Optimal Dividends under a Stochastic Interest Rate

JULIA EISENBERG *

Institute of Mathematical Methods in Economics, Vienna University of Technology.

Abstract

We consider an insurance entity endowed with an initial capital and an income, modeled as a Brownian motion with drift. The discounting factor is modeled as a stochastic process: at first as a geometric Brownian motion, then as an exponential function of an integrated Ornstein-Uhlenbeck process. It is assumed that the insurance company seeks to maximize the cumulated value of expected discounted dividends up to the ruin time. We find an explicit expression for the value function and for the optimal strategy in the first but not in the second case, where one has to switch to the viscosity ansatz.

Key words: optimal control, Hamilton–Jacobi–Bellman equation, Vasicek model, geometric Brownian motion, interest rate, short rate, dividends

2010 Mathematical Subject Classification: Primary 93B05
Secondary 49L20, 49L25

1 Introduction

In fact, the present paper deals with the excessively investigated problem of dividend maximization in the Brownian risk model, however under a stochastic interest rate independent of the surplus. That is to say, in the previous studies the discounting rate, which undoubtedly plays a crucial role in the choice of the optimal strategy, is assumed to be a positive constant. Since as already mentioned, the dividend optimization problems have been investigated for a quite long time, we omit listing the existing literature and just refer to a survey on the dividend problems in insurance by Albrecher and Thonhauser [1] and references therein.

Interest rates build an integral part of market economy, influencing huge firm investment as well as small households spending decisions. Random perturbations on financial markets can cardinally change the monetary behaviour of an investor, leading to a totally different result than expected under the assumption of a constant interest rate. Intuitively, it is clear that a stochastic interest rate reflects fluctuations on the market much better than a deterministic one. Thus, it is clear that in order to describe the behaviour of an economic agent possibly realistic, we should use a stochastic interest

*email: jeisenbe@fam.tuwien.ac.at

rate. Rather, the question is which short rate (rates of interest for a short period of time) models should be used and whether any explicit results can be obtained.

In the following, we discuss the problem of dividend maximization under the two following models: the Dothan and the Vasicek model. The Dothan model assumes that the discounting factor follows a geometric Brownian motion. The advantage of such a model is that the expected value of the discounting factor is given by an exponential function with an exponential linearly depending on time. On the other hand, its disadvantage is that it does not have the mean reversion property and the discounting factor can become arbitrarily high. In contrast, the Vasicek model incorporates mean reversion, the short rate is defined as an Ornstein-Uhlenbeck process. A huge disadvantage of this model is that the short rate can become negative. We correct this defect by defining the discounting factor as an exponential function applied on the integrated Ornstein-Uhlenbeck process. Due to a complex structure of the preference function, the calculations become much more complicated as well. For more details about interest rate models in general and in particular about the two mentioned above, confer for example [7].

To the best of our knowledge, the problem of dividend maximization under a stochastic interest rate has not been previously addressed. In [9], stochastic interest rate models were applied to an economic agent whose surplus was assumed to be deterministic. There, it was possible to find explicit expressions for the value function and for the optimal strategy under the stochastic interest rate models mentioned above. In the case of a geometric Brownian motion, there was no difference to the case with a constant interest rate in the form of the value function or the optimal strategy. Just the drift and the volatility of the geometric Brownian motion found its way into some constant parameters in the value function.

In the present paper, we model the surplus process as a Brownian motion with drift. In the setting with a constant interest rate and a general diffusion with special parameters as a surplus process, the problem of dividend maximization was considered by Shreve et al. in [12]. Also, in [3] Asmussen and Taksar solved the problem for the special case of Brownian motion. They found out that the optimal strategy is a constant barrier. In Section 2, we investigate the problem of dividend maximization under the Dothan model and find, similar to the case with deterministic income, that the optimal strategy does not change (compared to the Asmussen-Taksar case) in its form, but in the parameters. In Section 3, we consider the Vasicek model. Here, the situation changes significantly. It is not that easy any more to calculate a return function corresponding to some strategy. Even in the case of the strategy "always pay out on the maximal possible rate" we are not able to show the concavity of the corresponding return function in the x-component if the initial interest rate is negative. We show that the value function is a viscosity solution to the corresponding Hamilton–Jacobi-Bellman equation, for the methods confer for example Bardi and Capuzzo-Dolcetta [5].

2 Geometric Brownian Motion as a Discounting Factor

Consider an insurance company whose surplus is given by a Brownian motion with drift $X_t = x + \mu t + \sigma W_t$, where $\{W_t\}$ is a standard Brownian motion. The insurance company is allowed to pay out dividends, where the accumulated dividends until t are given by $\mathbf{C}_t$, the dividend rate process by $C = \{c_t\}$, yielding for the ex-dividend surplus X^C:

$$X_t^C = x + \mu t + \sigma W_t - \mathbf{C}_t \; .$$

The consideration will be stopped at the ruin time of X^C. Let further $\{B_t\}$ be a standard Brownian motion independent of $\{W_t\}$. We let the underlying filtration $\{\mathcal{F}_t\}$ be generated by $\{W_t, B_t\}$. At first, we allow just for the strategies $C = \{c_s\}$ with accumulated dividends up to time t given by $\mathbf{C}_t = \int_0^t c_s \, \mathrm{d}s$ with $c_s \in [0, \xi]$ for some $\xi > 0$, and call such a strategy admissible if additionally C is cadlag, adapted to $\{\mathcal{F}_t\}$ and for the ex-dividend process it holds $X_t^C \geq 0$ up to the ruin time.
As a risk measure, we consider the value of expected discounted dividends, where the dividends are discounted by a geometric Brownian motion

$$\exp\{-r - mt - \delta B_t\} \; , \quad \text{if} \quad m > \frac{\delta^2}{2} \; .$$

Denoting by τ^C the ruin time of the surplus process under some admissible strategy $C = \{c_s\}$, we define the return function corresponding to C to be

$$V^C(r, x) = \mathbb{E}\Big[\int_0^{\tau^C} e^{-r - ms - \delta B_s} c_s \, \mathrm{d}s \Big] \; .$$

Note that

$$\begin{aligned} V^C(t, x) &= \mathbb{E}\Big[\int_0^{\tau^C} e^{-r - ms - \delta B_s} c_s \, \mathrm{d}s \Big] \\ &\leq \mathbb{E}\Big[\int_0^{\infty} e^{-r - ms - \delta B_s} \xi \, \mathrm{d}s \Big] \\ &= \frac{\xi e^{-r}}{m - \frac{\delta^2}{2}} \; . \end{aligned}$$

The HJB equation corresponding to the problem is given by

$$\begin{aligned} \mu V_x + \frac{\sigma^2}{2} V_{xx} - m V_r + \frac{\delta^2}{2} V_{rr} + \sup_{0 \leq c \leq \xi} c\{e^{-r} - V_x\} \\ = 0 \; . \end{aligned} \tag{1}$$

For the sake of convenience, we define the following quantities:

$$\begin{aligned}
\eta &:= \frac{\xi-\mu-\sqrt{(\xi-\mu)^2+2\sigma^2\left(m-\frac{\delta^2}{2}\right)}}{\sigma^2}<0,\\
\theta &:= \frac{-\mu+\sqrt{\mu^2+2\sigma^2\left(m-\frac{\delta^2}{2}\right)}}{\sigma^2}>0,\\
\zeta &:= \frac{-\mu-\sqrt{\mu^2+2\sigma^2\left(m-\frac{\delta^2}{2}\right)}}{\sigma^2}<0.
\end{aligned} \tag{2}$$

Consider at first the case where we pay on the maximal possible rate ξ up to ruin. Let $\tau_x^{\xi,0} := \inf\{t\ge 0:\ x+(\mu-\xi)t+\sigma W_t=0\}$. Note that $\tau_x^{\xi,0}$ is independent of the process $\{B_t\}$. Then via Fubini's theorem, the return function corresponding to the strategy $c_s \equiv \xi$ is

$$\begin{aligned}
V^\xi(r,x) &= \xi\mathbb{E}\Big[\int_0^{\tau_x^{\xi,0}} e^{-r-mt-\delta B_t}\,\mathrm{d}t\Big]\\
&= \xi\mathbb{E}\Big[\int_0^{\tau_x^{\xi,0}} \mathbb{E}[e^{-r-mt-\delta B_t}]\,\mathrm{d}t\Big]\\
&= \frac{\xi e^{-r}}{m-\frac{\delta^2}{2}}\mathbb{E}\Big[1-e^{-\left(m-\frac{\delta^2}{2}\right)\tau_x^{\xi,0}}\Big]\\
&= \frac{\xi e^{-r}}{m-\frac{\delta^2}{2}}\Big\{1-e^{\eta x}\Big\}\,.
\end{aligned}$$

The last step in the above calculation is due to the formula 2.0.1 in [6, p. 295]. Further, due to $\eta<0$ we have

$$V_x^\xi(r,x) = -\frac{\xi e^{-r}}{m-\frac{\delta^2}{2}}\eta e^{\eta x}>0\,.$$

In particular, $V_x^\xi(t,x)<e^{-r}$ if $-\frac{\xi\eta}{m-\frac{\delta^2}{2}}\le 1$. This means that the function V^ξ solves the HJB equation (1) if $-\frac{\xi\eta}{m-\frac{\delta^2}{2}}\le 1$ and becomes a candidate for the value function.

Let now $-\frac{\xi\eta}{m-\frac{\delta^2}{2}}>1$. We conjecture that the value function can be written in the form $V(r,x)=e^{-r}F(x)$. In this case, the HJB equation becomes

$$\begin{aligned}
\mu F'(x)+\frac{\sigma^2}{2}F''(x)-mF(x)+\frac{\delta^2}{2}F(x)&\\
+\sup_{0\le c\le\xi} c\{1-F'(x)\}&=0\,.
\end{aligned}$$

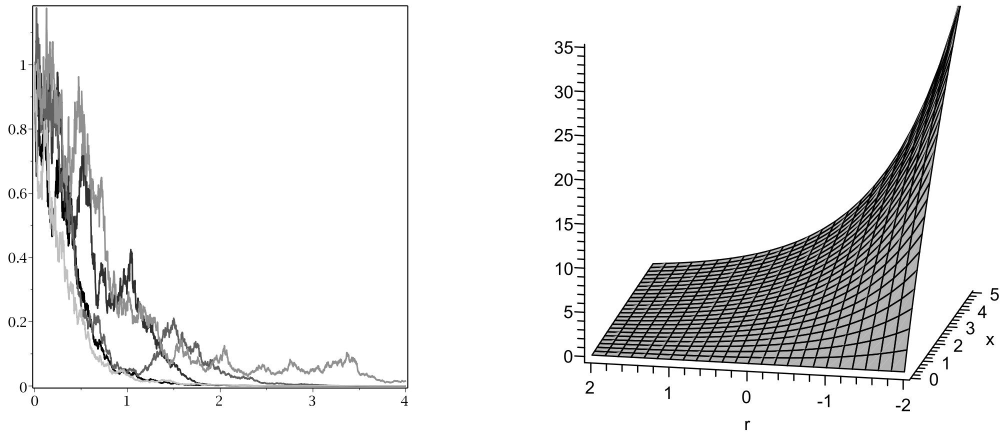

Figure 1: Left picture: Realisations of $\{e^{-r-ms-\delta B_s}\}$, $r=1$, for $m=2$ and $\delta=1$. Right picture: The value function.

Proceeding like in Schmidli [11, p. 99], we have to solve the following two differential equations:

$$\begin{aligned}\mu F_1'(x) + \frac{\sigma^2}{2} F_1''(x) - \big(m - \frac{\delta^2}{2}\big) F_1(x) \\ + \xi\{1 - F_1'(x)\} = 0\end{aligned}$$

and

$$\mu F_2'(x) + \frac{\sigma^2}{2} F_2''(x) - \big(m - \frac{\delta^2}{2}\big) F_2(x) = 0\ .$$

We conjecture that the value function is given by $e^{-r}F_2(x)$ below some boundary and by $e^{-r}F_1(x)$ above this boundary. Since the value function is bounded in x for a fixed r and $V(r,0)=0$, we can conclude

$$F_1(x) = Ce^{\eta x} \quad \text{and} \quad F_2(x) = B\big(e^{\theta x} - e^{\zeta x}\big)\ .$$

Our target is to determine $\hat{x} > 0$, C and B such that $F_1(\hat{x}) = F_2(\hat{x})$, $F_1'(\hat{x}) = F_2'(\hat{x})$ and $F_1''(\hat{x}) = F_2''(\hat{x})$. For $F(x) = F_1(x)$ for $x > \hat{x}$ and $F(x) = F_2(x)$ for $x < \hat{x}$ we obtain

$$F(x) = \\ \begin{cases} \frac{\xi\eta}{m-\frac{\delta^2}{2}} \cdot \frac{e^{\theta x} - e^{\zeta x}}{\eta\left(e^{\theta\hat{x}} - e^{-\zeta\hat{x}}\right) - \theta e^{\theta\hat{x}} + \zeta e^{\zeta\hat{x}}} & x \le \hat{x} \\ \frac{\xi}{m-\frac{\delta^2}{2}} \cdot \left(1 + \frac{e^{\eta(x-\hat{x})}\left(\theta e^{\theta\hat{x}} - \zeta e^{\zeta\hat{x}}\right)}{\eta\left(e^{\theta\hat{x}} - e^{\zeta\hat{x}}\right) - \theta e^{\theta\hat{x}} + \zeta e^{\zeta\hat{x}}}\right) & x > \hat{x}\ , \end{cases}$$

where $\hat{x}$ is given by

$$\hat{x} = \frac{\log\left(\frac{\zeta^2 - \eta\zeta}{\theta^2 - \eta\theta}\right)}{\theta - \zeta}\ .$$

The function $e^{-r}F(x)$ is twice continuously differentiable with respect to r and to x, increasing and concave in x, decreasing and convex in r.

Proposition 2.1
The value function is given by $V(r,x) = e^{-r}F(x)$. *The optimal strategy* $C^* = \{c_s^*\}$ *is*

$$c_s^*(x) = \xi \mathbb{1}_{[X_s^{C^*} > \hat{x}]}$$

Proof: Let C be an arbitrary admissible strategy and τ^C be the ruin time of $\{X_t^C\}$. Then it holds via Ito's formula, using that $e^{-r}F(x)$ fulfils (1):

$$\begin{aligned}
e^{-r_{t\wedge\tau^C}} F(X_{t\wedge\tau^C}^C) &= e^{-r}F(x) \\
&\quad + \int_0^{t\wedge\tau^C} e^{-r-ms-\delta B_s} F'(X_s^C)(\mu - c_s)\,\mathrm{d}s \\
&\quad + \frac{\sigma^2}{2}\int_0^{t\wedge\tau^C} e^{-r-ms-\delta B_s} F''(X_s^C)\,\mathrm{d}s \\
&\quad - \int_0^{t\wedge\tau^C} e^{-r-ms-\delta B_s}\big(m - \frac{\delta^2}{2}\big) F(X_s^C)\,\mathrm{d}s \\
&\quad + \sigma\int_0^{t\wedge\tau^C} e^{-r-ms-\delta B_s} F'(X_s^C)\,\mathrm{d}W_s \\
&\quad + \delta\int_0^{t\wedge\tau^C} e^{-r-ms-\delta B_s} F(X_s^C)\,\mathrm{d}B_s \\
&\le e^{-r}F(x) - \int_0^{t\wedge\tau^C} e^{-r-ms-\delta B_s} c_s\,\mathrm{d}s \\
&\quad + \sigma\int_0^{t\wedge\tau^C} e^{-r-ms-\delta B_s} F'(X_s^C)\,\mathrm{d}W_s \\
&\quad + \delta\int_0^{t\wedge\tau^C} e^{-r-ms-\delta B_s} F(X_s^C)\,\mathrm{d}B_s\,.
\end{aligned}$$

It is easy to see that $F'(x)$ is bounded. Further, $\{e^{-r-ms-\delta B_s}\}$ is a continuous process fulfilling $\int_0^t \mathbb{E}[e^{-2r-2ms-2\delta B_s}]\,\mathrm{d}s < \infty$. According to this, we can conclude that the last two stochastic integrals above are martingales with expectation 0. Taking expectations on the both sides of the above inequality yields

$$\begin{aligned}
\mathbb{E}\big[e^{-r-m(t\wedge\tau^C)-\delta B_{t\wedge\tau^C}} F(X_{t\wedge\tau^C}^C)\big] &\le e^{-r}F(x) \\
&\quad - \mathbb{E}\Big[\int_0^{t\wedge\tau^C} e^{-r-ms-\delta B_s} c_s\,\mathrm{d}s\Big]\,.
\end{aligned}$$

Note that for $C = C^*$ equality holds. In the case $\tau^C \leq t$ one obviously has $F(X^C_{\tau^C}) = F(0) = 0$. Since F is bounded by $\frac{\xi}{m - \frac{\delta^2}{2}}$, one has

$$\begin{aligned}
&\mathbb{E}\big[e^{-r-m(t\wedge\tau^C)-\delta B_{t\wedge\tau^C}} F(X^C_{t\wedge\tau^C})\big] \\
&\quad = \mathbb{E}\big[e^{-r-mt-\delta B_t} F(X^C_t) 1\!\!1_{[\tau^C > t]}\big] \\
&\quad \leq \mathbb{E}\big[e^{-r-mt-\delta B_t} F(X^C_t)\big] \\
&\quad \leq \mathbb{E}\big[e^{-r-mt-\delta B_t}\big] \frac{\xi}{m - \frac{\delta^2}{2}} \\
&\quad = e^{-r-\left(m-\frac{\delta^2}{2}\right)t} \frac{\xi}{m - \frac{\delta^2}{2}} \; .
\end{aligned}$$

Since we assumed $m > \frac{\delta^2}{2}$, it holds

$$\lim_{t\to\infty} \mathbb{E}\big[e^{-r-m(t\wedge\tau^C)-\delta B_{t\wedge\tau^C}} F(X^C_{t\wedge\tau^C})\big] = 0 \; .$$

Thus,

$$e^{-r} F(x) \begin{cases} \geq V^C(r,x) & : C \text{ arbitrary,} \\ = V^{C^*}(r,x) & : C = C^* \; . \end{cases}$$

And we can conclude that $V(r,x) \geq V^{C^*}(r,x) = e^{-r}F(x) \geq \sup\limits_C V^C(r,x) = V(r,x)$. □

Example 2.2
Let $m = 2$, $\delta = 1$, $\mu = 2$, $\sigma = 4$ and $\xi = 5$. Then the value function is illustrated in the right picture in Figure 1. The optimal barrier is in this case given by $\hat{x} = 1.7216$. In the left picture of Figure 1 one sees 5 realisations of a geometric Brownian motion with initial value $r = 1$, $m = 2$ and $\delta = 1$. The volatility of the paths is quite high, but yet the trend is similar to the one of $e^{-\lambda s}$, where λ is a positive constant. Such a behaviour corresponds to a normal yield curve, which we will discuss in more detail in the next section. ■

The optimal strategy is like in the case with a constant deterministic interest rate a barrier strategy with a constant barrier. The barrier depends on the parameters of the geometric Brownian motion describing the preference function. Similar to the case with deterministic discounting, one can show that if $\xi < \mu$ with a positive probability the process remains positive up to infinity. Also, for the optimal strategy the ruin occurs almost surely, confer for example [11, p. 102].

Remark 2.3 (Unrestricted Dividends)
In the case of unrestricted dividends, the proofs are similar to those in Schmidli [11, pp. 102-104]. In order to incorporate the interest rate, we need to modify some parameters

in the value function from [11] and to multiply it by e^{-r}. For that reason, we just list the most important results, for details please confer [11] .

• *The value function $V(r,x)$ can be obtained as the limit over the value functions $V_\xi(r,x)$ for by ξ restricted dividend rates, where the boundary ξ goes to infinity, and is given by*

$$V(r,x) = e^{-r} \begin{cases} \frac{e^{\theta x} - e^{\zeta x}}{\theta e^{\theta \hat{x}} - \zeta e^{\zeta \hat{x}}} & x \le \hat{x} \\ \frac{\mu}{m - \frac{\delta^2}{2}} + x - \hat{x} & x > \hat{x} \end{cases},$$

where

$$\hat{x} = \frac{\ln\left(\frac{\zeta^2}{\theta^2}\right)}{\theta - \zeta} .$$

• *The optimal strategy C^* is to pay out the excess to $\hat{x}$, i.e. $c_t^* = \max\{\sup\limits_{0 \le s \le \tau \wedge t} X_s^0 - \hat{x}, 0\}$, where $X_s^0 = x + \mu s + \sigma W_t$ and τ denotes the ruin time under C^*.* ■

3 Ornstein-Uhlenbeck Process as a Preference Rate

In this subsection, we denote by $\{r_s\}$ an Ornstein-Uhlenbeck process

$$r_s = re^{-as} + \tilde{b}(1 - e^{-as}) + \tilde{\delta} e^{-as} \int_0^s e^{au} \, \mathrm{d}B_u$$

or as a stochastic differential equation

$$\mathrm{d}r_s = a(\tilde{b} - r_s) \, \mathrm{d}t + \tilde{\delta} \, \mathrm{d}B_t ,$$

where $\{B_u\}$ is a standard Brownian motion, $a, \tilde{\delta} > 0$, and let $U_s = \int_0^s r_u \, \mathrm{d}u$. In the following, we manifest the fact $r_0 = y$ by writing U_s^y. Here, $\tilde{b}$ is the long-term mean of the process $\{r_s\}$, i.e. all trajectories of the interest rate process $\{r_s\}$ will evolve around $\tilde{b}$ in the long run. Under the assumption of a normal yield curve (yields rise as maturity lengthens), we assume that $\tilde{b} > \frac{\tilde{\delta}^2}{2a^2}$. In the case $\tilde{b} < 0$, $\{U_s^r\}$ converges to $-\infty$ for $s \to \infty$, i.e. we would have an inverted yield curve. In Figure 2 one sees 5 realisations of the process $\exp\{-U_s^1\}$, i.e. $r_0 = 1$, $\tilde{\delta} = 1$, $a = 3$, $\tilde{b} = 4$ (the left picture), $\tilde{b} = 0$ (the picture in the centre) and $\tilde{b} = -1$ (the right picture). The left picture reflects the aforesaid normal behaviour of the yield curve. Our target is to maximize the expected discounted dividends if the interest rate is given by $\{r_t\}$. It means, we discount a dividend rate at time s, say $c_s > 0$, by the factor $e^{-U_s^r}$. However, considering the structure of U_s^r it becomes evident that

$$U_s^r = \tilde{b}s + \tilde{U}_s^{r-\tilde{b}}$$

where $\tilde{U}^{r-\tilde{b}} = \int_0^s \tilde{r}_u \, \mathrm{d}u$ and $\tilde{r}_u = (r - \tilde{b})e^{-au} + \delta e^{-au} \int_0^u e^{at} \, \mathrm{d}t$. Note that $\{\tilde{r}\}$ is an Ornstein-Uhlenbeck process with the long-term mean equal 0. Thus, the term $e^{-\tilde{b}s}$ represents the usual deterministic discounting factor and the term $e^{-\tilde{U}_s^{r-\tilde{b}}}$ adds stochasticity

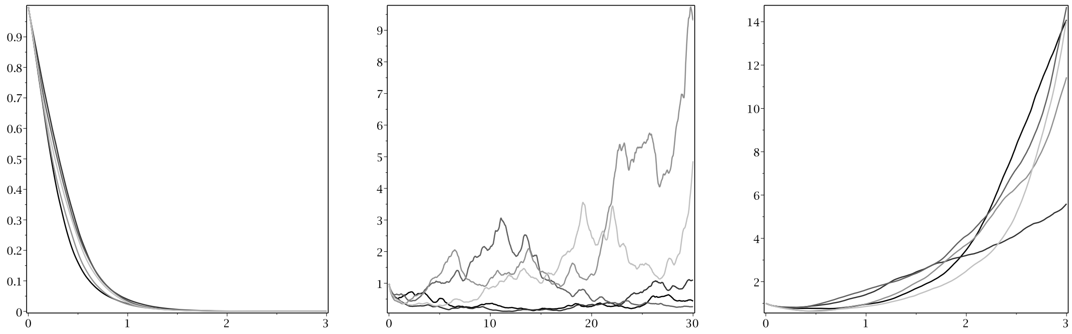

Figure 2: Realisations of $\exp\{-U_s^1\}$, $r_0 = 1$, for $\tilde{b} = 4$, $\tilde{b} = 0$ and $\tilde{b} = -1$.

and consequently uncertainty to the discounting dynamics.
We let the surplus process be modeled like in the section before by a Brownian motion with drift

$$X_t = x + \mu t + \sigma W_t \ ,$$

where $\{W\}$ is independent of $\{B\}$. For a dividend rate strategy $C = \{c_s\}$, the ex-dividend surplus X^C is then

$$X_t^C = x + \mu t + \sigma W_t - \mathbf{C}_t \ ,$$

where $\mathbf{C}$ denotes the cumulated dividend process corresponding to C. A dividend rate strategy $C = \{c_s\}$ is called admissible if $c_s \in [0, \xi]$, for some fixed $\xi > 0$, is cadlag, adapted to the filtration $\{\mathcal{F}_s\}$, generated by $\{B_s, W_s\}$ and fulfilling $X_t^C \geq 0$ up to the ruin time. We denote the set of admissible strategies by $\mathcal{C}$. Let $C = \{c_s\}$ be an admissible strategy and τ_x^C the ruin time of $\{X_t^C\}$, where $X_0^C = x$. The return function corresponding to C and the value function are defined as

$$\begin{aligned} V^C(r,x) &= \mathbb{E}\Big[\int_0^{\tau_x^C} c_s e^{-U_s^r}\,\mathrm{d}s\Big], \quad (r,x) \in \mathbb{R} \times \mathbb{R}_+ \ , \\ V(r,x) &= \sup_{C \in \mathcal{C}} V^C(r,x), \quad (r,x) \in \mathbb{R} \times \mathbb{R}_+ \ . \end{aligned}$$

Here, confer for example Brigo and Mercurio, $\mathbb{E}\big[e^{-U_s^r}\big]$ is the price at zero of a zero-coupon bond (or pure-discount bond) with maturity s. But we are interested in the price at s. In Borodin and Salminen [6, p. 525] one finds $\mathbb{E}[e^{-U_s^r}] = e^{f(r,s)}$, where

$$\begin{aligned} f(r,s) := -\tilde{b}s + \frac{\tilde{\delta}^2}{2a^2}s - \frac{r-\tilde{b}}{a}\big(1 - e^{-as}\big) \\ + \frac{\tilde{\delta}^2}{4a^3}\big(1 - (2 - e^{-as})^2\big) \ . \end{aligned}$$

Notation 3.1
In order to get $\lim_{s\to\infty} f(r,s) < \infty$ *for all* $r \in \mathbb{R}$, *we additionally assume* $\tilde{b} > \frac{\tilde{\delta}^2}{2a^2}$ *and let*

$$b := \tilde{b} - \frac{\tilde{\delta}^2}{2a^2} \quad \text{and} \quad \delta := \frac{\tilde{\delta}}{\sqrt{2a}}.$$

■

Now, we can rewrite the function f as follows

$$f(r,s) = -bs - \frac{r-b}{a}\big(1-e^{-as}\big) - \frac{\delta^2}{2a^2}(1-e^{-as})^2. \tag{3}$$

Note that we can estimate the function f as follows

$$\begin{aligned} f(r,s) &\ge -bs - \frac{\delta^2}{2a^2} - \max\left\{\frac{r-b}{a},0\right\} \\ f(r,s) &\le -bs - \min\left\{\frac{r-b}{a},0\right\}. \end{aligned} \tag{4}$$

Due to the properties of f, the value function V is bounded

$$\begin{aligned} V(r,x) &= \sup_{C\in\mathcal{C}} V^C(r,x) \le \xi\mathbb{E}\Big[\int_0^\infty e^{-U_t^r}\,\mathrm{d}t\Big] \\ &= \xi\int_0^\infty e^{f(t,r)}\,\mathrm{d}t \\ &\begin{cases} \le \xi\frac{e^{-\min\left\{\frac{r-b}{a},0\right\}}}{b} \\ \ge \xi\frac{e^{-\max\left\{\frac{r-b}{a},0\right\}-\frac{\delta^2}{2a^2}}}{b} \end{cases}. \end{aligned} \tag{5}$$

The HJB equation corresponding to the problem is

$$\begin{aligned} \mu V_x + \frac{\sigma^2}{2}V_{xx} + a(\tilde{b}-r)V_r + \frac{\tilde{\delta}^2}{2}V_{rr} - rV \\ + \sup_{0\le c\le\xi} c\Big\{1-V_x\Big\} = 0. \end{aligned} \tag{6}$$

We start by considering the constant strategy $c_s \equiv \xi$.

3.1 Payout on the Maximal Rate

Denote the ruin time of the process

$$X_t^\xi = x + (\mu-\xi)t + \sigma W_t,$$

starting at x by $\tau_x^{\xi,0}$. The return function corresponding to the strategy "pay out on the maximal rate until ruin" is given by

$$\begin{aligned} V^\xi(r,x) &= \xi\mathbb{E}\Big[\int_0^{\tau_x^{\xi,0}} e^{-U_s^r}\,\mathrm{d}s\Big] \\ &= \xi\mathbb{E}\Big[\int_0^{\tau_x^{\xi,0}} e^{f(r,s)}\,\mathrm{d}s\Big], \end{aligned}$$

where we used the independence of U^r and $\tau_x^{\xi,0}$ and Fubini's theorem. In order to calculate $V^\xi(r,x)$ more precisely, consider the function $f(r,s)$, defined in (3). Let $\Theta_0 = 1$, $z(r) := \frac{r-b}{a} + \frac{\delta^2}{a^2}$,

$$\Theta_k := \begin{cases} (-1)^{k/2} \frac{\delta^k}{2^{k/2} a^k (k/2)!} & : k \text{ even} \\ 0 & : \text{otherwise} \end{cases},$$

$$A_n(z(r)) := \sum_{k=0}^{n} \frac{z(r)^k}{k!} \Theta_{n-k}$$

then

$$e^{f(r,s)} = e^{-z(r) + \frac{\delta^2}{2a^2}} \sum_{n=0}^{\infty} A_n(z(r)) e^{-(b+an)s} .$$

The power series above is well defined for all $(r,s) \in \mathbb{R} \times \mathbb{R}_+$ because

$$\begin{aligned} \big|A_n(z(r))\big| &\leq \sum_{k=0}^{n} \frac{|z(r)|^k}{k!} |\Theta_{n-k}| \\ &\leq \sum_{k=0}^{n} \frac{|z(r)|^k}{k!} \frac{\delta^{2(n-k)}}{2^{n-k} a^{2(n-k)} (n-k)!} \\ &= \frac{(|z(r)| + \frac{\delta^2}{2a^2})^n}{n!} . \end{aligned}$$

Define further

$$\eta_n := \frac{\xi - \mu - \sqrt{(\xi-\mu)^2 + 2\sigma^2(b+an)}}{\sigma^2}. \tag{7}$$

The Laplace transform of $\tau_x^{\xi,0}$ is well known, so that using $\mathbb{E}[e^{-(b+an)\tau_x^{\xi,0}}] = e^{\eta_n x}$, confer for example [6, p. 295], we obtain

$$\begin{aligned} &V^\xi(r,x) \\ &= e^{-z(r) + \frac{\delta^2}{2a^2}} \sum_{n=0}^{\infty} A_n(z(r)) \xi \mathbb{E}\Big[\int_0^{\tau_x^{\xi,0}} e^{-(b+an)s} \,\mathrm{d}s\Big] \\ &= e^{-z(r) + \frac{\delta^2}{2a^2}} \sum_{n=0}^{\infty} A_n(z(r)) \xi \frac{1 - e^{\eta_n x}}{b + an} . \end{aligned}$$

Note that for $x, h > 0$ the process $\{X_t^\xi\}$ starting at $x+h$ has to cross the level x before the ruin occurs and $\tau_{x+h}^{\xi,0} > \tau_x^{\xi,0}$ a.s. Due to this, the function $V^\xi(r,x)$ is strictly increasing in x:

$$\begin{aligned} V^\xi(r, x+h) &= \mathbb{E}\Big[\xi \int_0^{\tau_{x+h}^{\xi,0}} e^{f(r,s)} \,\mathrm{d}s\Big] \\ &> \mathbb{E}\Big[\xi \int_0^{\tau_x^{\xi,0}} e^{f(r,s)} \,\mathrm{d}s\Big] = V^\xi(r,x) . \end{aligned}$$

Obviously, V^ξ is twice continuously differentiable with respect to r and to x. In particular, by the definition of η_n in (7) it holds

$$\eta_n^2 = \frac{2(\xi-\mu)}{\sigma^2}\eta_n + \frac{2(b+an)}{\sigma^2},$$

giving

$$\begin{aligned}
V^\xi_{xx}(r,x) &= e^{-z(r)+\frac{\delta^2}{2a^2}} \sum_{n=0}^\infty A_n(z(r))\xi \frac{-e^{\eta_n x}\eta_n^2}{b+an} \\
&= \frac{2(\xi-\mu)}{\sigma^2} V^\xi_x(r,x) \\
&\quad - \frac{2\xi}{\sigma^2} e^{-z(r)+\frac{\delta^2}{2a^2}} \sum_{n=0}^\infty A_n(z(r)) \frac{e^{\eta_n x}(b+an)}{b+an} \\
&= \frac{2(\xi-\mu)}{\sigma^2} V^\xi_x(r,x) \\
&\quad - \frac{2\xi}{\sigma^2} e^{-z(r)+\frac{\delta^2}{2a^2}} \sum_{n=0}^\infty A_n(z(r)) \mathbb{E}\big[e^{-(b+an)\tau_x^{\xi,0}}\big] \\
&= \frac{2(\xi-\mu)}{\sigma^2} V^\xi_x(r,x) - \frac{2\xi}{\sigma^2}\mathbb{E}\big[e^{f(r,\tau_x^{\xi,0})}\big]\,.
\end{aligned}$$

Lemma 3.2
Let $\tau_x^{\xi,0}$ be defined like before. The function $\Delta(r,x) := \mathbb{E}\big[e^{f(r,\tau_x^{\xi,0})}\big]$ is positive and strictly decreasing in x for all $(r,x) \in \mathbb{R}_+^2$.

Proof: It is obvious, that Δ is positive. Since we assumed $\tilde{b} > \frac{\tilde{\delta}^2}{2a^2}$, it holds $b > 0$. Verbatim from Subsection 2.1 in [9], for a fixed $r \in \mathbb{R}_+$ the function $f_s(r,s)$ can have at most one zero at $s = w_1(r) = -\frac{1}{a}\ln(u_1(r))$ with

$$u_1(r) = \frac{r - b + \frac{\delta^2}{a} + \sqrt{\big(r-b+\frac{\delta^2}{a}\big)^2 + 4b\frac{\delta^2}{a}}}{2\delta^2/a} > 0\,.$$

This means that for a fixed r it holds either $f_s(r,s) \le 0$ for all $s \in [0,\infty)$, if $u_1(r) \ge 1$, or $f_s(r,s) > 0$ for $s \in [0, w_1(r))$ and $f_s(r,s) < 0$ for $s \in (w_1(r),\infty)$, if $u_1(r) < 1$. Consequently, we consider just the case where the function f is decreasing in s for $s \in \mathbb{R}_+$ and the case where f is at first increasing and then decreasing in s.
It is easy to see that the function $u_1(r)$ is increasing in r and $u_1(0) = 1$. It means that $f_s(r,s) < 0$ for all $(r,s) \in \mathbb{R}_+^2$.
Since for $y > x$ it obviously holds $\tau_y^{\xi,0} > \tau_x^{\xi,0}$ a.s., we can conclude that for $r \in \mathbb{R}_+$ the function $\Delta(r,x)$ is strictly decreasing in x. □

Lemma 3.3
For $(r,x) \in \mathbb{R}_+^2$ it holds $V^\xi_{xx}(r,x) < 0$.

Proof: Assume first $\xi \leq \mu$. Then due to the representation

$$V_{xx}^{\xi}(r,x) = \frac{2(\xi-\mu)}{\sigma^2} V_x^{\xi}(r,x) - \frac{2\xi}{\sigma^2}\Delta(r,x) \ ,$$

and positivity of Δ, the claim is trivially fulfilled even for all $(r,x) \in \mathbb{R} \times \mathbb{R}_+$.
Assume for $\xi > \mu$ that there is an $(r^*, x^*) \in \mathbb{R}_+^2$ such that $V_{xx}^{\xi}(r^*, x^*) > 0$. Then the r.h.s. of the above representation of V_{xx}^{ξ} is also non-negative. Furthermore, the function $\frac{2(\xi-\mu)}{\sigma^2} V_x^{\xi}(r,x) - \frac{2\xi}{\sigma^2}\Delta(r,x)$ is increasing in x at (r^*, x^*) due to Lemma (3.2). It means that V_{xx}^{ξ} becomes/remains positive for all $x > x^*$, which implies that V_x^{ξ} is increasing in x for $x \in (x^*, \infty)$, contradicting $V_x^{\xi} > 0$ and

$$\begin{aligned}
&\lim_{x\to\infty} V_x^{\xi}(r,x) = \\
&\qquad \lim_{x\to\infty} e^{-z(r)+\frac{\delta^2}{2a^2}} \sum_{n=0}^{\infty} A_n(z(r))\xi \frac{-e^{\eta_n x}\eta_n}{b+an} = 0 \ .
\end{aligned}$$

In the case $V_{xx}(r^*, x^*) = 0$, we can use the same arguments like above due to the fact that $\Delta(r,x)$ is strictly increasing increasing in x if $r \in \mathbb{R}_+$. $\square$

It means in particular, that the function $1 - V_x^{\xi}$ is strictly increasing in x for all $(r,x) \in \mathbb{R}_+^2$. On the other hand, with similar arguments like above one has

$$\begin{aligned}
V_{xr}^{\xi}(r,x) &= V_{rx}^{\xi}(r,x) \\
&= -\frac{\mathrm{d}}{\mathrm{d}x}\xi\mathbb{E}\Big[\int_0^{\tau_x^{\xi,0}} \frac{1-e^{-as}}{a} e^{f(r,s)}\,\mathrm{d}s\Big] < 0 \ ,
\end{aligned}$$

i.e. $1 - V_x^{\xi}(r,x)$ is increasing in r for all $x \in \mathbb{R}_+$. This allows the following conclusion: If $1 - V_x^{\xi}(0,0) \leq 0$, there is an $r^* \in \mathbb{R}_+$ such that $1 - V_x^{\xi}(r^*, 0) = 0$ and

$$1 - V_x^{\xi}(r,x) > 1 - V_x^{\xi}(r,0) > 1 - V_x^{\xi}(r^*,0) = 0$$

for $r > r^*$ and $x > 0$. For $0 \leq r < r^*$, because $V_{xx}^{\xi} < 0$, $(r,x) \in \mathbb{R}_+^2$, the function V_x^{ξ} is invertible with respect to x for all $x \in \mathbb{R}_+$. By the implicit function theorem, there is a continuously differentiable function $\alpha(r)$, defined on $[0, r^*]$, such that

$$1 - V_x^{\xi}(r, \alpha(r)) \equiv 0$$

and $1 - V_x^{\xi}(r,x) < 0$ for $x < \alpha(r)$, $1 - V_x^{\xi}(r,x) > 0$ for $x > \alpha(r)$. Obviously,

$$\begin{aligned}
0 &= -\frac{\mathrm{d}}{\mathrm{d}r} V_x^{\xi}(r, \alpha(r)) \\
&= V_{xr}^{\xi}(r,\alpha(r)) + \alpha'(r) V_{xx}^{\xi}(r, \alpha(r)) \ .
\end{aligned}$$

The fact $V_{xx}^{\xi}, V_{xr}^{\xi} < 0$ yields $\alpha'(r) < 0$.

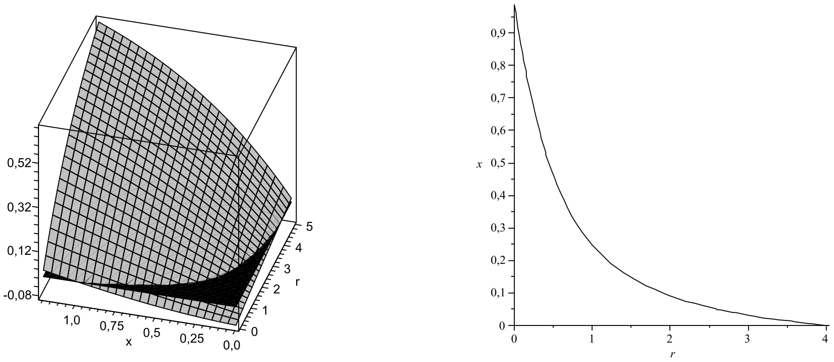

Figure 3: Left picture: Function $1 - V_x^\xi(r, x)$ (gray) cutting the 0 surface (black). Right picture: The curve $\alpha(r)$.

Example 3.4
Consider the following parameters $b = 2$, $a = 1$, $\sigma = 1$, $\delta = 1$, $\mu = 0.4$ and $\xi = 5$. Then the corresponding function $1 - V_x^\xi$ cutting the $z = 0$ surface can be seen in the left picture in Figure 3. In the right picture one finds the separating curve $\alpha(r)$. For $r^* = 4.04$ the curve cuts $x = 0$. In particular, it is not optimal to always pay out on the maximal rate. ■

3.2 Value Function and Viscosity Solutions

Here, we again consider $(r, x) \in \mathbb{R} \times \mathbb{R}_+$. At first, we investigate the properties of the value function V.

Lemma 3.5 (Basic properties of V)
The value function $V(r, x)$ is locally Lipschitz continuous, increasing in x and decreasing in r.

Proof: The value function is obviously strictly increasing in x and decreasing in r. Let further $x, h \in \mathbb{R}_+$, $r \in \mathbb{R}$ and C be an ε-optimal strategy for the initial point $(r + h, x)$. Then C is also an admissible strategy for (r, x). In particular, denoting the ruin time of the surplus process X^C by τ^C one has

$$\begin{aligned}
0 &\ge V(r+h, x) - V(r, x) \\
&\ge \mathbb{E}\Big[\int_0^{\tau^C} e^{-U_s^{r+h}} c_s \,\mathrm{d}s - \int_0^{\tau^C} e^{-U_s^r} c_s \,\mathrm{d}s\Big] - \varepsilon \\
&= \mathbb{E}\Big[\int_0^{\tau^C} e^{-U_s^r} c_s \big(e^{-h(1-e^{-as})} - 1\big) \,\mathrm{d}s\Big] - \varepsilon \\
&\ge -h\mathbb{E}\Big[\int_0^{\tau^C} e^{-U_s^r} c_s \big(1 - e^{-as}\big) \,\mathrm{d}s\Big] - \varepsilon \\
&\ge -hV(r, x) - \varepsilon \ge -h\xi \frac{e^{-\min\left\{\frac{r-b}{a}, 0\right\}}}{b} - \varepsilon\,.
\end{aligned}$$

Let now C be an ε-optimal strategy for the initial point $(r, x+h)$, $\tau := \inf\{t \geq 0 : \ X_t^0 \notin (0, x+h), \ X_0 = x\}$ and define $\tilde{C} = \{\tilde{c}_t\}$ to be

$$\tilde{c}_t = \begin{cases} 0 & t \leq \tau \\ c_{t-\tau} & t > \tau \text{ and } X_\tau^0 = x+h \ . \end{cases}$$

Then,

$$\begin{aligned}
0 &\leq V(r, x+h) - V(r, x) \\
&\leq V^C(r, x+h) + \varepsilon - V^{\tilde{C}}(r, x+h) \\
&= V^C(r, x+h) + \varepsilon \\
&\quad - \mathbb{E}\Big[e^{-U_\tau^r} \int_0^{\tau^C} e^{-\tilde{U}_s^{r_\tau}} c_s \,\mathrm{d}s \mathbb{1}_{[X_\tau^0 = x+h]}\Big] \\
&= V^C(r, x+h) + \varepsilon \\
&\quad - \mathbb{E}\Big[e^{-U_\tau^r} \int_0^{\tau^C} e^{-(r_\tau - r)(1-e^{-as})} e^{-\tilde{U}_s^r} c_s \,\mathrm{d}s \\
&\quad \times \mathbb{1}_{[X_\tau^0 = x+h]}\Big] \\
&\leq V^C(r, x+h) + \varepsilon \\
&\quad - \mathbb{E}\Big[\mathbb{1}_{[r_\tau \leq r]} e^{-U_\tau^r}; X_\tau^0 = x+h\Big] V^C(r, x+h) \\
&\quad - \mathbb{E}\Big[\mathbb{1}_{[r_\tau > r]} \mathbb{1}_{[X_\tau^0 = x+h]} e^{-U_\tau^r} \\
&\quad \times \int_0^{\tau^C} \big(1 + (r - r_\tau)(1 - e^{-as})\big) e^{-\tilde{U}_s^r} c_s \,\mathrm{d}s\Big] \\
&\leq \Big\{1 - \mathbb{E}\Big[e^{-U_\tau^r} \mathbb{1}_{[X_\tau^0 = x+h]}\Big] \\
&\quad + \mathbb{E}\Big[\mathbb{1}_{[r_\tau > r]} e^{-U_\tau^r} (r_\tau - r) \mathbb{1}_{[X_\tau^0 = x+h]}\Big]\Big\} \\
&\quad \times V^C(r, x+h) + \varepsilon \ .
\end{aligned}$$

Since, $\{r_t\}$ and $\{X_t^0\}$ are independent, the function in the curly brackets in the last line of the above inequality depends on h just via the stopping time τ. In Borodin and Salminen [6, p. 525] one finds the expectations of $e^{-U_t^r}$ and $e^{-U_t^r} r_t$. The density function of τ is well known, so that the expectations in the curly brackets can be easily calculated. In particular, the obtained functions are locally Lipschitz continuous, and we can conclude that there is a constant L such that

$$\begin{aligned}
0 &\leq V(r, x+h) - V(r, x) \leq hLV(r, x+h) + \varepsilon \\
&\leq hL \frac{\xi e^{-\min\left\{\frac{r-b}{a}, 0\right\}}}{b} + \varepsilon \ .
\end{aligned}$$

Thus, the value function is locally Lipschitz continuous. $\square$

Since the value function is locally Lipschitz, Rademacher's Theorem ensures the existence of the derivatives almost everywhere. Furthermore, the Lipschitz continuity on compact sets implies that V is absolutely continuous.
Here, we cannot expect that the value function is twice continuously differentiable with respect to x and to r, i.e. that V will be a classical solution to the HJB equation (6). For that reason, the concept of viscosity solutions has to be considered.

Definition 3.6
We say that a continuous function $\underline{u} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ is a viscosity subsolution to (6) at $(\bar{r}, \bar{x}) \in \mathbb{R} \times \mathbb{R}_+$ if any function $\psi \in C^{2,1}\big(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}_+\big)$ with $\psi(\bar{r}, \bar{x}) = \underline{u}(\bar{r}, \bar{x})$ such that $\underline{u} - \psi$ reaches the maximum at $(\bar{r}, \bar{x})$ satisfies

$$\mu\psi_x + \frac{\sigma^2}{2}\psi_{xx} + a(\tilde{b} - r)\psi_r + \frac{\tilde{\delta}^2}{2}\psi_{rr} - r\psi \\ + \sup_{0 \le c \le \xi} c\big(1 - \psi_x\big) \ge 0$$

and we say that a continuous function $\bar{u} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ is a viscosity supersolution to (6) at $(\bar{r}, \bar{x}) \in \mathbb{R} \times \mathbb{R}_+$ if any function $\phi \in C^{2,1}\big(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}_+\big)$ with $\phi(\bar{r}, \bar{x}) = \bar{u}(\bar{r}, \bar{x})$ such that $\bar{u} - \phi$ reaches the minimum at $(\bar{r}, \bar{x})$ satisfies

$$\mu\phi_x + \frac{\sigma^2}{2}\phi_{xx} + a(\tilde{b} - r)\phi_r + \frac{\tilde{\delta}^2}{2}\phi_{rr} - r\phi \\ + \sup_{0 \le c \le \xi} c\big(1 - \phi_x\big) \le 0 \; .$$

A viscosity solution to (6) is a continuous function $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ if it is both a viscosity subsolution and a viscosity supersolution at any $(r, x) \in \mathbb{R} \times \mathbb{R}_+$.

Since, we are dealing with a Brownian motion with drift as a surplus process, the proof methods are standard. So that we prove as an illustration that the value function is a viscosity supersolution. The remaining proofs can be can be obtained from the author on request.

Proposition 3.7
The value function $V(r, x)$ is a viscosity solution to (6).

Proof: Here, we just show that V is a supersolution. For this purpose let $c, h \in \mathbb{R}_+$, $(r, x) \in \mathbb{R} \times \mathbb{R}_+$, $x > 0$, be fixed and $\tau_x^{\xi,0} = \inf\{t \ge 0 : \, x + (\mu - \xi)t + \sigma W_t = 0\}$.
Similar to Azcue and Muler [4], see also Eisenberg [9], one can show that V fulfils the

dynamic programming principle. This yields for an admissible strategy $C = \{c_s\}$

$$\begin{aligned} V(r,x) &= \sup_{C\in\mathcal{C}} \Big\{ \mathbb{E}\Big[\int_0^{\tau_x^{\xi,0}\wedge h} e^{-U_s^r} c_s \,\mathrm{d}s \Big] \\ &\quad + \mathbb{E}\Big[e^{-U^r_{\tau_x^{\xi,0}\wedge h}} V\big(r_{\tau_x^{\xi,0}\wedge h}, X^C_{\tau_0^{\xi,0}\wedge h}\big)\Big]\Big\} \\ &\geq \mathbb{E}\Big[c \int_0^{\tau_x^{\xi,0}\wedge h} e^{-U_s^r} \,\mathrm{d}s \Big] \\ &\quad + \mathbb{E}\Big[e^{-U^r_{\tau_x^{\xi,0}\wedge h}} V\big(r_{\tau_x^{\xi,0}\wedge h}, X^u_{\tau_x^{\xi,0}\wedge h}\big)\Big] \\ &= c\mathbb{E}\Big[\int_0^{\tau_x^{\xi,0}\wedge h} e^{f(r,s)} \,\mathrm{d}s \Big] \\ &\quad + \mathbb{E}\Big[e^{-U^r_{\tau_x^{\xi,0}\wedge h}} V\big(r_{\tau^{\xi}\wedge h}, X^u_{\tau_x^{\xi,0}\wedge h}\big)\Big] \,. \end{aligned}$$

Now let ϕ be a twice continuously differentiable with respect to x and to r test function with $V(r,x) = \phi(r,x)$ and $V - \phi$ attaining a minimum at (r,x). Then, we obtain

$$\begin{aligned} 0 &\geq c\mathbb{E}\Big[\int_0^{\tau_x^{\xi,0}\wedge h} e^{f(r,s)} \,\mathrm{d}s \Big] \\ &\quad + \mathbb{E}\Big[e^{-U^r_{\tau_x^{\xi,0}}} V\big(r_{\tau_x^{\xi,0}}, (\xi - c)\tau_x^{\xi,0}\big) 1\!\!1_{[\tau_x^{\xi,0}\leq h]}\Big] \\ &\quad - \mathbb{E}\Big[V(r,x) 1\!\!1_{[\tau_x^{\xi,0}\leq h]}\Big] \\ &\quad + \mathbb{E}\Big[\Big(e^{-U_h^r}\phi\big(r_h, X_h^u\big) - \phi(r,x)\Big) 1\!\!1_{[\tau_x^{\xi,0}> h]}\Big] \,. \end{aligned}$$

In the next step, we divide the above expression by h and let h go to 0.
Since V is locally Lipschitz, the expectation $\mathbb{E}\Big[e^{-U_s^r} V\big(r_s, (\xi - c)s\big) - V(r,x)\Big]$ is bounded, giving together with $\lim_{h\to 0} h^{-1}\mathbb{P}[\tau_x^{\xi,0} \leq h] = 0$:

$$\begin{aligned} \lim_{h\to 0} \frac{\mathbb{E}\Big[\Big(e^{-U^r_{\tau_x^{\xi,0}}} V\big(r_{\tau_x^{\xi,0}}, X^u_{\tau_x^{\xi,0}}\big) - V(r,x)\Big) 1\!\!1_{[\tau_x^{\xi,0}\leq h]}\Big]}{h} \\ = 0 \,. \end{aligned}$$

For the remaining terms one has

$$\begin{aligned} \mu\phi_x + \frac{\sigma^2}{2}\phi_{xx} + a(\tilde{b} - r)\phi_r + \frac{\tilde{\delta}}{2}\phi_{rr} - r\phi \\ c - c\phi_x \leq 0 \,. \end{aligned}$$

This inequality holds for all $c \in [0,\xi]$. Therefore, V is a supersolution to (6). $\square$

Proposition 3.8
Let $v(r,x)$ be a super- and $u(r,x)$ a subsolution to (6)*, satisfying the conditions fulfilled by the value function: locally Lipschitz continuous, increasing in x, decreasing in r, bounded like described in* (5) *and $u(r,0) = v(r,0) = 0$. If $\lim\limits_{r\to\infty} u(r,x) \le \lim\limits_{r\to\infty} v(r,x)$, then $u(r,x) \le v(r,x)$ for all $(r,x) \in \mathbb{R} \times \mathbb{R}_+$.*

Outlook

For the discounting function given by a geometric Brownian motion, it was an easy exercise to prove the existence of the optimal strategy for the both cases: for restricted and unrestricted dividend rates. This lies partly on the fact that the surplus and the discounting processes are assumed to be independent, and partly on the fact that the expected value of a geometric Brownian motion is given by an exponential function, whose exponential linearly depends on time.

In the last section, we have considered the problem of consumption maximization under a stochastic interest rate described by an Ornstein-Uhlenbeck process. It was impossible to find explicit expressions for the optimal strategy and the value function. In contrast to the case of a deterministic surplus linearly depending on time, we could not even prove that the value function is concave in the x component. We can just conjecture that the optimal strategy would be given by a barrier function depending on $\{r_t\}$. Further investigation of this problem will be the subject of our future research.

Acknowledgments

The author would like to thank the anonymous reviewer for his/her helpful and constructive comments that greatly contributed to improving the final version of the paper. This research was supported by the Austrian Science Fund, grant P26487.

References

[1] Albrecher, H. and Thonhauser, S.: Optimality results for dividend problems in insurance. *RACSAM* 103(2), 295–320, (2009)

[2] Amann, H. and Escher, J.: *Analysis II.* Birkhäuser Verlag AG, Basel-Boston-Berlin. (1998)

[3] Asmussen, S. and Taksar, M.: Controlled diffusion models for optimal dividend pay-out. *Insurance Math. Econ.* 20, 1–15, (1997)

[4] Azcue, P. and Muler, N.: Optimal reinsurance and dividend distribution policies in the Cramér–Lundberg model. Math. Finance 15, 261–308 (2005)

[5] Bardi, M. and Capuzzo-Dolcetta, I. (1997). *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations.* Birkhäuser Boston Inc.

[6] Borodin, A.N. and Salminen, P.: *Handbook of Brownian Motion – Facts and Formulae.* Birkhäuser Verlag, Basel. (1998)

[7] Brigo, D. and Mercurio, F.: *Interest Rate Models - Theory and Practice.* Springer, Heidelberg. 2 edition (2006)

[8] Eisenberg, J., Grandits, P. and Thonhauser, S.: Optimal consumption under deterministic income. Journal of Optimization Theory and Applications, 160(1), 255–279, (2014)

[9] Eisenberg J.: Deterministic income with deterministic and stochastic interest rates. Preprint, TU Wien.

[10] Fleming, W.H. and Rishel, R.W.: *Deterministic and Stochastic Optimal Control*. Springer-Verlag, Berlin-New York. (1975)

[11] Schmidli, H. (2008). *Stochastic Control in Insurance*. Springer-Verlag, London.

[12] Shreve, S.E., Lehotzky, J.P. and Gaver D.P.: Optimal consumption for general diffusions with absorbing and reflecting barriers. SIAM J. Control Optim., 22(1), 55–75, (1984)