{n=1}^{N} D_n e^{-\delta tn} \frac{e^{\theta_n x} - e^{\zeta_n x}}{e^{\theta_n q} - e^{\zeta_n q}}, \tag{13}$$

for $x, t \geq 0$ where $\eta_0 := 0$. In order to achieve a $\mathcal{C}^{(1,1)}$ fit we choose $D_n := \sum_{k=0}^{n} A_{n,k}$ and

$$A_{n,n} := \frac{\sum_{k=0}^{n-1} (\nu_n - \eta_k) A_{n,k}}{\eta_n - \nu_n}, \quad \nu_n := \frac{\theta_n e^{\theta_n q} - \zeta_n e^{\zeta_n q}}{e^{\theta_n q} - e^{\zeta_n q}}.$$

This leaves the choice for $A_{n,0}, \ldots, A_{n,k-1}$ open which we now motivate by inspecting the dynamics equation for F, G which should be:

$$G_t(t,x) + \mu G_x(t,x) + \frac{\sigma^2}{2} G_{xx}(t,x) = 0,$$
$$F_t(t,x) + \mu F_x(t,x) + \frac{\sigma^2}{2} F_{xx}(t,x) = \xi \Big(F_x(t,x) + e^{-\delta t} (\gamma F(t,x) - 1) \Big)$$

with boundary condition $G(t,0) = 0$ for $t, x \geq 0$.
It is easy to verify that $G^N(t,0) = 0$ and $G_t^N(t,x) + \mu G_x^N(t,x) + \frac{\sigma^2}{2} G_{xx}^N(t,x) = 0$.

However, since $H_k(x) := e^{\eta_k x}$ solves the equation

$$\delta k H_k(x) = (\mu - \xi)\partial_x H_k(x) + \frac{\sigma^2}{2}\partial_x^2 H_k(x)$$

we find that

$$F_t^N(t,x) + (\mu-\xi)F_x^N(t,x) + \frac{\sigma^2}{2}F_{xx}^N(t,x) = \sum_{n=1}^{N} e^{-\delta t n}\sum_{k=0}^{n-1}\delta(k-n)A_{n,k}e^{\eta_k(x-q)},$$
$$e^{-\delta t}\xi(\gamma F^N(t,x)-1) = -e^{-\delta t}\xi + \sum_{n=2}^{N+1} e^{-\delta t n}\sum_{k=0}^{n-1}\gamma\xi A_{n-1,k}e^{\eta_k(x-q)}$$

We will treat the term $e^{-\delta t(N+1)}\xi\gamma\sum_{k=0}^{N} A_{N,k}e^{\eta_k(x-q)}$ as an error term and otherwise equate the two expressions above. This allows to define the remaining coefficients which are given by:

$$A_{n,k} := \frac{\gamma\xi A_{n-1,k}}{\delta(k-n)} = (-\frac{\gamma\xi}{\delta})^{n-k}\frac{A_{k,k}}{(n-k)!} = (-\Delta)^{n-k}\frac{A_{k,k}}{(n-k)!},$$
$$A_{n,0} := \Big(-\frac{\gamma\xi}{\delta}\Big)^{n-1}\frac{\xi}{\delta n!} = \frac{(-\gamma)^{n-1}\xi^n}{\delta^n n!} = \frac{(-\Delta)^n}{-\gamma n!}$$

for $n \geq k \geq 1$ and the last line also for $n = 0$.
The following lemma shows that F^N solves "almost" the same equation as F is thought to solve. Instead of being zero we see an error term which converges for time to infinity faster than $e^{-\delta t N}$.

Lemma 4.1
We have

$$G_t^N(t,x) + \mu G_x^N(t,x) + \frac{\sigma^2}{2}G_{xx}^N(t,x) = 0,$$
$$F_t^N(t,x) + \mu F_x^N(t,x) + \frac{\sigma^2}{2}F_{xx}^N(t,x) + \xi\psi^N(e^{-\delta t},x) = -e^{-\delta t(N+1)}\xi\gamma\sum_{k=0}^{N}A_{N,k}e^{\eta_k(x-q)},$$

for any $t \geq 0, x \geq q$ where

$$\psi^N(e^{-\delta t},x) := -F_x^N(t,x) + e^{-\delta t}(1-\gamma F^N(t,x)).$$

Proof: The claim follows by inserting the definitions of G^N and F^N. □

We define

$$V^N(t,x) := \mathbb{1}_{\{x\geq q\}}F^N(t,x) + \mathbb{1}_{\{x<q\}}G^N(t,x), \tag{14}$$
$$\psi^N(e^{-\delta t},x) := -V_x^N(t,x) + e^{-\delta t}(1-\gamma V^N(t,x))$$

for any $t, x \geq 0$. We now want to compare the approximate performance function V^N for the barrier strategy with level q to the unknown value function. We proceed by first bounding ψ^N in terms of a double power series in $e^{-\delta t}$ and x-dependent exponentials.

Lemma 4.2
With the preceding definitions we have for $x \geq q$

$$-\psi^N(e^{-\delta t}, x)\mathbb{1}_{\{\psi^N(e^{-\delta t},x)<0\}}$$
$$\leq \sum_{n=1}^{N+1} e^{-\delta tn} \left(\sum_{k=0}^{n} e^{\eta_k(x-q)} \Big\{ \mathbb{1}_{\{n=1,k=0\}} - \mathbb{1}_{\{n\neq N+1\}}\eta_k A_{n,k} - \mathbb{1}_{\{n\neq 1,k\neq n\}}\gamma A_{n-1,k} \Big\} \right)^+$$

and for $0 \leq x < q$ *we have*

$$\psi^N(e^{-\delta t}, x)\mathbb{1}_{\{\psi^N(e^{-\delta t},x)>0\}} \leq \sum_{n=1}^{N+1} e^{-\delta tn} \Big(\mathbb{1}_{\{n=1\}} - D_n h_n'(x)\mathbb{1}_{\{n\neq N+1\}}$$
$$-\gamma D_{n-1}h_{n-1}(x)\mathbb{1}_{\{n\neq 1\}} \Big)^+$$

where $h_n(x) := \frac{e^{\theta_n x} - e^{\zeta_n x}}{e^{\theta_n q} - e^{\zeta_n q}}$.

Proof: Inserting the definition of ψ^N and the definitions of F^N and G^N found in Equation (12) resp. Equation (13) yields for $x \geq q$ where $\eta_0 = 0$

$$\psi^N(e^{-\delta t}, x)$$
$$= \sum_{n=1}^{N+1} e^{-\delta tn} \sum_{k=1}^{n} e^{\eta_k(x-q)} \left(\mathbb{1}_{\{n=1,k=0\}} - \mathbb{1}_{\{n\neq N+1\}}\eta_k A_{n,k} - \mathbb{1}_{\{n\neq 1,k\neq n\}}\gamma A_{n-1,k} \right)$$

and for $0 \leq x < q$ we obtain

$$\psi^N(e^{-\delta t}, x) = \sum_{n=1}^{N+1} e^{-\delta tn} \left(\mathbb{1}_{\{n=1\}} - D_n h_n'(x)\mathbb{1}_{\{n\neq N+1\}} - \gamma D_{n-1}h_{n-1}(x)\mathbb{1}_{\{n\neq 1\}} \right).$$

Using the inequality $(\sum_{n=1}^{N} e^{-\delta tn} c_n)^+ \leq \sum_{n=1}^{N} e^{-\delta tn}(c_n)^+$ for $c \in \mathbb{R}^N$ we obtain for $x \geq q$

$$-\psi^N(e^{-\delta t}, x)\mathbb{1}_{\{\psi^N(e^{-\delta t},x)<0\}}$$
$$\leq \sum_{n=1}^{N+1} e^{-\delta tn} \left(\sum_{k=1}^{n} e^{\eta_k(x-q)} \Big\{ \mathbb{1}_{\{n=1,k=0\}} - \mathbb{1}_{\{n\neq N+1\}}\eta_k A_{n,k} - \mathbb{1}_{\{n\neq 1,k\neq n\}}\gamma A_{n-1,k} \Big\} \right)^+$$

and for $0 \leq x < q$ we obtain

$$\psi^N(e^{-\delta t}, x)\mathbb{1}_{\{\psi^N(e^{-\delta t},x)>0\}} \leq \sum_{n=1}^{N+1} e^{-\delta tn} \Big(\mathbb{1}_{\{n=1\}} - D_n h_n'(x)\mathbb{1}_{\{n\neq N+1\}}$$
$$-\gamma D_{n-1}h_{n-1}(x)\mathbb{1}_{\{n\neq 1\}} \Big)^+$$

as claimed. $\square$

We will employ the same method like in Section 3.2 and rely on the occupation bounds from Theorem A.1. We have in mind that $V^N \approx V^C \leq V$. The three error terms appearing on the right-hand side of the following proposition are in this order the error for behaving suboptimal above the barrier, the error for behaving suboptimal below the barrier and the approximation error.

Proposition 4.3
We have

$$\begin{aligned}
&V(t,x) \leq V^N(t,x) \\
&+ \sum_{n=1}^{N+1} e^{-\delta t n} \xi \Bigg[\Bigg(\sum_{k=0}^{n} \Big\{ \mathbb{1}_{\{n=1,k=0\}} - \mathbb{1}_{\{n \neq N+1\}} \eta_k A_{n,k} - \mathbb{1}_{\{n \neq 1, k \neq n\}} \gamma A_{n-1,k} \Big\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times \int_q^\infty e^{\eta_k (y-q)} f_n(x,y)\, \mathrm{d}y \Bigg)^+ \\
&+ \int_0^q \Big(-D_n \frac{\theta_n e^{\theta_n y} - \zeta_n e^{\zeta_n y}}{e^{\theta_n q} - e^{\zeta_n q}} + \Big(\mathbb{1}_{\{n=1\}} - \gamma \mathbb{1}_{\{n \neq 1\}} D_{n-1} \frac{e^{\theta_{n-1} y} - e^{\zeta_{n-1} y}}{e^{\theta_{n-1} q} - e^{\zeta_{n-1} q}} \Big)^+ f_n(x,y)\, \mathrm{d}y \Bigg] \\
&+ e^{-\delta t (N+1)} \xi \gamma \int_0^\infty \sum_{k=0}^{N} |A_{N,k}| e^{\eta_k (y-q)} f_{N+1}(x,y)\, \mathrm{d}y
\end{aligned}$$

for any $t, x \geq 0$ where f_k are defined in Proposition 3.3.

Proof: Observe that V^N is analytic outside the barrier q and $\mathcal{C}^{(1,\infty)}$ on $\mathbb{R}_+ \times \mathbb{R}_+$ and the second space derivative is a bounded function. Thus, we can apply the change of variables formula, confer [16].
Choose an arbitrary strategy $\bar{C}$ and denote its ruin time by τ. Following the heuristics from Section 2.1 up to Step 4 with $H = V^N$ for the strategy $\bar{C}$ yields

$$\begin{aligned}
&V^{\bar{C}}(t,x) = V^N(t,x) + \mathbb{E}_{(t,x)} \Big[\int_t^\tau e^{-\gamma \int_t^r e^{-\delta u} \bar{C}_u \, \mathrm{d}u} (\bar{C}_r - \xi \mathbb{1}_{\{X_r^{\bar{C}} > q\}}) \psi^N(e^{-\delta r}, X_r^{\bar{C}})\, \mathrm{d}r \Big] \\
&- \xi \gamma \mathbb{E}_{(t,x)} \Big[\int_t^\tau e^{-\gamma \int_t^r e^{-\delta u} \bar{C}_u \, \mathrm{d}u} e^{-\delta r (N+1)} \mathbb{1}_{\{X_r^{\bar{C}} > q\}} \sum_{k=0}^{N} A_{N,k} e^{\eta_k (X_r^C - q)}\, \mathrm{d}r \Big] \\
&\leq V^N(t,x) + \xi \gamma \mathbb{E}_{(t,x)} \Big[\int_t^\tau e^{-\delta r (N+1)} \mathbb{1}_{\{X_r^{\bar{C}} > q\}} \sum_{k=0}^{N} |A_{N,k}| e^{\eta_k (X_r^C - q)}\, \mathrm{d}r \Big] \\
&+ \mathbb{E}_{(t,x)} \Big[\int_t^\tau \Big(-\xi \mathbb{1}_{\{X_r^{\bar{C}} > q, \psi^N(e^{-\delta r}, X_r^{\bar{C}}) < 0\}} + \xi \mathbb{1}_{\{X_r^{\bar{C}} < q, \psi^N(e^{-\delta r}, X_r^{\bar{C}}) > 0\}} \Big) \psi^N(e^{-\delta r}, X_r^{\bar{C}})\, \mathrm{d}r \Big],
\end{aligned}$$

where we used that $0 \leq \bar{C}_r \leq \xi$. Applying Lemma 4.2 to the last summand, pulling out the sum and applying Theorem A.1 yields

$$
\begin{aligned}
V^{\bar{C}}(t,x) \le V^N(t,x) \\
&+ \sum_{n=1}^{N+1} e^{-\delta t n}\xi\Bigg[\Bigg(\sum_{k=0}^{n}\Big\{\mathbb{1}_{\{n=1,k=0\}} - \mathbb{1}_{\{n\neq N+1\}}\eta_k A_{n,k} - \mathbb{1}_{\{n\neq 1,k\neq n\}}\gamma A_{n-1,k}\Big\} \\
&\qquad\qquad \times \int_q^\infty e^{\eta_k(y-q)} f_n(x,y)\,\mathrm{d}y\Bigg)^+ \\
&+ \int_0^q \left(\mathbb{1}_{\{n=1\}} - D_n h_n'(y)\mathbb{1}_{\{n\neq N+1\}} - \gamma D_{n-1}h_{n-1}(y)\mathbb{1}_{\{n\neq 1\}}\right)^+ f_n(x,y)\,\mathrm{d}y\Bigg] \\
&+ e^{-\delta t(N+1)}\xi\gamma\int_q^\infty \sum_{k=0}^N |A_{N,k}| e^{\eta_k(y-q)} f_{N+1}(x,y)\,\mathrm{d}y\ ,
\end{aligned}
$$

where $h_n(y) := \frac{e^{\theta_n y} - e^{\zeta_n y}}{e^{\theta_n q} - e^{\zeta_n q}}$. Since $\bar{C}$ was an arbitrary strategy and the right hand side does not depend on $\bar{C}$, the claim follows. □

Now we quantify the notion $V^N \approx V^C$. Here, we see a single error term which corresponds to the approximation error (third summand) in Proposition 4.3.

Lemma 4.4
Let $t, x \ge 0$. Then we have

$$
|V^N(t,x) - V^C(t,x)| \le e^{-\delta t(N+1)}\xi\gamma \int_q^\infty \sum_{k=0}^N |A_{N,k}| e^{\eta_k(y-q)} f_{N+1}(x,y)\,\mathrm{d}y\ .
$$

Proof: By following the lines of the proof of Proposition 4.3 with the specific strategy $\bar{C}_t = C_t = \xi\mathbb{1}_{\{X_t^C > q\}}$ until estimates are used yields

$$
\begin{aligned}
V^C(t,x) &= V^N(t,x) + \mathbb{E}_{(t,x)}\Big[\int_t^\tau e^{-\gamma\int_t^r e^{-\delta u}C_u\,\mathrm{d}u}(C_r - \xi\mathbb{1}_{\{X_r^C > q\}})\psi^N(r, X_r^C)\,\mathrm{d}r\Big] \\
&\quad - \xi\gamma\mathbb{E}_{(t,x)}\Big[\int_t^\tau e^{-\gamma\int_t^r e^{-\delta u}C_u\,\mathrm{d}u} e^{-\delta r(N+1)}\mathbb{1}_{\{X_r^C > q\}}\sum_{k=0}^N A_{N,k}e^{\eta_k(X_r^C - q)}\,\mathrm{d}r\Big] \\
&= V^N(t,x) \\
&\quad - \xi\gamma\mathbb{E}_{(t,x)}\Big[\int_t^\tau e^{-\gamma\int_t^r e^{-\delta u}C_u\,\mathrm{d}u} e^{-\delta r(N+1)}\mathbb{1}_{\{X_r^C > q\}}\sum_{k=0}^N A_{N,k}e^{\eta_k(X_r^C - q)}\,\mathrm{d}r\Big]\ .
\end{aligned}
$$

Hence, we find

$$
\begin{aligned}
|V^C(t,x) - V^N(t,x)| &\le \xi\gamma\mathbb{E}_{(t,x)}\Big[\int_t^\tau e^{-\delta r(N+1)} 1\!\!1_{\{X_r^C>q\}} \sum_{k=0}^N |A_{N,k}| e^{\eta_k (X_r^C - q)}\,\mathrm{d}r\Big] \\
&= \xi\gamma e^{-\delta t(N+1)} \int_{\mathbb{R}} 1\!\!1_{\{X_r^C>q\}} \sum_{k=0}^N |A_{N,k}| e^{\eta_k (y_r - q)} f_{N+1}(x,y)\,\mathrm{d}y
\end{aligned}
$$

due to Theorem A.1. □

5 Examples

Here, we consider two examples. The first one will illustrate how the value function and the optimal strategy can be calculated using a straightforward approach under various unproven assumptions. In fact, we will assume (without proof) that the value function is smooth enough, the optimal strategy is of barrier type and that the barrier, the value function above the barrier and the value function below the barrier have suitable power series representations. In [14] has been observed that similar power series – if exist – have very large coefficients for certain parameter choices. This could mean that the power series doesn't converge or that insufficient computing power was at hand.
In the second subsection, we will illustrate the new approach and calculate the distance of the performance function of a constant barrier strategy to the value function. The key advantages of this approach are that we do not rely on properties of the value function, nor do we need to know how it looks like. From a practical perspective, if the value function cannot be found, one should simply choose any strategy with an easy-to-calculate return function. Then, it is good to know how large the error to the optimal strategy is.

5.1 The straightforward approach

In this example we let $\mu = 0.15$, $\delta = 0.05$, $\gamma = 0.2$ and $\sigma = 1$. We try to find the value function numerically. However, we do not know whether the assumptions which we will make do actually hold true for any possible parameters — or, even for the parameters we chose.
We conjecture and assume that the optimal strategy is of a barrier type where the barrier is given by a time-dependent curve, say α; the value function $V(t,x)$ is assumed to be a $\mathcal{C}^{1,2}(\mathbb{R}_+^2)$ function and we define

$$
\begin{aligned}
h(t,x) &:= V(t,x), \quad t \ge 0, x \in [\alpha(t), \infty), \\
g(t,x) &:= V(t,x), \quad t \ge 0, x \in [0, \alpha(t)],
\end{aligned}
$$

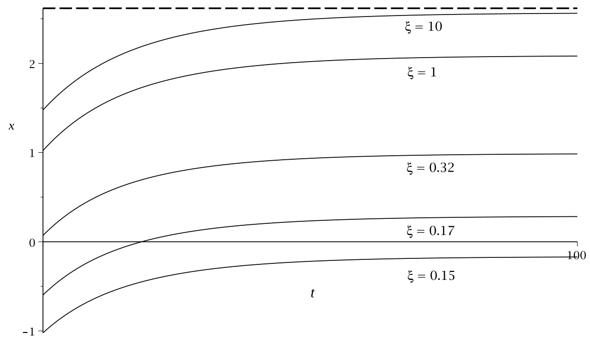

Figure 2: The optimal strategies for different values of ξ. The dashed line corresponds to the Asmussen-Taksar strategy [5] (unrestricted dividend case).

This means, we assume that h solves HJB equation (1) on $\mathbb{R}_+ \times [\alpha(t), \infty)$ and g solves (1) on $\mathbb{R}_+ \times [0, \alpha(t)]$. In particular, the functions h and g fulfil

$$h_t + (\mu - \xi) h_x + \frac{\sigma^2}{2} h_{xx} + \xi e^{-\delta t}\big(1 - \gamma h\big) = 0 \,, \qquad \lim_{x\to\infty} h(t,x) = U\Big(\frac{\xi e^{-\delta t}}{\delta}\Big) \,,$$
$$g_t + \mu g_x + \frac{\sigma^2}{2} g_{xx} = 0 \,, \qquad g(t,0) \equiv 0 \,.$$

Similar to the derivations of the functions F and G in Section 4, we assume that

$$\begin{aligned} h(t,x) &:= \frac{1}{\gamma} - \frac{1}{\gamma} e^{-\Delta e^{-\delta t}} + e^{-\Delta e^{-\delta t}} \sum_{n=1}^{\infty} J_n e^{-\delta t n} e^{\eta_n x} \,, \\ g(t,x) &:= \sum_{n=1}^{\infty} L_n e^{-\delta t n}\big(e^{\theta_n x} - e^{\zeta_n x}\big) \,, \\ \alpha(t) &:= \sum_{n=0}^{\infty} \frac{a_n}{n!} e^{-\delta t n} \,, \end{aligned}$$

for some coefficients. Note that we do not investigate the question whether the functions h, g and α have a power series representation. We define further auxiliary coefficients $b_{k,n}, p_{k,n}$ and $q_{k,n}$:

$$e^{\eta_n \alpha(t)} =: \sum_{k=0}^{\infty} \frac{b_{k,n}}{k!} e^{-\delta t k}, \qquad e^{\theta_n \alpha(t)} =: \sum_{k=0}^{\infty} \frac{p_{k,n}}{k!} e^{-\delta t k}, \qquad e^{\zeta_n \alpha(t)} =: \sum_{k=0}^{\infty} \frac{q_{k,n}}{k!} e^{-\delta t k} \,,$$

Since we assume that the value function is twice continuously differentiable with respect to x, we have using smooth fit

$$h(t,\alpha(t)) = g(t,\alpha(t)), \quad g_x(t,\alpha(t)) = h_x(t,\alpha(t)), \quad g_{xx}(t,\alpha(t)) = h_{xx}(t,\alpha(t)) \,. \tag{15}$$

Note that (15) yields $h_t(t,\alpha(t)) = g_t(t,\alpha(t))$. Therefore, we can conclude $h_x(t,\alpha(t)) = e^{-\delta t}\Big(1-\gamma h(t,\alpha(t))\Big)$. Alternatively to (15), one can consider at $(t,\alpha(t))$ the equations

$$-h_x + e^{-\delta t}(1-\gamma h) = 0, \quad -g_x + e^{-\delta t}(1-\gamma g) = 0, \quad h = g\,. \tag{16}$$

Thus, we can find the coefficients a_n, J_n and L_n from the three equations (16).
First, we calculate the coefficients of the power series resulting from the functions $e^{\eta_n\alpha(t)}$, $e^{\theta_n\alpha(t)}$, $e^{\zeta_n\alpha(t)}$. This is done using the general Leibniz rule:

$$\begin{aligned}
b_{k+1,n} &= \eta_n \sum_{j=0}^{k}\binom{k}{j} a_{k-j+1} b_{j,n}, \quad b_{0,n} = e^{\eta_n\alpha(0)},\\
p_{k+1,n} &= \theta_n \sum_{j=0}^{k}\binom{k}{j} a_{k-j+1} p_{j,n}, \quad p_{0,n} = e^{\theta_n\alpha(0)},\\
q_{k+1,n} &= \zeta_n \sum_{j=0}^{k}\binom{k}{j} a_{k-j+1} q_{j,n}, \quad q_{0,n} = e^{\zeta_n\alpha(0)}\,.
\end{aligned}$$

Now, in order to calculate the coefficients of the power series representations of $h(t,\alpha(t))$ and $g(t,\alpha(t))$ and their derivatives, we define auxiliary coefficients for $m\in\{1,2\}$:

$$X_{m,j} := \sum_{n=1}^{j} J_n \eta_n^{m-1}\frac{b_{j-n,n}}{(j-n)!}, \qquad Z_{m,k} := \sum_{j=1}^{k}\frac{\Delta^{k-j}}{(k-j)!}X_{m,j},$$

$$W_{m,k,j} := L_j\big(\theta_j^{m-1}p_{k,j} - \zeta_j^{m-1}q_{k,j}\big), \qquad Y_{m,k} := \sum_{n=1}^{k}\frac{W_{m,k-n,n}}{(k-n)!}.$$

Then, we can write the functions g and h along with their derivatives as power series:

$$g(t,\alpha(t)) = \sum_{k=1}^{\infty} e^{-\delta tk}Y_{1,k}, \qquad h(t,\alpha(t)) = \sum_{k=1}^{\infty} e^{-\delta tk}Z_{1,k} - \frac{1}{\gamma}\sum_{k=1}^{\infty}\big(-\Delta\big)^k\frac{e^{-\delta tk}}{k!},$$

$$g_x(t,\alpha(t)) = \sum_{k=1}^{\infty} e^{-\delta tk}Y_{2,k}, \qquad h_x(t,\alpha(t)) = \sum_{k=1}^{\infty} e^{-\delta tk}Z_{2,k}\,.$$

Equating coefficients yields $a_0 = \log\left(\frac{\eta_1-\zeta_1}{\eta_1-\theta_1}\cdot\frac{\zeta_1}{\theta_1}\right)/(\theta_1-\zeta_1)$, $L_1 = \frac{1}{\theta_1 e^{\theta_1 a_0} - \zeta_1 e^{\zeta_1 a_0}}$, $J_1 = \frac{e^{-\eta_1 a_0}}{\eta_1}$ and for $k\geq 2$:

$$X_{2,k} = -\gamma X_{1,k-1}, \qquad Y_{2,k} = -\gamma Y_{1,k-1}, \qquad Y_{1,k} = Z_{1,k} - \frac{\big(-\Delta\big)^k}{\gamma k!}\,. \tag{17}$$

Note that Equations (17) specify L_k, J_k and a_{k-1} in kth step. The coefficients given above have a recursive structure. Due to this fact the presented method turns out to be very time- and memory-consuming. Numerical calculations show that the above procedure yields well-defined power series for relative small values of ξ. However, for big ξ the coefficients explode, which makes the calculations unstable and imprecise especially for t close to zero.

Mathematica code for the calculation of the coefficients J_n, L_n and a_n.

```
(*The optimal strategy for xi=1, (g=gamma)*)
mu = 0.15; xi = 1; sigma = 1; delta = 0.05; g = 0.2; n = 500;

w = -xi*g/delta;

eta[k_] = xi - mu - Sqrt[(xi - mu)^2 + 2*delta*k];
zeta[k_] = -mu - Sqrt[mu^2 + 2*delta*k];
theta[k_] = -mu + Sqrt[mu^2 + 2*delta*k];

(*The coefficients*)

Array[L, n]; Array[J, n]; Array[a, n];

(*Auxiliary functions*)

p[k_, m_] := p[k, m] = If[m == 0, Exp[theta[k]*a[0]],
                                  theta[k]*Sum[Binomial[m - 1, j]*a[m - j]*p[k, j], {j, 0, m - 1}]];
q[k_, m_] := q[k, m] = If[m == 0, Exp[zeta[k]*a[0]],
                                  zeta[k]*Sum[Binomial[m - 1, j]*a[m - j]*q[k, j], {j, 0, m - 1}]];
b[k_, m_] := b[k, m] =  If[m == 0, Exp[eta[k]*a[0]],
                                   eta[k]*Sum[Binomial[m - 1, j]*a[m - j]*b[k, j], {j, 0, m - 1}]];

X1[k_] := X1[k] = Sum[J[j]*b[j, k - j]/((k - j)!), {j, 1, k}];
X2[k_] := X2[k] = Sum[J[j]*b[j, k - j]/((k - j)!), {j, 2, k - 1}];
X3[k_] := X3[k] = Sum[J[j]*eta[j]*b[j, k - j]/((k - j)!), {j, 2, k - 1}];
X4[k_] := X4[k] = Sum[J[j]*eta[j]*b[j, k - j]/((k - j)!), {j, 1, k - 1}];
Z[k_]  := Z[k]  = Sum[w^(k - j)/(k - j)!*X1[j], {j, 1, k - 1}];

Y1[k_] := Y1[k] = Sum[L[j]*(p[j, k - j] - q[j, k - j])/((k - j)!), {j, 1, k}];
Y2[k_] := Y2[k] = Sum[L[j]*(p[j, k - j] - q[j, k - j])/((k - j)!), {j, 2, k - 1}];
Y3[k_] := Y3[k] = Sum[L[j]*(theta[j]*p[j, k - j] - zeta[j]*q[j, k - j])/((k - j)!), {j, 2, k - 1}];
Y4[k_] := Y4[k] = Sum[L[j]*(theta[j]*p[j, k - j] - zeta[j]*q[j, k - j])/((k - j)!), {j, 1, k - 1}];

r1[k_,j_] := r1[k, j] = (p[k, j] - q[k, j]);
r2[k_,j_] := r2[k, j] = (theta[k]*p[k, j] - zeta[k]*q[k, j]);
r3[k_,j_] := r3[k, j] = (theta[k]^2*p[k, j] - zeta[k]^2*q[k, j]);

(*The first coefficients*)

a[0] = 1/(theta[1] - zeta[1])*Log[(eta[1] - zeta[1])/(eta[1] -theta[1])*zeta[1]/theta[1]];
L[1] = 1/(theta[1]*p[1, 0] - zeta[1]*q[1, 0]);
J[1] = 1/(eta[1]*b[1, 0]);
L[2] = (w - eta[2]/eta[1]*w + w^2/(2*g)*eta[2])/(r2[2,0] - eta[2]*r1[2,0]);
J[2] = (L[2]*r2[2,0] - w)/(eta[2]*b[2, 0]);
a[1] = (-g*J[1]*b[1, 0] - J[2]*eta[2]*b[2, 0])/eta[1];

(*Recursions for the calculation of the remaining coefficients*)

f[k_] := f[k] = (-g*Y1[k - 1] - Y4[k])/r2[k,0];
h[k_] := h[k] = (-g*X1[k - 1] - X4[k])/(eta[k]*b[k, 0]);
v[k_] := v[k] = (k - 1)!/(L[1]*(r2[1, 0] - r1[k, 0]/r2[k, 0]*r3[1, 0])
                         - J[1]*eta[1]*b[1, 0]*(1 - eta[1]/eta [k]))
               *(Z[k] - w^k/(g*k!) + X2[k] - g/eta[k]*X1[k - 1] - 1/eta[k]*X3[k]
                 - Y2[k] + g*r1[k, 0]/r2[k, 0]*Y1[k - 1] + r1[k, 0]/r2[k, 0]*Y3[k]
                 - Sum[Binomial[k - 2, j]*a[k - 1 - j]/(k - 1)!*(L[1]*(r2[1, j] - r1[k, 0]/r2[k, 0]*r3[1, j])
                                                    - J[1]*b[1, j]*(eta[1] - eta[1]^2/eta[k]))
                                                                                    , {j, 1, k - 2}]);


For[i = 3, i <= n, i++, a[i - 1] = v[i]; L[i] = f[i]; J[i] = h[i]];
```

In Figure 3 we see the functions h (black) and g (gray) meeting at the barrier $\alpha(t)$ in the left picture. The right picture illustrates the crucial functions $-h_x + e^{-\delta t}(1-\gamma h)$ (black), $-g_x + e^{-\delta t}(1-\gamma g)$ (gray) along with the zero-plane (white). One sees that the zero-plane cuts $-h_x + e^{-\delta t}(1-\gamma h)$ and $-g_x + e^{-\delta t}(1-\gamma g)$ exactly along the curve α. Note that the numerical procedure used here works well just for small values of ξ. Due to the recursive structure of the coefficients, the bigger ξ-values let the coefficients explode and enforce an early truncation of the power series representations.

It should be noted here once again that the obtained functions h and g do not represent the value function. And the optimal strategy cannot yet be claimed to be of a barrier type with the barrier given by α. First, one has to prove a verification theorem.

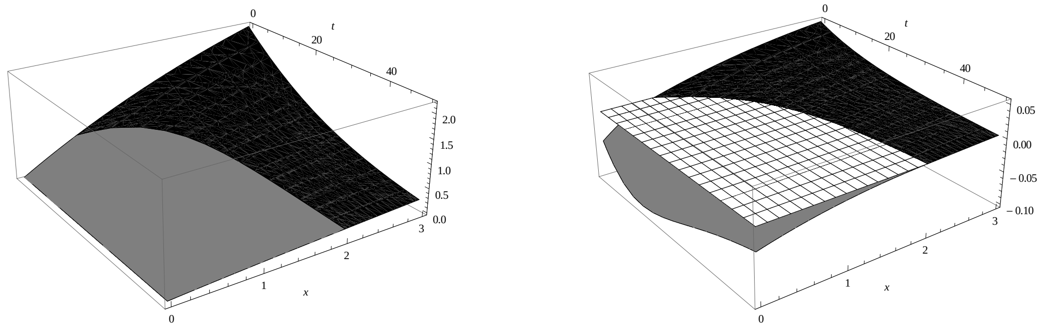

Figure 3: The functions $h(t,x)$ (black) & $g(t,x)$ (gray) in the left picture and the functions $-h_x + e^{-\delta t}(1-\gamma h)$ (black) & $-g_x + e^{-\delta t}(1-\gamma g)$ (gray) & 0 (white) in the right picture for $\xi = 1$.

5.2 The distance to the value function

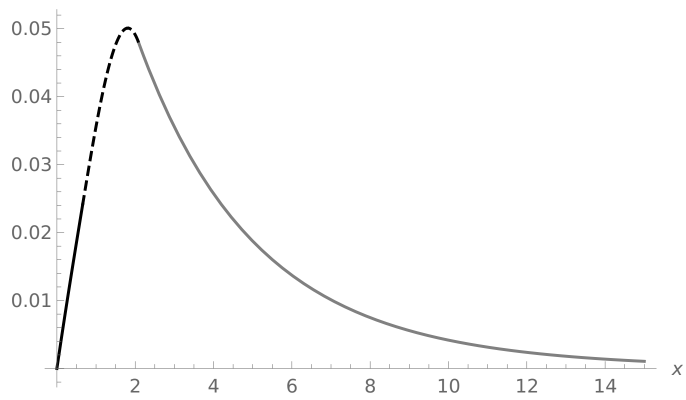

Figure 4: The difference of the value function and an approximation of the performance function corresponding to a constant barrier strategy at $t = 0$: $V(0,x) - V^N(0,x)$ for $\xi = 1$ with V^N given in (14) and the barrier q given in (18).

We use the same parameters as in the previous section, i.e. $\mu = 0.15$, $\delta = 0.05$, $\gamma = 0.2$ and $\sigma = 1$. We illustrate the error bound given by Proposition 4.3 for $N = 20$ summands and four different values for ξ, namely 0.15, 0.17, 0.32 and 1. We will compare the unknown value function to the performance of the barrier strategy with barrier at

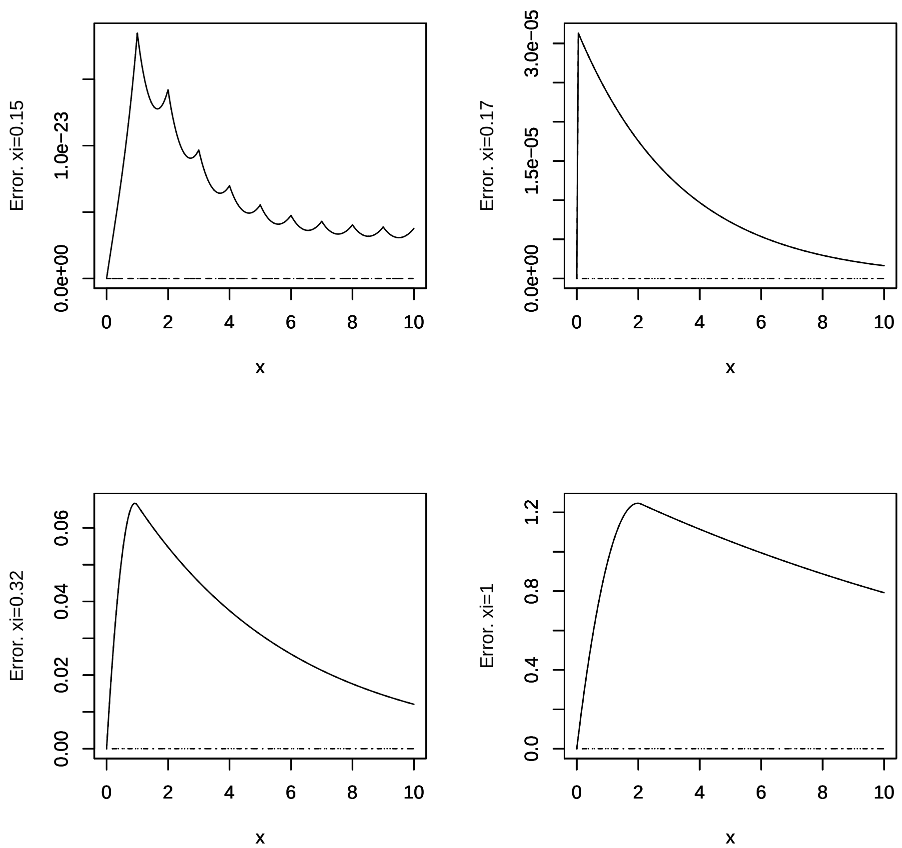

Figure 5: The plots show a numerical valuation of the error bounds given in Proposition 4.3 for the barrier strategy with parameters $\xi = 0.15$, $\xi = 0.17$, $\xi = 0.32$ and $\xi = 1$ respectively as indicated at the side of each plot. The error bound is shown at time $t = 0$ where it is largest across several values of x.

$$q = \left(\frac{\log(-\zeta_1) + \log(\zeta_1 + \eta_1) - \log(\theta_1) - \log(\theta_1 - \eta_1)}{\theta_1 - \zeta_1}\right)^+, \tag{18}$$

i.e. we employ the strategy $C_s = \xi \mathbb{1}_{\{X_s^C \geq q\}}$. Recall the definition of η_1, θ_1 and ζ_1 from (2) and (5).

The barrier strategy with the barrier q has been shown to be optimal if no utility function is applied, see [19, p. 97]. In the case of $\xi = 0.15$ one finds $q = 0$, i.e. we pay out at maximal rate all the time which is optimal due to Proposition 3.2. Therefore, this case is left with approximation error only. For the other values of ξ, it is non-optimal to follow a barrier strategy and, hence, we do have a substantial error which cannot disappear in the limit. The corresponding pictures in Figure 5.2 show this error as for $N = 20$ summands the approximation error is already several magnitudes smaller than the error incurred by following a suboptimal strategy.

Figure 4 illustrates for $\xi = 1$ the difference between the value function $V(x)$ and the approximation V^N, given in (14), of the performance function corresponding to the barrier strategy with the barrier q given in (18) at $t = 0$. Note that the difference

$V(0,x) - V^N(0,x)$ consists of 3 subfunctions:

$$V(0,x) - V^N(0,x) = \begin{cases} F(0,x) - F^N(0,x) & : x \geq q, \text{ gray line in Figure 4,} \\ F(0,x) - G^N(0,x) & : x \in [\alpha(0), q], \text{ black dashed line in Figure 4,} \\ G(0,x) - G^N(0,x) & : x \in [0, \alpha[0]], \text{ black line in Figure 4.} \end{cases}$$

It is clear that for any fixed x the maximal difference $V(t,x) - V^N(t,x)$ is attained at $t = 0$, as the curve α is increasing and converges to q for $t \to \infty$. Thus, the difference $q - \alpha(t)$ attains its maximum at $t = 0$ leading to a bigger difference between the performance functions.

A Appendix

In this section we provide deterministic upper bounds for the expected discounted occupation of a process whose drift is not precisely known. This allows to derive an upper bound for the expect discounted and cumulated positive functional of the process. These bounds are summarised in Theorem A.1.
Let $a, b \in \mathbb{R}$ with $a \leq b$, $I := [a,b]$, $\sigma > 0$, $\delta \geq 0$, W a standard Brownian motion and consider the process

$$\mathrm{d}X_t = C_t \,\mathrm{d}t + \sigma \,\mathrm{d}W_t$$

where C is some I-valued progressively measurable process. We recall that we denote by $\mathbb{P}_x$ a measure with $\mathbb{P}_x[X_0 = x]$. The local time of X at level y and time t is denoted by L_t^y and $\tau := \inf\{t \geq 0 : X_t = 0\}$. Further we define for $x, y \geq 0$

$$\alpha := \frac{a + \sqrt{a^2 + 2\delta\sigma^2}}{\sigma^2}, \qquad \beta_+ := \frac{\sqrt{b^2 + 2\delta\sigma^2} - b}{\sigma^2}, \qquad \beta_- := \frac{-\sqrt{b^2 + 2\delta\sigma^2} - b}{\sigma^2},$$

$$f(x,y) := \frac{2\left(e^{\beta_+(x\wedge y)} - e^{\beta_-(x\wedge y)}\right) e^{-\alpha(x-y)^+}}{\sigma^2\left((\beta_+ + \alpha)e^{y\beta_+} - (\beta_- + \alpha)e^{y\beta_-}\right)}.$$

Theorem A.1
We have $\mathbb{E}_x\left[\int_0^\tau e^{-\delta s}\,\mathrm{d}L_s^y\right] \leq \sigma^2 f(x,y)$. *In particular, for any measurable function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *we have*

$$\mathbb{E}_x\left[\int_0^\tau e^{-\delta s}\psi(X_s)\,\mathrm{d}s\right] \leq \int_0^\infty \psi(y) f(x,y)\,\mathrm{d}y.$$

The proof is given at the end of this section.

Lemma A.2
f is absolutely continuous in its first variable with derivative

$$f_x(x,y) := \begin{cases} \frac{2\left(\beta_+ e^{x\beta_+} - \beta_- e^{x\beta_-}\right)}{\sigma^2\left((\beta_+ + \alpha)e^{y\beta_+} - (\beta_- + \alpha)e^{y\beta_-}\right)} & x \leq y, \\ \frac{2\left(-\alpha e^{y\beta_+} + \alpha e^{y\beta_-}\right)e^{-\alpha(x-y)}}{\sigma^2\left((\beta_+ + \alpha)e^{y\beta_+} - (\beta_- + \alpha)e^{y\beta_-}\right)} & x > y. \end{cases}$$

For any $y \geq 0$ the function $f_x(\cdot, y)$ is of finite variation and

$$\mathrm{d}f_x(x,y) = -\frac{2}{\sigma^2}\delta_y(\mathrm{d}x) + \left(\frac{2\delta}{\sigma^2}f(x,y) - \frac{2(b\mathbb{1}_{\{x<y\}} + a\mathbb{1}_{\{x>y\}})}{\sigma^2}f_x(x,y)\right)\,\mathrm{d}x$$

where δ_y denotes the Dirac-measure in y. Moreover, if we denote by $f_{xx}(x,y)$ the second derivative of f with respect to the first variable for $x \neq y$, then we get

$$\sup_{u\in[a,b]}\left(\frac{\sigma^2}{2}f_{xx}(x,y) + uf_x(x,y) - \delta f(x,y)\right) = 0, \quad x \neq y.$$

Proof: Obtaining the derivative and the associated measure is straightforward. If $\delta = 0$, then the statement of the lemma is trivial. This is due to the fact that $\beta_+ = 0$, $\beta_- = -\frac{2b}{\sigma^2}$, $\alpha = \frac{2a}{\sigma^2}$. The function f in this case fulfils $f_x(x,y) > 0$ if $x < y$ and $f_x(x,y) < 0$ if $x > y$. Now, assume that $\delta > 0$. We have $\alpha, \beta_+ > 0 > \beta_-$ which immediately yields $f_x(x,y) > 0$ for $x < y$ and $f_x(x,y) < 0$ for $x > y$. The last equality follows. □

Lemma A.3
Let $y \geq 0$ and assume that $C_t = a\mathbb{1}_{\{X_t>y\}} + b\mathbb{1}_{\{X_t\leq y\}}$. Then

$$\mathbb{E}_x\Big[\int_0^\tau e^{-\delta s}\,\mathrm{d}L_s^y\Big] = \sigma^2 f(x,y).$$

Proof: Ito Tanaka's formula together with the occupation time formula yield

$$\begin{aligned}
f(X_{t\wedge\tau}, y) &= f(x,y) + \int_0^t \sigma f_x(X_{s\wedge\tau}, y)\,\mathrm{d}W_s - \frac{1}{\sigma^2}L^y_{t\wedge\tau} \\
&\quad + \int_0^t C_s f_x(X_{s\wedge\tau}, y) + \frac{\sigma^2}{2}f_{xx}(X_{s\wedge\tau}, y)\,\mathrm{d}s \\
&= f(x,y) + \int_0^t \sigma f_x(X_{s\wedge\tau}, y)\,\mathrm{d}W_s - \frac{1}{\sigma^2}L^y_{t\wedge\tau} + \delta\int_0^t f(X_{s\wedge\tau}, y)\,\mathrm{d}s.
\end{aligned}$$

Using the product formula yields

$$e^{-\delta t}f(X_{t\wedge\tau}, y) = f(x,y) + \int_0^t \sigma e^{-\delta s}f_x(X_{s\wedge\tau}, y)\,\mathrm{d}W_s - \frac{1}{\sigma^2}\int_0^{t\wedge\tau} e^{-\delta s}\,\mathrm{d}L_s^y.$$

Since $f_x(\cdot, y)$ is bounded we see that the second summand is a martingale. If $\delta > 0$, then we find that

$$\lim_{t\to\infty}\mathbb{E}_x[e^{-\delta t}f(X_{t\wedge\tau}, y)] = 0.$$

If $\delta = 0$ and $a \leq 0$, then $\tau < \infty$ $\mathbb{P}$-a.s. and boundedness of f yields

$$\lim_{t\to\infty}\mathbb{E}_x[f(X_{t\wedge\tau}, y)] = 0.$$

If $\delta = 0$ and $a > 0$, then $\lim_{t\to\infty} X_{t\wedge\tau}$ takes values in $\{0, \infty\}$ and $\lim_{x\to\infty} f(x,y) = 0$, thus boundedness of f yields again

$$\lim_{t\to\infty} \mathbb{E}_x[f(X_{t\wedge\tau}, y)] = 0.$$

Thus, we find by monotone convergence

$$0 = f(x,y) - \frac{1}{\sigma^2} \lim_{t\to\infty} \mathbb{E}_x \left[\int_0^{t\wedge\tau} e^{-\delta s} \,\mathrm{d}L_s^y \right] = f(x,y) - \frac{1}{\sigma^2} \mathbb{E}_x \left[\int_0^{\tau} e^{-\delta s} \,\mathrm{d}L_s^y \right].$$

□

The next lemma is a simple variation of the occupation times formula.

Lemma A.4
Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous, τ be a random time and $\psi : \mathbb{R} \to \mathbb{R}_+$ be Borel measurable. Then

$$\int_0^\tau g(s)\psi(X_s)\sigma^2 \,\mathrm{d}s = \int_{\mathbb{R}} \psi(y) Z^y \,\mathrm{d}y$$

where $Z^y := \int_0^\tau g(s) \,\mathrm{d}L_s^y$.

Proof: If the claim is proved for bounded stopping times, then an arbitrary stopping time τ can be approximated by bounded stopping times via $\tau = \lim_{N\to\infty} \min\{N, \tau\}$ and monotone convergence yields the claim. For the remainder of the proof we assume that τ is a bounded stopping time. Additionally, we start with bounded and Lebesgue-integrable ψ. Once the claim is proved for bounded and Lebesgue-integrable ψ, it follows for the remaining ψ by monotone convergence.
Let $\epsilon > 0$. Since g is continuous it is uniform continuous on $[0, \tau]$. Hence, there is $\delta > 0$ such that $|g(x) - g(y)| < \epsilon$ for any $x, y \in [0, \tau]$ with $|x - y| < \delta$. For an integer $N > \tau/\delta$ we find with $F_N := \sum_{k=1}^N \int_{(k-1)\frac{\tau}{N}}^{k\frac{\tau}{N}} (g(s) - g((k-1)\tau/N))\, \psi(X_s)\sigma^2 \,\mathrm{d}s$ that

$$\begin{aligned}
\int_0^\tau g(s)\psi(X_s)\sigma^2 \,\mathrm{d}s &= \sum_{k=1}^N \int_{(k-1)\frac{\tau}{N}}^{k\frac{\tau}{N}} g((k-1)\tau/N)\psi(X_s)\sigma^2 \,\mathrm{d}s \\
&\quad + \sum_{k=1}^N \int_{(k-1)\frac{\tau}{N}}^{k\frac{\tau}{N}} (g(s) - g((k-1)\tau/N))\, \psi(X_s)\sigma^2 \,\mathrm{d}s \\
&= \int_{\mathbb{R}} \psi(y) \sum_{k=1}^N g((k-1)\tau/N)(L^y_{k\frac{\tau}{N}} - L^y_{(k-1)\frac{\tau}{N}}) \,\mathrm{d}y + F_N \,,
\end{aligned}$$

where we used [18, Corollary VI.1.6] for the second equality.
We have $|F_N| \le \epsilon \int_0^\tau \psi(X_s)\sigma^2 \,\mathrm{d}s$ by choice of δ and

$$\sum_{k=1}^N g((k-1)\tau/N)(L^y_{k\frac{\tau}{N}} - L^y_{(k-1)\frac{\tau}{N}}) \to \int_0^\tau g(s) \,\mathrm{d}L_s^y, \quad N \to \infty.$$

It holds

$$\left|\sum_{k=1}^{N} g((k-1)\tau/N)(L^y_{k\frac{\tau}{N}} - L^y_{(k-1)\frac{\tau}{N}})\right| \le \sup_{a\in[0,\tau]} g(a)\cdot L^y_\tau$$

and, hence, Lebesgue's dominated convergence result yields

$$\int_{\mathbb{R}} \psi(y) \sum_{k=1}^{N} g((k-1)\tau/N)(L^y_{k\frac{\tau}{N}} - L^y_{(k-1)\frac{\tau}{N}})\, \mathrm{d}y \to \int_{\mathbb{R}} \psi(y) Z^y\, \mathrm{d}y$$

as required. □

Proof:[Proof of Theorem A.1] Fix $y \ge 0$. For any progressively measurable process η with values in I we define

$$Y^\eta_t := X_0 + \int_0^t \eta_s\, \mathrm{d}s + \sigma W_t \quad \text{and} \quad V(x) := \sup_\eta \mathbb{E}_x\left[\int_0^\tau e^{-\delta s}\, \mathrm{d}L^{y,\eta}_s\right],$$

where $\tau^\eta := \inf\{t \ge 0 : Y^\eta_t = 0\}$ and $L^{\cdot,\eta}$ denotes a continuous version of the local time of Y^η. Clearly, we have

$$\mathbb{E}_x\left[\int_0^\tau e^{-\delta s}\, \mathrm{d}L^{y,\eta}_s\right] \le V(x).$$

Moreover, the previous two lemmas yield that Y^{η^*} with

$$\eta^*_t = a1\!\!1_{\{Y^{\eta^*}_t > y\}} + b1\!\!1_{\{Y^{\eta^*}_t \le y\}}$$

is the optimally controlled process and we get $V(x) = \sigma^2 f(x,y)$. (The process η^* exists because the corresponding SDE admits pathwise uniqueness, see [18, Thm IX.3.5].) This proves the inequality for the local time. The additional inequality follows now from Lemma A.4. □

Acknowledgments

The research of the first author was funded by the Austrian Science Fund (FWF), Project number V 603-N35. We would like to thank the unknown referees for their helpful comments and suggestions.

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