Transforming public pensions: A mixed scheme with a credit granted by the state

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Abstract

Birth rates have dramatically decreased and, with continuous improvements in life expectancy, pension expenditure is on an irreversibly increasing path. This will raise serious concerns for the sustainability of the public pension systems usually financed on a pay-as-you-go (PAYG) basis where current contributions cover current pension expenditure. In this paper, we propose, as an alternative solution, that the deficit of the scheme is immediately covered by the state but in return the individuals need to invest an amount of money into a fund. This investment is designed so that the individuals can repay to the state the deficit of the PAYG scheme at a particular level of probability and at the same time provides, on expectation, some gains to individuals. Two different strategies of debt repayment depending on the amount invested and the timing of the repayment to the state are analysed. We compare our results with the direct payment of the contribution that makes the system balanced by the individual. If the investment period is long enough, the optimal strategy tends to be a lump sum debt repayment. Directly covering the deficit of the PAYG is the optimal strategy if the investment period is short and the amount invested is relatively small. For shorter investment intervals and higher investment amounts it might be optimal to use a continuous repayment scheme.

Keywords: Pensions, Financial sustainability, Adequacy, Risk, Geometric Brownian motion, Stochastic control

2010 Mathematical Subject Classification: Primary 93E20
Secondary 91B30, 60K10

JEL classification: C61, G22, G52, H55, J26

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1 Introduction

The decline in fertility rates, the increase in longevity and the current forecasts for the ageing of the baby-boom generation all point to a substantial increase in the age dependency ratio, and this will raise serious concerns for the sustainability of PAYG pension systems. In particular, the life expectancy at birth is expected to increase by 5.3 for males and 5.1 years for females when comparing 2016 with 2070 (European Commission [11]). This is a worldwide problem, and consequently, many European countries (European Commission [6, 7]) have already carried out some parametric reforms, or even structural reforms, to ensure the financial sustainability of their pension systems.

In Europe, the common trend of the pension crisis is a wave of parametric adjustments including countries, among others, France, Greece, Hungary, Romania and Spain, see Whitehouse [28, 27] and OECD [17, 18, 19, 20].

In Latin America, since the 1980's, most of the countries in the region made structural reforms replacing completely or partially their PAYG system with programmes containing a fully funded component of individually capitalised accounts (Rofman et al. [24]). As a result, a transfer of financial market and volatility risk from the state to the individual happened.

The PAYG rate of return can be lower than the rate of return of funding schemes, especially in countries where the working population is not growing. In this case, the individual might consider that there is an implicit cost equivalent to the difference in return; see Robalino and Bodor [23] and Valdés-Prieto [26] for taxes implicit to PAYG schemes. However, the high-variability of the funding rate of return makes the choice between PAYG and funding less obvious and there might be advantages of mixing PAYG and funded schemes (De Menil et al. [8], Persson [22]). Also, PAYG is a useful social security financing technique which ensures income redistribution at both the inter and intra-generational levels.

On the other hand, Fajnzylber and Robalino [12] state that the transition from a PAYG to a fully funded scheme has a high transition cost where current contributors pay twice: first to finance their own retirement capital and second to finance pension benefits of current retirees. Currently, countries, such as Australia, Canada, Norway, Sweden, Latvia and Poland, amongst others, combine funded and PAYG elements within the mandatory pension system to improve the pension sustainability of the PAYG part and at the same time to increase the amount of the pensions. These mixed systems have been advocated, particularly by the World Bank, as a practical way to reconcile the higher financial market returns compared with GDP growth with the costs of a scheme with a greater funded element.

The academic literature has extensively focused on studying the optimal allocation between a funding (defined contribution) scheme and a PAYG in mixed pension system under a portfolio theory framework.[1] However, these studies do not deal with the problems in the financial sustainability of the PAYG component.

Robalino and Bodor [23] propose the use of government indexed bonds - first introduced by Buchanan [4] to support the sustainability of PAYG schemes. In particular, Robalino and Bodor [23] analyse the case of a cash-surplus of the pension fund being invested every six or twelve months in GDP indexed government bonds. However this approach does not guarantee the financial sustainability of the system, and the application is done to notional account systems. Other papers, Auerbach and Kotlikoff [2], Sinn [25], Palacios and Sinn [21], also provide some discussion about the use of government bonds.

Our research proposes a different method to ensure the financial sustainability of PAYG schemes based on mixed pension systems. In our framework, the deficit of the scheme is immediately covered by the state but in return the individuals have to invest an amount of money into a fund.Two different types of debt repayment to the state – depending on the amount invested and the timing of the repayment of the deficit – are analysed. The investment should be designed so that in expectation the individuals can repay the debt to the state and receive some return on investment. We prove that the optimal strategy tends to be a lump sum if long-term investments are allowed. However, directly covering the deficit of the PAYG is the optimal strategy if the allowed investment period is short and the amount invested in the fund is relatively small.

Following this introduction, the next section describes the model together with the assumptions used. In our proposed modelling framework we take on the view of a prototypical customer (PC) that earns the average salary and has the average age. This can be regarded as a standardization approach comparable to the representative agent in finance. Our framework avoids any demographic modelling (i.e. longevity and/or fertility risks) for the PC as these risks are immediately translated into changes in the balanced contribution rate of the PAYG. Further, our model is based on a credit granted by the state to the PC in the sense that the state covers the deficit as soon as it happens and the PC will be paying to the state in the future. In Section 3, some variants are presented where the PC invests the corresponding money into a fund and needs to repay the credit either at the end of the year or after a longer specified period. A comparison between the annual approach and the long-term approach is given along. In Section 4, we assume that the individuals need to transfer any excess of return above some particular level that needs to be optimised. We compare the results with the corresponding discrete time approach of Section 3. Section 5 concludes and makes suggestions for further research.

[1] See, for example, Matsen and Thogersen [15], Devolder and Melis [9], and Alonso-García and Devolder [1].

2 The Model Setting

To introduce a swift transformation of PAYG into a mixed pension system which is socially accepted, we suggest that the contributions of the individuals go into a PAYG and a funding scheme. While this will not immediately solve the problem of increasing contributions, the willingness of the state to take over possible losses or to formally grant credits combined with the higher return potential of the fund investment shall make the mixed system more attractive to the contributors. It shall give them the feeling that they get more for the higher contribution.

For the fund evolution we consider standard models such as a geometric Brownian motion. The costs of PAYG are assumed to be known for a certain time span. Our justification for this is that the state can control the level of the contributions that have to be paid out of the contributors' wages. The above mentioned time span will be the time for which we suggest different mixed strategies for the contributors. In Section 3 we will present our modeling approach in a discrete-time framework while Section 4 is embedded in a continuous-time setting.

To eliminate the explicit consideration of the evolution of wages and mortality, we introduce a representative or a prototypical contributor (PC), i.e. *the average contributor*, with an average age, an average salary and average salary increases. This PC contributes an amount C_0 – expressed in percentage of his salary – at time $t = 0$ into the PAYG. Let assume that the state anticipates the deficit and it is known that the contributions that make the PAYG system sustainable over the next T years are given by

$$C = (C_1, ..., C_T)$$

with

$$C_j > C_0,\ j = 1, ..., T\ .$$

We assume that the state targets to transform the classical PAYG scheme into a mixed pension scheme. For this, the state is formally taking over the payment of the differences $C_j - C_0$ for $j = 1, ..., T$. However, these payments represent some kind of a credit granted to the contributor, because, as soon as the deficit occurs the PC has to invest some pre-specified amount of money in a fund, in addition to the regular contribution of C_0 to PAYG. We are considering the same allocation for all individuals, i.e. for the PC. Such a limited choice of different investment opportunities is in line with a lot of current initiatives in the public pension sector that call for standard products that are cost-efficient due to only limited possible choices.

A certain part of the return on fund investment will be used to repay the debt amount of $C_j - C_0$ in the (near) future. Once the deficit is paid, the gains, if any, belong to the PC and could be used for instance for the retirement phase. In contrast, if the investment is not enough to cover the deficit, the government will bear the risk of not full debt repayment in one particular year.

Note in particular that the reduction of the PAYG contribution and the possible gains of the PC will highly depend on the amount invested.

The following sections specify two types of investment strategies depending on the amount invested and the time horizon for the debt repayment. We calculate the probability of full payback of the debt, the expected loss of the state and the expected gains of the PC. We also compare our results with the direct payment of C_j to the PAYG scheme by the PC.

We will always act on some filtered space $(\Omega, \mathfrak{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ where $\mathcal{F}$ is assumed to be the right-continuous filtration generated by a given geometric Brownian motion F.

3 Lump Sum Repayment in a Discrete-Time Setting

In this section, we consider an annual fund investment approach where (an agreed multiple of) the increase of the contribution paid by the PC is invested for a year. Depending on the the actual way how the risk of a shortfall is shared between PC and the state, we suggest two different models.

3.1 Variant A: Annual Payback

We assume that **at time** $t = 0$ it is agreed that the contribution to PAYG of the PC is fixed to be C_0 for the next $T + 1$ years. In return, the PC invests an amount of $\alpha(C_j - C_0)$ into a fund with value dynamics given by

$$F_t = F_0 e^{\mu t + \sigma W_t}$$

at time $j = 1, ..., T$ for some $\alpha > 0$. Then, at time $j+1$, the PC pays $C_j - C_0$ to the state if

$$\alpha(C_j - C_0)e^{\mu+\sigma(W_{j+1}-W_j)} \geq C_j - C_0 \ .$$

If this is the case, then the remaining value of the fund position stays with the PC and PC makes a gain of

$$G_j := (C_j - C_0)\left(\alpha e^{\mu+\sigma(W_{j+1}-W_j)} - 1\right)^+ \ .$$

Otherwise, the PC liquidates the fund position and pays

$$\alpha(C_j - C_0)e^{\mu+\sigma(W_{j+1}-W_j)}$$

to the state that in addition covers the amount of money needed for PAYG,

$$L_j := (C_j - C_0) \cdot \left(1 - \alpha e^{\mu+\sigma(W_{j+1}-W_j)}\right)^+ \ .$$

This procedure is repeated every year.[2]

It should be stated explicitly that in the second case above, for $\alpha > 1$ the PC realizes a loss compared to the payment of C_j to PAYG directly. Note also the following relation between gain of the PC and loss of the state given by

$$C_j - C_0 - L_j = \alpha e^{\mu+\sigma(W_{j+1}-W_j)} - G_j \tag{1}$$

which can easily be seen by the fact that we either have $L_j = 0$ or $G_j = 0$ (or both are zero in the unlikely event of $C_j - C_0 = \alpha e^{\mu+\sigma(W_{j+1}-W_j)}$).

Certainly, the state wants to keep the probability of losses small, while the whole procedure is only attractive for the PC if there is a possible gain at least on the level of expectations. It is thus clear that the fund characteristics play an important role. Therefore, we summarize some properties in the following proposition.

Proposition 3.1

For a value of $\alpha > 0$ and $C_j > C_0 > 0$, we obtain:

a) The probability P_j that the full payment of $C_j - C_0$ is made to the state at time $j+1$ is given by

$$P_j := \mathbb{P}\left[\alpha e^{\mu+\sigma(W_{j+1}-W_j)} \geq 1\right] = \Phi\left(\frac{\mu + \ln(\alpha)}{\sigma}\right).$$

b) The expected loss $\mathbb{E}[L_j]$ of the state at time $j+1$ is given by

$$\begin{aligned}\mathbb{E}[L_j] &= \mathbb{E}\left[(C_j - C_0)\left(1 - \alpha e^{\mu+\sigma(W_{j+1}-W_j)}\right)^+\right] \\ &= (C_j - C_0)\left\{\Phi\left(-\frac{\mu + \ln(\alpha)}{\sigma}\right) - \alpha e^{\mu+\frac{\sigma^2}{2}}\Phi\left(-\frac{\mu+\sigma^2+\ln(\alpha)}{\sigma}\right)\right\}.\end{aligned}$$

c) The expected gain $\mathbb{E}[G_j]$ of the PC at time $j+1$ is given by

$$\mathbb{E}[G_j] = (C_j - C_0)\left\{\alpha e^{\mu+\frac{\sigma^2}{2}}\Phi\left(\frac{\mu+\sigma^2+\ln(\alpha)}{\sigma}\right) - \Phi\left(\frac{\mu + \ln(\alpha)}{\sigma}\right)\right\}.$$

Proof: Assertion a) is implied by the fact that $W_{j+1} - W_j$ is standard normally distributed.

For Assertion b), the normal distribution of $W_{j+1} - W_j$ yields

$$\begin{aligned}\mathbb{E}[L_j] &= (C_j - C_0)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{\mu+\ln(\alpha)}{\sigma}} e^{-\frac{y^2}{2}}\,\mathrm{d}y - \frac{\alpha}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{\mu+\ln(\alpha)}{\sigma}} e^{\mu+\sigma y-\frac{y^2}{2}}\,\mathrm{d}y\right) \\ &= (C_j - C_0)\left\{\Phi\left(-\frac{\mu + \ln(\alpha)}{\sigma}\right) - \alpha e^{\mu+\frac{\sigma^2}{2}}\Phi\left(-\frac{\mu+\sigma^2+\ln(\alpha)}{\sigma}\right)\right\}.\end{aligned}$$

[2] As the PC invests at time j - in line with the necessary increase of the PAYG contribution - and pays back at time $j+1$, the procedure also includes a credit for a duration of one year.

Assertion c) is then implied by relation (1) between G_j and L_j. □

Remark 3.2 (Risk for the PC)
By construction of the strategy, the PC can only make a loss in year j of

$$L_j^{(PC)} = (\alpha - 1)(C_j - C_0)$$

in the case of $\alpha > 1$. Note that the loss equals a constant value, only its occurrence is random. By Part a) of Proposition 3.1, the probability for such a loss, its expected value and variance are given by

$$\begin{aligned}
\mathbb{P}\left[L_j^{(PC)}\right] &= \Phi\left(-\frac{\mu + \ln(\alpha)}{\sigma}\right) \\
\mathbb{E}[L_j^{(PC)}] &= (\alpha - 1)(C_j - C_0)\Phi\left(-\frac{\mu + \ln(\alpha)}{\sigma}\right) \\
\mathbb{V}ar[L_j^{(PC)}] &= (\alpha - 1)^2(C_j - C_0)^2\Phi\left(-\frac{\mu + \ln(\alpha)}{\sigma}\right)\Phi\left(\frac{\mu + \ln(\alpha)}{\sigma}\right)
\end{aligned}$$

Note that a high value of α reduces all those numbers, while a high volatility σ of the fund tends to increase them. ■

The probability of being able to fully pay back $C_j - C_0$ to the state at time $j+1$ is increasing in α. The same is true for the expected final fund position of the PC given that $\min\left\{C_j - C_0; \alpha e^{\mu+\sigma(W_{j+1}-W_j)}\right\}$ has been already paid to the state. However, a time horizon of just one year is very short. Consider for instance the case of

$$\alpha = 1 ,$$

i.e. one invests exactly the debt amount $C_j - C_0$ at the beginning of year $j+1$ into the fund. Then, part a) of Proposition 3.1 implies that the probability for obtaining $C_j - C_0$ at time j is given by

$$\mathbb{P}\left[(C_j - C_0)e^{\mu+\sigma(W_{j+1}-W_j)} > C_j - C_0\right] = \Phi\left(\frac{\mu}{\sigma}\right) .$$

Note that for a fund with optimistic parameters $\mu = 0.04, \sigma = 0.2$ this probability is approximately 58% which might be considered as not satisfactory enough. In order to reduce the probability of an annual shortfall (i.e. no full debt repayment by the PC) significantly, the state should require

$$\alpha \gg 1 ,$$

i.e. the PC should contribute C_0 plus a fund investment such that the resulting sum is well-above the actually needed contribution of C_j at time j. At the same time, only a strong requirement for a small probability of shortfall will lead to building up a significant amount of money in a fund for the PC, as shown in the following example.

Example 3.3

We illustrate the issues raised above by looking at two settings of values for the fund parameters:

a) ***A standard fund****:* $\mu = 0.04,\ \sigma = 0.2$

In this setting, the probabilities for a full payment of $C_1 - C_0$ *at time 1 as a function of* α*, denoted by* P_1*, are computed. Further, we also calculate the expected loss for the state,* $\mathbb{E}[L_1]$*, and the expected gain of the PC after paying to the state,* $\mathbb{E}[G_1]$*. In addition we look at the following two quantities:*

$$\mathbb{E}[B_1] := \mathbb{E}\left[(C_1 - C_0)\left(\alpha e^{\mu+\sigma W_1} - 1\right)\right] ,$$
$$\mathbb{E}[G_1^A] := \mathbb{E}[G_1] - (\alpha - 1)(C_1 - C_0) .$$

$\mathbb{E}[B_1]$ *is the expected fund position after one year if always the full value* $C_1 - C_0$ *has to be paid back by the PC.* $\mathbb{E}[G_1^A]$ *describes the expected gain for the PC minus the additional investment of* $(\alpha - 1)(C_1 - C_0)$*. The above quantities for* $C_0 = 1,\ C_1 = 1.1$ *can be found in Table 1 below.*

α	0.9	1	1.05	1.1	1.25	2	3
P_1	0.37	0.58	0.67	0.75	0.91	> 0.99	1.00
$\mathbb{E}[L_1]$	0.01	0.005	0.004	0.003	0.001	$< 10^{-4}$	$< 10^{-4}$
$\mathbb{E}[G_1]$	0.006	0.0117	0.015	0.020	0.034	0.112	0.219
$\mathbb{E}[B_1]$	-0.004	0.006	0.011	0.017	0.033	0.112	0.219
$\mathbb{E}[G_1^A]$	0.0158	0.0117	0.0104	0.0095	0.0085	0.0124	0.0186

Table 1: The key quantities of the scheme as functions of α for $C_0 = 1$, $C_1 = 1.1$.

While the first four rows in Table 1 show an obvious behaviour (i.e. the full payback probability and the gains increase and the expected loss of the state decreases with increasing α*), it is actually the last row that should be noticed. The expected differences between the net gains and the additionally invested amount first decrease with increasing* α *and then increase for large values of* α*. In particular, for values of* $1 \le \alpha \le 1.25$ *there is no real incentive for PC to exceed* $\alpha = 1$*. The reason for that is that, initially,* $\mathbb{E}[G_1]$ *does not grow fast enough with increases in* α*. There is an initially large benefit for PC as the state takes over some significant investment risk via accepting an expected loss of* 5% *per unit of money given as a credit. If the state accepts an expected loss of* 10% *per unit of money in the case for* $\alpha = 0.9$ *then PC has already realised a sure gain of* 1% *plus a small expected net wealth after paying back* $C_j - C_0$*. In total, this leads to* $\mathbb{E}[G_1^A] = 0.0158$*, i.e. an expected gain of* 15.8% *compared to the payment of* $C_1 - C_0 = 0.1$ *to PAYG directly. However, such a (partially) safe gain can only be interpreted as a subsidy granted by the state to motivate fund investment.*

The effect of the state taking over the shortfall risk becomes insignificant *with increasing values of α. Even more, for very large values of α the probability that we end up with a zero fund position after paying back vanishes, and the fund investment can unfold its full potential.*

If the state required a probability for a full credit payback of 0.9, 0.95 *or* 0.99 *then corresponding values of α are given by* 1.24, 1.34 *and* 1.53. *Thus, to avoid a loss with a probability of* 99%, *the PC needs to invest* 53% *more in the fund than when paying the increase in the contribution directly into PAYG. On the positive side, the PC obtains an expected net fund wealth of* 6.25% *for additionally paying in* 5.3% *in our example*

This is an example for illustrative purposes, however, we think that even if α is (slightly) higher than 1, our framework, can be applicable in real world as this is a multiple of a potential small value, i.e. difference in the value of contributions. Let us consider, for example, an annual average salary of 25,000 and a contribution rate of 15% – which is in line with the average contribution rate in Europe. If the contribution rate that makes the system balanced is 16.5%, then the annual contributions needed are 4,125. One possibility would be for the PC to immediately increase his contributions to this new level. This would represent an annual increase of 375 with respect to the contribution rate of 15%. The alternative that we propose is the investment by the PC of a certain amount into a fund. If $\alpha = 1.25$, the PC would invest 469, which is 7.8 euros more per month compared to the case of paying directly the deficit of the PAYG. After this investment, the state, with a probability of 0.91, gets the full debt from the PC, i.e. 375 and the PC would get back, in expectation, 123. As the investment in the fund is slightly higher than the amount strictly needed to balance the PAYG scheme, i.e. 94, the positive difference for the PC with respect to covering the PAYG debt directly is 29.

b) ***A very well-diversified fund***: $\mu = 0.04,\ \sigma = 0.1$

From Table 2, we can now realize that the probability of the full payment P_1 and the expected loss $\mathbb{E}[L_1]$ are better for $\alpha \geq 1$ than in the case of the fund with $\sigma = 0.2$. However, the expected net gains for the PC $\mathbb{E}[G_1]$ are slightly smaller. ■

α	0.9	1	1.05	1.1	1.25	2	3
P_1	0.26	0.66	0.81	0.91	0.96	0.99	0.996
$\mathbb{E}[L_1]$	0.007	0.002	0.001	$4 \cdot 10^{-4}$	10^{-4}	$< 10^{-4}$	$< 10^{-4}$
$\mathbb{E}[G_1]$	0.002	0.007	0.011	0.015	0.020	0.026	0.031

Table 2: The probability for the full debt repayment P_1, the expected loss of the state $\mathbb{E}[L_1]$ and the expected gain of the PC $\mathbb{E}[G_1]$ as functions of α.

To summarise the above examples, it either needs a large value of α which might not be feasible for all contributors or a longer time of investment before the credit is paid back to the state. We will look at the second variant in Section 3.2.

We will here also have a quick look at a different variant of paying (back) the difference $C_j - C_0$ after the investment of $\alpha(C_j - C_0)$. This variant will be considered in detail in Section 4 in a continuous-time framework that allows a more flexible version.

Let us therefore assume that the state agrees that the PC can keep at least a certain return of up to $b \geq -1$ $\big($i.e. an amount of $(1+b)\alpha(C_j - C_0)\big)$ at the end of the investment period. Whatever exceeds this - but does not exceed $C_j - C_0$, goes to the state. If, however, the surplus investment result is less than $C_j - C_0$ then the possibly remaining part of $C_j - C_0$ will be taken over by the state. Note that now the loss L_j of the state gets the more complicated representation of

$$L_j := (C_j - C_0) \cdot \left(1 - \alpha\left(e^{\mu+\sigma(W_{j+1}-W_j)} - (1+b)\right)^+\right)^+$$

while relation (1) still stays valid. Further, from the form of L_j, we can compute

$$\mathbb{P}[L_j = 0] = \Phi\left(\frac{\mu - \ln(1 + b + \frac{1}{\alpha})}{\sigma}\right) .$$

Hence, for $b = 0$ the probability of no loss for the state is bounded by $\Phi(\mu/\sigma)$ which might not be high enough and will even decrease for bigger values of b. Further, a non-positive value of b can only be preferable for the PC in the case of $\alpha < 1$, but then the probability for a full debt repayment to the state decreases.

The probability of the full repayment for 1-year investment and the standard fund with $\mu = 0.04, \sigma = 0.2$ are summarised in Table 3. The values obtained

α / b	0.8	0.9	1	1.25	2	10
0.02	<0.001	<0.001	<0.001	0.003	0.0291	0.357
0	<0.001	<0.001	<0.001	0.003	0.0338	0.391
-0.5	0.005	0.014	0.034	0.133	0.5793	0.997
-0.75	0.034	0.090	0.018	0.482	0.9493	1
-0.9	0.097	0.224	0.391	0.770	0.9971	1
-1	0.180	0.372	0.579	0.906	0.9999	1

Table 3: Probability for the full debt repayment as a function of α and b.

in Table 3 show some clear indications:

- Even in the case of $b = -1$, a high probability (i.e. one above 80%) such that this investment result exceeds $C_1 - C_0$ is only obtained for the cases of $\alpha \geq 1.25$. In the latter case, the PC is better off by paying the difference $C_1 - C_0$ directly to PAYG.

- For values of $\alpha < 1$ the payback probabilities are far from being satisfactory for the state.

Further, by noting that with a similar, but slightly longer computation as in part b) of the proof of Proposition 3.1, we obtain

$$\begin{aligned} \mathbb{E}[L_j] &= (C_j - C_0)\Big[\Phi(W^{(u)}) + \alpha(1+b)\left(\Phi(W^{(u)}) - \Phi(W^{(d)})\right) \\ &\quad -\alpha e^{\mu+0.5\sigma^2}(\Phi(W^{(u)}+\sigma) - \Phi(W^{(d)}+\sigma))\Big], \end{aligned}$$

where we have used the abbreviations of

$$W^{(u)} = \frac{\ln(1+b+\frac{1}{\alpha}) - \mu}{\sigma}, \quad W^{(d)} = \frac{\ln(1+b) - \mu}{\sigma}.$$

We then also obtain the expected gain $\mathbb{E}[G_j]$ with the help of relation (1). For our standard fund with $\mu = 0.04, \sigma = 0.2$ we calculated the probability of a full payback, the expected losses of the state and the expected gains of the PC for the values of $b = -0.5, 0$ and large values of $\alpha = 2, 5, 10$.

α	2	2	5	5	10	10
b	-0.5	0	-0.5	0	-0.5	0
$P1$	0.58	0.03	0.98	0.24	0.997	0.39
$\mathbb{E}[L_1]$	0.011	0.078	0.0006	0.06	<0.0001	0.052
$\mathbb{E}[G_1]$	0.1233	0.1899	0.4315	0.4912	0.9619	0.9618

Table 4: The probability of full payback, the expected payment to the state and the expected net fund value of the PC after (partial) payback as functions of α and b.

Table 4 shows that only for the values with $b = -0.5$ (and $\alpha = 5, 10$) the probabilities for a full payback and the values of the expected losses are acceptable for the state. The value of $b = -0.5$ is, however, difficult to accept by the PC as it means for e.g. a value of $\alpha = 5$ the state can still claim the full amount of 0.1 even if the PC has lost 30% of the invested amount of 0.5. Of course, the value of $b = 0$, which would at least prevent the state from claiming money from the PC in all cases of an investment loss, is something that the PC might expect. However, only in the case of $\alpha = 10$ (or $\alpha = 5$) and $b = -0.5$ both parties should be able to live with the results. A very low expected loss paired with a success probability of nearly one would satisfy the state, while PC realizes an expected gain of 0.9619. This means that the expected annual return of the PC on the investment of $1 = 10 \times 0.1$ is higher than 6% (note that $C_1 - C_0$ has to be paid to the state otherwise). Still, there remains the question if the PC can afford investing such a comparably higher amount of money and also wants to take the over the investment risk.

One reason for the bad performance of this variant of paying back the credit can be the fact that we are only able to pay back the surplus at the end of the year. We will therefore in Section 4 consider a continuous-time model where we pay back parts of the credit as soon as sufficiently high surpluses have been realized.

3.2 Variant B: A granted credit

As a second approach, the state again pays the difference of $C_j - C_0$ at times $j = 1, ..., T$, where we assume that the values C_j are deterministic. However, the state considers these payments as a credit with zero interest rate[3] to the PC. This construction allows the PC to invest money in a fund for a longer time and delays the full payback to time T. The potential gains from the fund investment are then used

- to pay back the credit and
- to build up money as reserves for the own future.

Thus, it remains to find the optimal additional amount of money, that the PC invests into the fund, and the strategy to pay back the credits granted by the state. As in Variant A, we consider the following strategy:

At time $j-1$ the PC contributes $C_0 + \alpha(C_j - C_0)$ with $\alpha > 0$, $j = 1, ..., T$. C_0 directly goes to PAYG while $\alpha(C_j - C_0)$ is invested into the fund. Thus, at time j the PC has a fund position F_j (before payment of the new contribution), where

$$F_j := \alpha \sum_{k=1}^{j} \left((C_{j-k+1} - C_0) e^{\mu k + \sigma (W_{j+1} - W_{j-k+1})} \right)$$

$$\mathbb{E}\left[F_j\right] = \alpha \sum_{k=1}^{j} (C_{j-k+1} - C_0) e^{(\mu + \frac{\sigma^2}{2})k}$$

for $j = 1, ..., T$. In the special case of $C_j = \bar{C}$ for all $j = 1, ..., T$, we have

$$\mathbb{E}\left[F_j\right] = \alpha \sum_{k=1}^{j} (C_{j-k+1} - C_0) e^{(\mu + \frac{1}{2}\sigma^2)k} = \alpha(\bar{C} - C_0) e^{\mu + \frac{1}{2}\sigma^2} \frac{1 - e^{(\mu + \frac{1}{2}\sigma^2)j}}{1 - e^{\mu + \frac{1}{2}\sigma^2}}$$

We assume that the full credit sum, defined as

$$C_{\text{total}} := \sum_{j=1}^{T} (C_j - C_0)$$

[3] This is a reasonable assumption given the current ultra low interest rate environment in Europe.

will be paid to the state at time T.

We thus arrive at

Lemma 3.4
For an $\alpha > 0$ and the above investment strategy into the fund, we have:
a) In the case of payments of $\alpha(C_j - C_0)$ at time j, $j = 1, ..., T$ into the fund, the credit can be fully paid back in expectation at time T if we choose

$$\alpha = \alpha^* := \frac{\sum_{j=1}^{T} (C_j - C_0)}{\sum_{j=1}^{T} (C_j - C_0) e^{(\mu + \frac{1}{2}\sigma^2)(T+1-j)}} \ .$$

In the particular case of $C_j = \bar{C}$ for all $j = 1, ..., T$, this simplifies to

$$\alpha = \alpha^* = T \frac{1}{e^{\mu + \frac{1}{2}\sigma^2}} \frac{1 - e^{\mu + \frac{\sigma^2}{2}}}{1 - e^{(\mu + \frac{1}{2}\sigma^2)T}} \ .$$

b) The expected gain for the PC in nominal values by comparing the credit sum $\sum (C_j - C_0)$ with the invested amount $\alpha^* \sum (C_j - C_0)$ is strictly positive as we have $\mu + \frac{\sigma^2}{2} > 0$.

Proof: a) follows from the required equality between the sum of (expected) payments by the state and $\mathbb{E}[F_T]$ which then has to be solved for $\alpha^* C_0$.
b) is obvious. □

In contrast to the one period setting of Variant A, we cannot easily calculate the probability of a loss by the state or the expected net fund value for the PC at time T. The reason is a well-known fact that the distribution of the sum of log-normally distributed random variables is not explicitly known. We therefore illustrate the performance of this strategy and its differences to Variant A in the next section.

3.3 Comparison of Variant A and Variant B

As we do not have explicit distributional results for Variant B, we present a numerical example where we choose $T = 10$ and also $C_j = \bar{C} = 1.1 \cdot C_0$. We then simulate $10,000$ realisations of the fund performance and estimate the probability of a shortfall/loss (i.e. the event of $F_T < T \cdot (\bar{C} - C_0)$), the expected loss for the state and the expected net wealth for the PC that remains in the fund at time T. We again consider the two different fund parameter sets already used for illustrating Variant A. We especially want to compute (or more precisely, estimate via Monte Carlo simulation)

$$\begin{aligned} P_{\text{shortfall}} &:= \mathbb{P}\left[F_T \le T \cdot (\bar{C} - C_0)\right] = \mathbb{P}\left[F_T \le 1\right], \\ E_{\text{shortfall}} &:= \mathbb{E}\left[(1 - F_T)^+\right], \\ E_{\text{finalnetfund}} &:= \mathbb{E}\left[(F_T - 1)^+\right] \ . \end{aligned}$$

For this, note first that in our setting, we can explicitly calculate

$$\mathbb{E}[F_T] = \alpha \cdot 0.1 \cdot e^{\mu+\frac{1}{2}\sigma^2} \frac{e^{10(\mu+\frac{1}{2}\sigma^2)}}{1-e^{\mu+\frac{1}{2}\sigma^2}} .$$

By the well known relation

$$\mathbb{E}[X] = \mathbb{E}[X^+] - \mathbb{E}[X^-]$$

we can thus estimate/compute $E_{\text{finalnetfund}}$ from $\mathbb{E}[F_T]$ and the estimate for $E_{\text{totalloss}}$.

3.3.1 A standard fund: $\mu = 0.04, \sigma = 0.2$.

Under our above assumptions on the fund parameters and $\bar{C} - C_0 = 0.1 \cdot C_0 = 0.1$, $T = 10$ we obtain first:

α	1	1.05	1.1	1.15	1.2	1.25
$P_{\text{shortfall}}$	0.272	0.232	0.197	0.171	0.145	0.123
$E_{\text{shortfall}}$	0.054	0.045	0.036	0.030	0.024	0.020
$E_{\text{finalnetfund}}$	0.470	0.530	0.593	0.657	0.722	0.789

Table 5: Probability for a shortfall, expected shortfall, expected net wealth as functions of α.

To see the time effect of investment one can e.g. compare the additional total nominal payment of 0.25 with the net gain of 0.789 in the case of the choice of $\alpha = 1.25$. Of course, one should take interest rates into account. However, in the current ultra low interest rate environment in Europe, using nominal values is a good approximation. This is especially true in a country such as Germany with negative interest rates for short term investments.

To compare the performance of Variant B with that of Variant A, we now consider the corresponding values when we use

$$\alpha = \alpha_{A,90} \approx 1.242 ,$$

the α that leads to a 90% security level for a full payment to the state at each single payment time. It comes along with the probability of at least one loss of 0.35 and a total expected loss of 0.0087. It is clear that this loss probability is not comparable to those of Variant B. The break even value α_B such that the expected total loss of the state equals that of Variant A is given by $\alpha_B = 1.44$. While it is comparably high, it leads to an expected net fund wealth of 1.047 compared to the additional nominal payments of 0.44.

α	1	1.05	1.1	1.15	1.2	1.25
$P_{\text{shortfall}}$	0.131	0.087	0.053	0.035	0.020	0.015
$E_{\text{shortfall}}$	0.012	0.007	0.004	0.002	0.001	0.001
$E_{\text{finalnetfund}}$	0.304	0.364	0.425	0.488	0.552	0.616

Table 6: Probability for a shortfall, expected shortfall, expected net wealth as functions of α.

3.3.2 A very well diversified fund: $\mu = 0.04, \sigma = 0.1$.

For this set of fund parameters we obtain: As in the case of Variant A the loss figures now clearly improve while the net fund wealth performs slightly weaker due to the small variance.

Again, we can compare

$$\alpha = \alpha_{A,90} \approx 1.0925 \ ,$$

the alpha that leads to a 90% security level for a full payment to the state at each single payment time with its counter part in Variant B that leads to the same expected total loss of 0.004515. This time, the break even value $\alpha_B = 1.0927$ is very close to $\alpha_{A,90}$. It leads to an expected net fund wealth of 0.416 compared to the additional nominal payments of 0.0927.

4 Continuous Debt Repayment with Optimisation Features

In this part of the paper, we assume the same credit scheme described in the previous section. However, we consider a different set of strategies that can be applied in order to repay the debt to the state. We assume that the state requires the PC to share the profits from an investment continuously in time, i.e. the PC has to transfer continuously in time any excess above some level b (to be optimised) into a special bank account during a specified period. As the continuous withdrawal procedure will be crucial for understanding the whole following section, we give first an intuitive explanation of a discretised withdrawal mechanism. Let $\varepsilon > 0$ and $b > 0$ be arbitrary but fixed. If the return on investment (normalised to the initial value 1) hits the level $1 + b + \varepsilon$, then ε is shifted to a special bank account. Subsequently, the process re-starts at $1 + b$, see Figure 1. Once the level $1 + b + \varepsilon$ is attained again, ε is shifted to the special account. This pattern is repeated until we arrive at the chosen time horizon (for instance, 1 year, 10 years etc.). The continuous withdrawal can be envisioned as the discretised withdrawal procedure with ε converging to 0. Note that using processes as a geometric Brownian motion, an Ornstein-Uhlenbeck or Cox-Ingersoll-Ross process to model the investment will unavoidably lead to the change of path after the

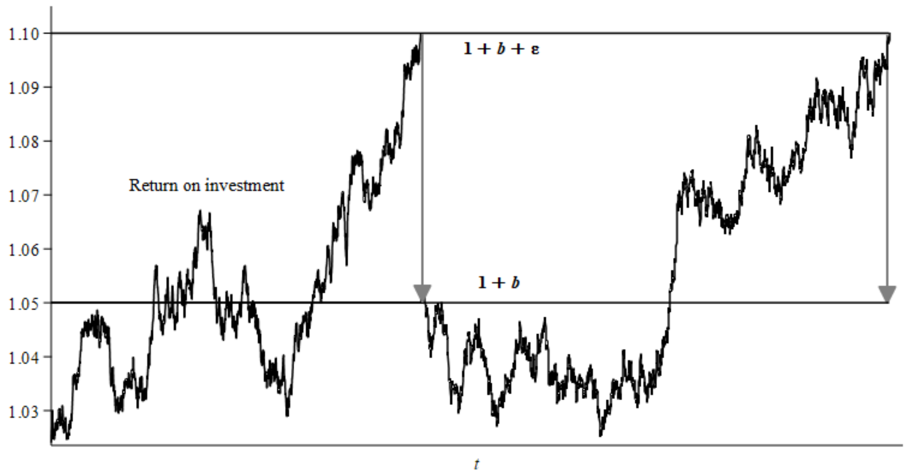

Figure 1: Return on investment with two hitting barriers: $1+b$ and $1+b+\varepsilon$ and withdrawal of ε by hitting $1+b+\varepsilon$.

withdrawal. All mentioned processes depend in their drift (and volatility) on the current value of the process, see for instance, Borodin and Salminen [3]. Hence, changing the current value through a withdrawal will change the entire future evolution of the process. The discretisation approach will be described mathematically in Lemma 4.2 below.

We propose the following credit scheme: once an increase in contribution becomes necessary, the PC has the choice to pay the increase in contribution or to invest in a pre-specified fund. Should the PC go for the second option, the state contractually agrees to pay the difference between the old and the new contributions in the following, say, 10 years. In return, the PC has to invest a certain amount of money at the beginning of every year so that following the continuous withdrawal mechanism described above at the end of every year the debt to the state can be repaid and the PC can get some positive return on investment.

Our objective is to maximise the amount of money remaining to the PC after the debt repayment. The emphasis lies on the investment of the minimal possible amount such that the probability that the debt can be fully repaid to the state stays above some pre-specified level. Note that the PC carries the investment risk where the state has the risk that the PC will not be able to repay the debt. It means in particular that we allow for negative values of the level b.

As before we model the price of the fund under consideration by a geometric Brownian motion

$$F_t = F_0 e^{\mu t + \sigma W_t} \ .$$

We denote by $D_t(b)$ the part of the gains, depending on the chosen barrier b, that is transferred to the debt account and by $R_t(b)$ the part remaining to the PC for the time horizon 1 year unless otherwise specified.

Remark 4.1
Let us clarify the relation between $F(t)$, $D_t(b)$ and $R_t(b)$ if we assume the above discretised withdrawal strategy: The fund starts with $F_0 = R_0(b)$. As long as the fund value does not exceed $(1+b+\varepsilon)F_0$, F_t and $R_t(b)$ are identical. Only at the time s_1, when the normalised (divided by F_0) fund attains the value $1+b+\varepsilon$ for the first time, we shift $\varepsilon \cdot F_0$ to the debt account D. It is then directly cashed in by the state and reduces the PC's debts, but is formally kept in $D_t(b)$. The remaining part $R_{s_1}(b) = (1+b)F_0$ remains invested in the fund until the next time s_2 when the fund starting in s_1 with the initial value $(1+b)F_0$ arrives at $(1+b+\varepsilon)F_0$. Then again, we shift $\varepsilon \cdot F_0$ to the debt account D and keep $R_{s_2}(b) = (1+b)F_0$ in the fund afterwards. Without the shifting we would have obtained:

$$R_{s_2}(b) = F_0 \cdot e^{\mu s_2 + \sigma W_{s_2}} = F_0 \frac{(1+b+\varepsilon)^2}{1+b}$$

while with the shifting, we have

$$R_{s_2}(b) + D_{s_2}(b) = F_0 \cdot (1+b+2\varepsilon)$$

which is smaller than $(1+b+\varepsilon)^2/(1+b) > 0$. Hence, we eliminate parts of the potential fund investment return by shifting money to the debt account. On the other hand, we keep the realised surplus for reducing the remaining debt and therefore ensure that at least parts of the debt are already repaid. ■

Let now $b \in [-1, 1]$ be arbitrary but fixed. If the return at time $t \in [0, 1]$ compared to the initial value F_0

$$\frac{F_t - F_0}{F_0} = e^{\mu t + \sigma W_t} - 1$$

exceeds the level b, the excess $\left(e^{\mu t + \sigma W_t} - 1 - b\right)$ is transferred to the debt bank account. Mathematically it means that the process $\{e^{\mu t + \sigma W_t}\}$ is downward-reflected at $1+b$. Since under the logarithm the reflection property remains preserved, we can consider the downward-reflection of $\{\mu t + \sigma W_t\}$ at $\ln(1+b)$. For a Brownian motion with drift we know that the reflected process, below the level $\ln(1+b)$, at time t is given by

$$\mu t + \sigma W_t - \left(\max_{0 \le s \le t} \{\mu s + \sigma W_s\} - \ln(1+b) \right)^+ .$$

Therefore, transforming the downward-reflection back, gives the part remaining to the PC

$$R_t(b) := e^{\mu t + \sigma W_t - \left(\max_{0 \le s \le t} \{\mu s + \sigma W_s\} - \ln(1+b) \right)^+},$$

confer for details for instance [3, p. 77]. It remains to find the expression for the debt part, denoted by $D_t(b)$, to be accumulated on the special account.

Lemma 4.2
Let b be arbitrary but fixed. For any time $t \geq 0$ it holds

$$D_t(b) = (1+b)\Big(\max_{0\leq s\leq t}\{\mu s + \sigma W_s\} - \ln(1+b)\Big)^+ .$$

Proof: Recall first that $D_t(b)$ describes the excess of the return from the fund above the pre-specified boundary of $1+b$.
In order to prove our claim we will stick to the following roadmap:

- We discretise the withdrawal procedure and calculate the corresponding amount transferred to the debt account until time t.
- Then, we calculate $D_t(b)$ as the limit of the discretised models.

Consider an $\varepsilon > 0$, define for $n \geq 1$

$$\begin{aligned} T_1 &:= \inf\{s \geq 0 : \ e^{\mu s + \sigma W_s} = 1+b+\varepsilon\} \\ &= \inf\{s \geq 0 : \ \mu s + \sigma W_s = \ln(1+b+\varepsilon)\}, \\ T_{n+1} &:= \inf\{s \geq T_n : e^{\ln(1+b)+\mu(s-T_n)+\sigma\big(W_s - W_{T_n}\big)} = 1+b+\varepsilon\} \\ &= \inf\Big\{s \geq T_n : \ln(1+b) + \mu(s-T_n) + \sigma\big(W_s - W_{T_n}\big) = \ln(1+b+\varepsilon)\Big\} \end{aligned}$$

and let

$$N_t(\varepsilon) := \sup\{n \geq 1 : \ T_n \leq t\} .$$

The stopping times T_n describe the times when the return on investment hits the level $1+b+\varepsilon$ with a subsequent withdrawal of ε. The random variable $N_t(\varepsilon)$ describes the number of withdrawals until a fixed time t for a given $\varepsilon > 0$. Note that $N_t(\varepsilon)$ only jumps at the stopping times T_n with $N_t(\varepsilon) = n$ for $t \in [T_n, T_{n+1})$. It means, we approximate the procedure of skimming off the return of the considered geometric Brownian motion by a process, depending on ε, where we first wait until $e^{\mu s+\sigma W_s}$ hits the level $1+b+\varepsilon$ and pay ε immediately into the debt account. Subsequently, we wait until the considered geometric Brownian motion (now with the start value $1+b$) again hits the level $1+b+\varepsilon$ and transfer ε into the debt account. The amount transferred into the debt account up to time t is given by $\varepsilon N_t(\varepsilon)$. The process after the withdrawals, denote it by F_t^ε, fulfils $F_t^\varepsilon = e^{\mu t+\sigma W_t}$ for $t \in [0, T_1)$ and

$$F_{T_1}^\varepsilon = 1+b = \Big(\frac{1+b}{1+b+\varepsilon}\Big) e^{\mu T_1 + \sigma W_{T_1}} = \Big(\frac{1+b}{1+b+\varepsilon}\Big)^{N_{T_1}(\varepsilon)} e^{\mu T_1 + \sigma W_{T_1}} .$$

Then, for $t \in (T_1, T_2)$ using $N_{T_1}(\varepsilon) = N_t(\varepsilon) = 1$ one gets

$$\begin{aligned} F_t^\varepsilon &= F_{T_1}^\varepsilon e^{\mu(t-T_1)+\sigma(W_t - W_{T_1})} = \Big(\frac{1+b}{1+b+\varepsilon}\Big) e^{\mu T_1 + \sigma W_{T_1}} e^{\mu(t-T_1)+\sigma(W_t - W_{T_1})} \\ &= \Big(\frac{1+b}{1+b+\varepsilon}\Big)^{N_t(\varepsilon)} e^{\mu t + \sigma W_t} . \end{aligned}$$

Iterating the above steps, yields for the process F_t^ε:

$$F_t^\varepsilon = \Big(\frac{1+b}{1+b+\varepsilon}\Big)^{N_t(\varepsilon)} e^{\mu t+\sigma W_t} = e^{N_t(\varepsilon)\ln\left(\frac{1+b}{1+b+\varepsilon}\right)} e^{\mu t+\sigma W_t} \ .$$

The structure of T_n as hitting times of an arithmetic Brownian motion yields, confer for instance [13, p. 95]:

$$N_t(\varepsilon) = \sup\left\{ n\in\mathbb{N} : \max_{0\le s\le t}\{\mu s+\sigma W_s\} - \ln(1+b) \ge n\cdot\ln\Big(\frac{1+b+\varepsilon}{1+b}\Big)\right\} .$$

Therefore, we can conclude

$$\lim_{\varepsilon\to 0} N_t(\varepsilon)\varepsilon = (1+b)\Big(\max_{0\le s\le t}\{\mu s+\sigma W_s\} - \ln(1+b)\Big)^+ \quad \text{a.s.}$$

Furthermore, it holds

$$\begin{aligned}\lim_{\varepsilon\to 0} N_t(\varepsilon)\ln\Big(\frac{1+b}{1+b+\varepsilon}\Big) &= \lim_{\varepsilon\to 0} N_t(\varepsilon)\varepsilon\cdot\frac{\ln(1+b)-\ln(1+b+\varepsilon)}{\varepsilon}\\ &= -\Big(\max_{0\le s\le t}\{\mu s+\sigma W_s\} - \ln(1+b)\Big)^+ \quad \text{a.s.}\end{aligned}$$

This means in particular that the process F_t^ε converges to $R_t(b)$ a.s. as $\varepsilon\to 0$, implying that $\varepsilon N_t(\varepsilon)$ converges to $D_t(b)$ a.s. for $\varepsilon\to 0$. Therefore:

$$D_t(b) = \lim_{\varepsilon\to 0}\varepsilon N_t(\varepsilon) = (1+b)\Big(\max_{0\le s\le t}\{\mu s+\sigma W_s\} - \ln(1+b)\Big)^+ .$$

□

Like in the previous section, we assume that the initial investment is given by $F_0 = \alpha(C_1 - C_0)$ with some positive α, and require for every period

$$\mathbb{P}\Big[F_0 D_t(b) \ge C_1 - C_0\Big] \ge p \ \Leftrightarrow\ \mathbb{P}\Big[D_t(b) \ge \frac{1}{\alpha}\Big] \ge p, \tag{2}$$

for some given $p\in[0,1]$. This credibility condition ensures that the level b is not chosen too high and the debt to the state will be paid with at least probability $p\cdot 100\%$.

Assume the percentage p is fixed contractually, then we can find the optimal pair (b,α) such that the expected loss of the PC is minimised. Let

$$\begin{aligned} V(b) :&= \mathbb{E}\big[R_1(b)\big] = \mathbb{E}\left[e^{\mu+\sigma W_1 - \left(\max\limits_{0\le s\le 1}\{\mu s+\sigma W_s\}-\ln(1+b)\right)^+}\right] ,\\ U(b) :&= \mathbb{E}[D_1(b)] = (1+b)\mathbb{E}\Big[\Big(\max_{0\le s\le 1}\{\mu s+\sigma W_s\} - \ln(1+b)\Big)^+\Big] ,\end{aligned}$$

i.e. $V(b)$ is the expected return on investment of the PC and $U(b)$ is the expected value of the accumulated debt account.

In order to proceed with our derivations and also for numerical calculations we will need to consider the density function of $\max_{0\le s\le t}\{\mu s+\sigma W_s\}$. This density, $g(y;t)$, is given by

$$g(y;t,\mu)=\frac{2}{\sqrt{2\pi t}\sigma}e^{-\frac{(y-\mu t)^2}{2\sigma^2 t}}-\frac{e^{\frac{2\mu y}{\sigma^2}}\mu}{\sigma^2}\mathrm{Erfc}\Big(\frac{y+\mu t}{\sqrt{2t}\sigma}\Big), \tag{3}$$

with $\mathrm{Erfc(x)}=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-z^2}\,\mathrm{d}z$, confer for instance Borodin & Salminen, p. 250 formula 1.1.4 for the distribution function of $\max_{0\le s\le 1}\{\mu s+\sigma W_s\}$.

For simplicity we write $g(y;\mu)$ if $t=1$.

The normalised loss (the spent amount exceeding the required payments of C_1-C_0, i.e. the real loss divided by C_1-C_0) is given by

$$L(b,\alpha):=\alpha-\alpha V(b)-1\to\min! \tag{4}$$

It is clear that $V(b)$ is strictly increasing in b, meaning that the maximum of $V(b)$, and the minimum of the loss, is attained at the maximal b, allowed by the credibility condition (2). However, this maximal b will again depend on α. Thus, we have to specify the set of admissible pairs (b,α) before we can solve the optimisation problem (4). Note first that the PC does not have an infinite amount of money on her/his disposal. Therefore, we have to restrict the set of admissible α to some reasonable values and introduce a liquidity restriction boundary $\alpha^*>0$ with $0<\alpha\le\alpha^*$.

Lemma 4.3
Assume p in (2) is fixed, $\tilde p$ is given by

$$\int_{\tilde p}^\infty g(y;t,\mu)\,\mathrm{d}y=p$$

and $\alpha^*<\infty$ is the liquidity restriction boundary.
If $\alpha^*<e^{1-\tilde p}$ then the PC should prefer to pay the required increase in contribution directly into the PAYG account.
If $\alpha^*\ge e^{1-\tilde p}$, the pair minimising the loss (4) is (b^*,α^*), where b^* is implicitly given by

$$\frac{1}{(1+b^*)(\tilde p-\ln(1+b^*))}=\alpha^*$$

Proof: Consider the credibility condition (2). Using the density (3), we get

$$\begin{aligned}\mathbb{P}\Big[D_t(b)\ge\frac{1}{\alpha}\Big]&=\mathbb{P}\Big[(1+b)\Big(\max_{0\le s\le t}\{\mu s+\sigma W_s\}-\ln(1+b)\Big)^+\ge\frac{1}{\alpha}\Big]\\&=\mathbb{P}\Big[\max_{0\le s\le t}\{\mu s+\sigma W_s\}\ge\frac{1}{(1+b)\alpha}+\ln(1+b)\Big]\\&=\int_{\ln(1+b)+\frac{1}{\alpha(1+b)}}^\infty g(y;t,\mu)\,\mathrm{d}y\ .\end{aligned}$$

Since $g(y; t, \mu) > 0$ for all $y \in \mathbb{R}_+$, for every $p \in [0, 1]$ there is a unique lower integral boundary $\tilde{p} \in \mathbb{R}_+$ such that

$$\int_{\tilde{p}}^{\infty} g(y; t, \mu)\, \mathrm{d}y = p \; .$$

Assume p in (2) is fixed and $\tilde{p}$ is the corresponding lower integral boundary. Then the set of admissible pairs (b, α) is described by the inequality

$$\tilde{p} \geq \ln(1 + b) + \frac{1}{\alpha(1 + b)} \; ,$$

which is equivalent to $\alpha \geq \frac{1}{(1+b)(\tilde{p}-\ln(1+b))} =: \Delta(b)$. Further, it holds

$$\Delta'(b) \begin{cases} \leq 0 : e^{\tilde{p}-1} - 1 \geq b, \\ > 0 : e^{\tilde{p}-1} - 1 < b, \end{cases}$$

i.e. $\Delta(b)$ attains its global minimum at $b = e^{\tilde{p}-1} - 1 > -1$ with $\Delta(e^{\tilde{p}-1} - 1) = e^{-\tilde{p}+1}$.

Since our target is to minimise the loss defined in (4), and it is done by the biggest allowed value of b, we can restrict our considerations to the set $b \geq e^{\tilde{p}-1} - 1$, the area where $\Delta(b)$ is strictly increasing.

Noting that $e^{\tilde{p}} - 1 > e^{\tilde{p}-1} - 1$ and $\Delta(e^{\tilde{p}} - 1) = \infty$, we define the set of allowed pairs (b, α) to be

$$e^{\tilde{p}-1} - 1 \leq b < e^{\tilde{p}} - 1 \;\text{ and }\; \alpha \geq \frac{1}{(1 + b)(\tilde{p} - \ln(1 + b))} \; .$$

Note that if $\alpha^* < \Delta(e^{\tilde{p}-1} - 1) = e^{-\tilde{p}+1}$, then the set of admissible pairs is empty.

If $\alpha^* \geq e^{-\tilde{p}+1}$, there is a unique $b^* \in [e^{\tilde{p}-1} - 1, e^{\tilde{p}} - 1)$ such that $\Delta(b^*) = \alpha^*$. And the set of admissible pairs shrinks to

$$e^{\tilde{p}-1} - 1 \leq b < b^* \;\text{ and }\; \frac{1}{(1 + b)(\tilde{p} - \ln(1 + b))} \leq \alpha \leq \alpha^* \; .$$

□

The above lemma ensures that if we require $\alpha^* \geq e^{1-\tilde{p}}$, the minimal loss will be attained at (b^*, α^*). If $L(b^*, \alpha^*)$ is bigger than zero, this is a clear indicator that the funding strategy is not working, and paying the increase in contribution immediately into the PAYG account is more preferable for the PC.

Note that if the accumulated amount exceeds the debt then the PC has an additional gain. Therefore, the entire expected normalised loss function L_e is defined as follows:

$$L_e(b, \alpha) := \alpha - \alpha V(b) - 1 - \alpha U(b) + 1 = \alpha - \alpha V(b) - \alpha U(b) \; .$$

Lemma 4.4
The function L_e is decreasing in b, if $\alpha > 0$.

Proof: W.l.o.g we assume $\alpha = 1$. Using (3), we get the following representation

$$
\begin{aligned}
V(b) + U(b) = e^{\mu+\frac{\sigma^2}{2}} &\int_0^{\ln(1+b)} g(y; \mu + \sigma^2)\,\mathrm{d}y \\
&+ (1+b)e^{\mu+\frac{\sigma^2}{2}} \int_{\ln(1+b)}^{\infty} e^{-y} g(y; \mu + \sigma^2)\,\mathrm{d}y \\
&+ (1+b) \int_{\ln(1+b)}^{\infty} (y - \ln(1+b)) g(y; \mu)\,\mathrm{d}y \,.
\end{aligned}
$$

Deriving with respect to b and using $e^z - 1 \geq z$ yields

$$
\begin{aligned}
V'(b) + U'(b) &= \mathbb{E}\Big[e^{\mu+\sigma W_1 - M_1} \mathbb{1}_{[M_1 > \ln(1+b)]}\Big] \\
&\quad + \mathbb{E}\Big[\Big(M_1 - \ln(1+b) - 1\Big) \mathbb{1}_{[M_1 > \ln(1+b)]}\Big] \\
&\geq \mathbb{E}\Big[\Big(\mu + \sigma W_1 - \ln(1+b)\Big) \mathbb{1}_{[M_1 > \ln(1+b)]}\Big] \,.
\end{aligned}
$$

In order to calculate the expectation in the last line above, we use the Markovian property of the Brownian motion W. Define additionally $\tau_b := \inf\{t \geq 0 : \mu t + \sigma W_t = \ln(1+b)\}$, then

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(\mu + \sigma W_1 - \ln(1+b)\Big) \mathbb{1}_{[M_1 > \ln(1+b)]}\Big] \\
&\quad = \mathbb{E}\Big[\Big(\mu\tau_b + \sigma W_\tau - \ln(1+b) + \mu(1-\tau_b) + \sigma\big(W_1 - W_{\tau_b}\big)\Big) \mathbb{1}_{[\tau_b < 1]}\Big] \\
&\quad = \mathbb{E}\Big[\Big(\mu(1-\tau_b) + \sigma \tilde{W}_{1-\tau_b}\Big) \mathbb{1}_{[\tau_b < 1]}\Big] = \mu\mathbb{E}[(1-\tau_b)\mathbb{1}_{[\tau_b < 1]}] > 0 \,,
\end{aligned}
$$

where $\tilde{W}$ is an independent copy of W. □

This means in particular that L_e attains its minimum at the biggest admissible b. Of course, one might minimise the function L_e instead of L and redefine the set of admissible barriers in (2) accordingly. However, we keep the credibility condition (2) and consider the additional (positive) expectation $\alpha U(b) - 1$ as a buffer similar to the net profit condition in risk theory, confer for instance [10, p. 130], where the expected premia should be strictly bigger than the expected loss in order to avoid the almost sure ruin. In our model, this means that we keep $\alpha\mathbb{E}[D_1(b)] > C_1 - C_0$ in order to make the repayment of the debt more probable even if the probability p in (2) is relatively small.

Since it does not make sense for the PC to prefer the funding scheme to paying the increase in contribution directly if the expected loss is positive, we formalise the following (normalised by dividing through $C_1 - C_0$) profitability condition

$$\alpha - \alpha V(b) = \alpha - \alpha \mathbb{E}\left[e^{\mu+\sigma W_1 - \left(\max\limits_{0\le s\le 1}\{\mu s+\sigma W_s\}-\ln(1+b)\right)^+}\right] < 1\,. \tag{5}$$

The above condition means that the normalised loss of the PC $\alpha - \alpha V(b)$ should be smaller than the amount of 1 that could have been paid directly into the PAYG. Recall that in order to create a buffer for the state in the sense that the debt will be fully repaid in expectation, we do not take into account the possible gain if the debt account exceeds the $C_1 - C_0$.
In order to rewrite the above conditions in terms of integrals, we introduce a new measure Q by the following Radon-Nicodym density $\frac{\mathrm{d}Q}{\mathrm{d}\mathbb{P}} = e^{\sigma W_1 - \frac{\sigma^2}{2}}$ and $\tilde{W}_s = W_s - \sigma s$ a standard Brownian motion under Q. Setting for simplicity $M_1 := \max\limits_{0\le s\le 1}\{\mu s + \sigma W_s\}$, we can rewrite the expected return on investment for the PC as follows using the density introduced in (3)

$$\begin{aligned} V(b) &= \mathbb{E}\Big[e^{\mu+\sigma W_1} e^{-(M_1-\ln(1+b))^+}\Big] = e^{\mu+\frac{\sigma^2}{2}}\mathbb{E}_Q\Big[e^{-(M_1-\ln(1+b))^+}\Big] \\ &= e^{\mu+\frac{\sigma^2}{2}}\Big\{\int_0^{\ln(1+b)} g(y;\mu+\sigma^2)\,\mathrm{d}y + (1+b)\int_{\ln(1+b)}^{\infty} e^{-y} g(y;\mu+\sigma^2)\,\mathrm{d}y\Big\}\,. \end{aligned}$$

Remark 4.5
Consider the derivatives of $V(b)$:

$$\begin{aligned} V'(b) &= e^{\mu+\frac{\sigma^2}{2}}\int_{\ln(1+b)}^{\infty} e^{-y} g(y;\mu+\sigma^2)\,\mathrm{d}y \\ &= \mathbb{E}\Big[e^{\mu+\sigma W_1 - \max\limits_{0\le s\le 1}\{\mu s+\sigma W_s\}}\,\mathbb{I}_{[\max\limits_{0\le s\le 1}\{\mu s+\sigma W_s\}\ge \ln(1+b)]}\Big] > 0, \\ V''(b) &= -\frac{e^{\mu+\frac{\sigma^2}{2}}}{(1+b)^2}\cdot g(\ln(1+b);\mu+\sigma^2) < 0\,. \end{aligned}$$

By maximising just the expected value, we do not take into account the variance and consequently the risk for return on investment to stay considerably below the expected value. One might consider the following target functional instead of $V(b)$.

$$\tilde{V}(b) := \mathbb{E}\Big[e^{\mu+\sigma W_1-(M_1-\ln(1+b))^+}\Big] - \lambda\mathbb{E}\Big[e^{2\mu+2\sigma W_1 - 2(M_1-\ln(1+b))^+}\Big] \to \max!$$

with some weight $\lambda > 0$. Using change of measure technique mentioned

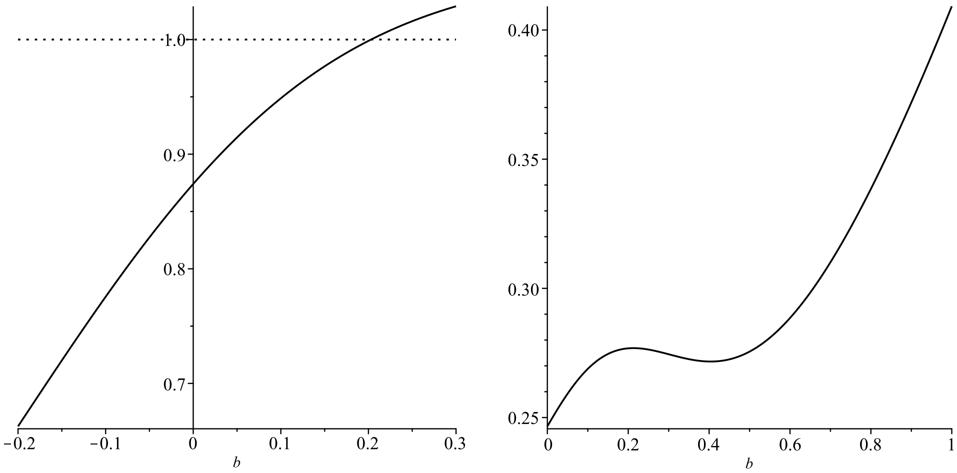

Figure 2: Expected **1-year** return on investment $V(b)$ and modified functional $\tilde{V}(b)$ with $\lambda = 0.85$.

above, the derivative fulfils

$$\begin{aligned}\tilde{V}'(b) &= \mathbb{E}\Big[\Big\{e^{\mu+\sigma W_1 - M_1} - 2\lambda(1+b)e^{2\mu+2\sigma W_1 - 2M_1}\Big\}\mathbb{1}_{[M_1 > \ln(1+b)]}\Big] \\ &= e^{\mu+\frac{\sigma^2}{2}} \int_{\ln(1+b)}^{\infty} e^{-y} g(y; \mu + \sigma^2)\,\mathrm{d}y \\ &\quad - 2\lambda(1+b)e^{2\mu+2\sigma^2} \int_{\ln(1+b)}^{\infty} e^{-2y} g(y; \mu + 2\sigma^2)\,\mathrm{d}y\end{aligned}$$

Since $\mu + \sigma W_1 \leq M_1$ a.s. we can conclude that $e^{\mu+\sigma W_1 - M_1} \geq e^{2\mu+2\sigma W_1 - 2M_1}$ a.s., meaning that for $2\lambda(1+b) \leq 1$ the first derivative $\tilde{V}'(b)$ stays positive, meaning that the value $\tilde{V}(b)$ is increasing. However, the global behaviour of $\tilde{V}'(b)$ is highly sensitive to the choice of λ - a variable which cannot be clearly justified from the economical point of view and creates in this way another source of uncertainty and an opportunity for manipulations.
Also, one should not forget that the derivative $\tilde{V}'(b)$ can not be considered globally, but just on the interval allowed by the credibility condition (2) and the boundary α^* defined in Remark (4.3). In Figure 2 we see the functions $V(b)$ (left) and $\tilde{V}(b)$ (right) for $\mu = 0.04$, $\sigma = 0.2$ and $\lambda = 0.85$. Depending on the b^* the maximum of $\tilde{V}(b)$ will be attained at different values of b. Also, the curve $\tilde{V}(b)$ changes its form depending on λ. ■

In the examples below we demonstrate how the credibility and the profitability conditions work for a fund with realistic parameters.

Example 4.6
Assume like above $\mu = 0.04$, $\sigma = 0.2$ and $\alpha^* = 10$.

In the table below, we compute several values of $\mathbb{P}\big[D_1(b) \geq \frac{1}{\alpha}\big]$ for different pairs (b, α).

α / b	1	2	3	4	5
-0.2	$7.7 \cdot 10^{-7}$	0.06553328	0.40025361	0.71208305	0.91564864
-0.1	$1.32 \cdot 10^{-6}$	0.03766247	0.23837807	0.45738957	0.62378926
-0.05	$1.48 \cdot 10^{-6}$	0.02776680	0.17865039	0.35389069	0.49496449
0	$1.53 \cdot 10^{-6}$	0.02014832	0.13156264	0.26808831	0.38351917
0.05	$1.49 \cdot 10^{-6}$	0.01441359	0.09534295	0.19919441	0.29075048

Table 7: $\mathbb{P}\big[D_1(b) \geq \frac{1}{\alpha}\big]$ for different values of α and b for 1 year time horizon.

We see that if the state requires a relatively high p, the values of α are high and b becomes negative.

• *Choose now $p = 0.7$, i.e. the credibility condition is $\mathbb{P}\big[D_1(b) \geq \frac{1}{\alpha}\big] \geq 0.7$. Then: $\tilde{p} = 0.093078333$, α^* should be bigger than 2.4766867, the biggest possible b leading to $\alpha = \infty$ equals 0.0975477 and $b^* = -0.00768$. It holds*

$$L(b^*, \alpha^*) = \alpha^* - \alpha^* V(b^*) - 1 = 0.327634 > 0 \ .$$

Thus, if the state requires $p = 0.7$, the PC should prefer to pay the increase in contribution $C_1 - C_0$ into the PAYG her-/himself and not to use the funding possibility.

• *If the state requires $\mathbb{P}\big[D_1(b) \geq \frac{1}{\alpha}\big] \geq 0.5$ then $\tilde{p} = 0.15750112$. The minimal possible α^* preventing the set of admissible pairs (b, α) to be non-empty equals $e^{-\tilde{p}+1} = 2.3221625$. The biggest possible b leading to $\alpha = \infty$ equals 0.1705821 and the maximal admissible b corresponding to $\alpha^* = 10$ is given by $b^* = 0.06574$. The expected loss is then*

$$L(b^*, \alpha^*) = -0.2603 < 0 \ .$$

What does this result mean for the PC? The function $V(b)$ (expected return on investment for the PC) is given in Figure 2, left picture. If $b < 0.2030$ then the expected return on investment will be smaller than 100%, meaning that in expectation the PC will not get her/his full investments back. However, the primary target of adding a funding component is to reduce the increase in contribution $C_1 - C_0$ and not to entirely avoid any payments. Below we demonstrate how this might work in practice.
In Germany, it is planned to increase the monthly contribution from 18.6% to 19.5% starting from the year 2024. Assume, for an average contributor the monthly increase will approximately amount to 20 Euro, i.e. 240 Euro per year. Thus, investing $240 \cdot \alpha^* = 2,400$ Euro into a fund with parameters given like above, the PC will pay the debt of 240 Euro with probability

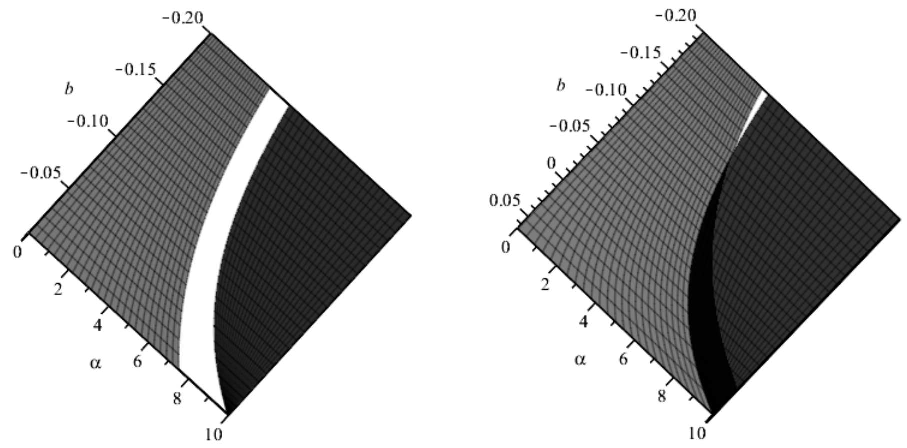

Figure 3: Sets of (b, α) fulfilling the credibility (2) (dark grey areas) and profitability (5) (light grey areas) conditions for $p = 0.7$ (left) and $p = 0.5$ (right).

50% and, compared to the direct payment to the PAYG, have a gain of $240 \cdot 0.2603 \approx 62.5$ *Euro. This means, the PC will pay in expectation* $240 - 62.5 = 177.5$ *instead of* 240 *Euro per year. The expected loss of the state is then given by*

$$(C_1 - C_0) - (C_1 - C_0)\alpha^* \cdot U(b^*) = -75.6,$$

meaning that in expectation the PC gets back an additional amount of 75.6. *Thus, the PC has an expected gain of* $62.5 + 75.6 = 138.1$ *compared to paying the increase of contribution immediately. And the state gets the debt in expectation fully back.*

In Figure 3 we plotted the sets of (b, α) *fulfilling the credibility* (2) *and the profitability* (5) *conditions.*
On the left picture of Figure 3 we see the area with (b, α) *such that* $V(b, \alpha) \geq 0.7$ *(dark grey) and the area where* $L(b, \alpha) \leq 0$ *(light grey). It is obvious that these two sets are disjoint, meaning that there is no combination of* (b, α) *for* $\alpha^* = 10$ *such that the debt is repaid with at least probability* 70% *and simultaneously the PC's loss is less than* $C_1 - C_0$. *In the right picture in Figure 3, we see that the set of pairs* (b, α) *fulfilling the credibility condition with* $p = 0.5$ *(dark grey) and simultaneously the profitability condition (light grey) is given by the narrow black area lying between the dark and light grey ones.* ■

4.1 A 10-years investment period

It is clear that investing for a longer period brings higher returns in expectation. Therefore, in this section, the state still pays the increase in

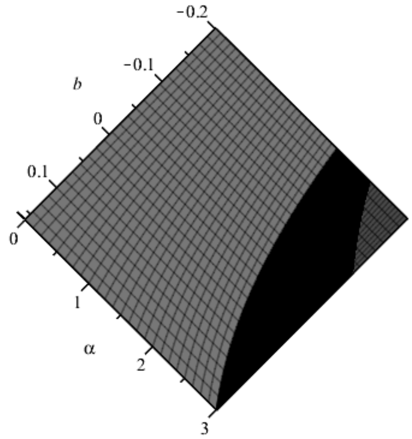

Figure 4: Sets of (b, α) fulfilling the credibility (2) (dark grey area) and profitability (5) (light grey area) conditions for **10 years** and $p = 0.7$.

contributions, but requires its money back after a period of 10 years. The PC invests some amount of money for 10 years in a pre-specified fund. At the end of the investment period, the PC has to repay the debt and will hopefully get some gain.

The invested amount should be a multiple, α, from the total anticipated increase of contributions over the next ten years. Considering the numbers given in Example 4.6, the forecast of increase in contribution over the next 10 years amounts to $2,400$ Euro. Like in the previous section, we require the credibility condition (2) and the profitability condition (5) while the time interval changes from 1 to 10.

Example 4.7

Assume again $\mu = 0.04$ and $\sigma = 0.2$. In the Table 8 below we calculate $\mathbb{P}[D_{10}(b) \geq \frac{1}{\alpha}]$ for different pars of (b, α). The difference to a 1 year investment is considerable. For instance, in order to keep the probability of repayment above 70% it suffices to set $\alpha = 3$ and $b = 0.05$. Therefore, we set $\alpha^* = 3$, i.e. the highest possible α the state is willing to adopt is equal to 3.

b \ α	1	2	3	4	5
−0.2	0.2546296	0.7276311	0.8842143	0.9508487	0.9860533
−0.1	0.2671423	0.6884180	0.8334411	0.8984640	0.9341821
−0.05	0.2698034	0.6680602	0.8076430	0.8715453	0.9071802
0	0.2706137	0.6474819	0.7817909	0.8444029	0.8797703
0.05	0.2699046	0.6268430	0.7560225	0.8172035	0.8521476

Table 8: $\mathbb{P}\Big[D_{10}(b) \geq \frac{1}{\alpha}\Big]$ for different values of α and b for 10 years time horizon.

Figure 4 showcases the pairs (b, α) allowed by the credibility condition (2) (dark grey area), the pairs (b, α) where the profitability condition is fulfilled (light grey area) and the intersection area (black). It is clear that the 10-year investment should be preferred if the financial situation allows. ■

5 Comparison of the Two Investment Types

This section compares the two strategies introduced above in Sections 3 and 4: continuous withdrawal and a lump sum repayment at the end of the period.

For that purpose, we look at the loss functions corresponding to each strategy in dependence on time. Let again α^* be the liquidity restriction. Define

$$L_d(t) := \alpha^* - \alpha^* \mathbb{E}[e^{\mu t + \sigma W_t}] + 1,$$

$$L_c(t,b) := \alpha^* - \alpha^* \mathbb{E}\big[e^{\mu t + \sigma W_t - \left(\max\limits_{0 \le s \le t} \{\mu s + \sigma W_s\} - \ln(1+b)\right)^+}\big] - 1 \; .$$

i.e. L_d is the normalised loss of the PC in the model with the lump sum debt repayment and L_c is the normalised loss in the continuous withdrawal model. Both functions depend now on the time interval under consideration. Let further

$$\begin{aligned}\Lambda(t,b) &:= L_c(t,b) - L_d(t) \\ &= \alpha^* \mathbb{E}[e^{\mu t + \sigma W_t}] - \alpha^* \mathbb{E}\big[e^{\mu t + \sigma W_t - \left(\max\limits_{0 \le s \le t} \{\mu s + \sigma W_s\} - \ln(1+b)\right)^+}\big] - 2 \; .\end{aligned}$$

It is easy to see that Λ is strictly increasing in t and strictly decreasing in b. Because

$$\Lambda(0,b) = \begin{cases} -2 & : \; b \ge 0 \\ -\alpha^* b - 2 & : \; b < 0, \end{cases}$$

and $\lim\limits_{t \to \infty} \Lambda(t,b) = \infty$ we conclude that for every b with $b \ge -\frac{2}{\alpha^*}$ there is an t such that $L(t,b) = 0$. For $b < -\frac{2}{\alpha^*}$ the function Λ stays positive meaning that the continuous withdrawal strategy is definitely not optimal.

Thus, for $b \ge -\frac{2}{\alpha^*}$ we can conclude that by implicit function theorem there is a curve $\beta : [0, \infty) \to [-\frac{2}{\alpha^*} \vee -1, 1]$, $\beta' > 0$ such that $\Lambda(t, \beta(t)) \equiv 0$, $\Lambda(t,b) > 0$ for $b < \beta(t)$ and $\Lambda(t,b) < 0$ for $b > \beta(t)$.

The choice of the investment strategy will depend on the parameters of the underlying fund, the liquidity restriction α^*, the investment time horizon t and credibility condition (2):

- If $\Lambda(t, b^*) > 0$, choose the lump sum repayment strategy;
- If $\Lambda(t, b^*) \le 0$ choose the continuous withdrawal strategy;
- If $L_d(t)$,$L_c(t, b^*) \ge 0$ - pay directly into the PAYG system.

Assuming again $\mu = 0.04$, $\sigma = 0.2$, $\alpha^* = 10$, $t = 1$ and $p = 0.5$ we get $b^* = 0.06574$, confer Example 4.6. It holds that

$$\begin{aligned} L_d(1) &= 10 - 10e^{\mu+\frac{\sigma^2}{2}} + 1 = 0.3816 > 0, \\ L_c(1, b^*) &= -0.2603 < 0 \ . \end{aligned}$$

Therefore, for one year time horizon and $\alpha^* = 10$ one should prefer the continuous withdrawal.
For the time horizon of 10 years one gets for the same parameters $b^* = 0.8707$

$$\begin{aligned} L_d(10) &= 10 - 10e^{(\mu+\frac{\sigma^2}{2})10} + 1 = -7.2211 < 0, \\ L_c(10, b^*) &= -3.5291 < 0, \\ \Lambda(10, b^*) &= 3.6921 > 0, \end{aligned}$$

meaning that one should definitely prefer the lump sum repayment strategy. This can be explained by the fact that by withdrawing money from the investment we miss possible gains. Even taking into account the amount on the debt account exceeding the actual debt will not make the continuous withdrawal more attractive:

$$L_c(10, b^*) - \alpha^* \mathbb{E}[D_{10}(b^*)] + 1 = -7.088 > -7.2211 \ .$$

In the table below we sum up the optimal choice of a strategy for different values of α and time horizons t in years. Let C denote the continuous withdrawal and LS the lump sum debt repayment strategy. If it is optimal not to use any of the funding strategies but just to pay the increase in contribution into the PAYG, we write PAYG.

t \ α	1	2	3	4	5	6	7	8	9	10
1	PAYG	PAYG	PAYG	C	C	C	C	C	C	C
2	PAYG	PAYG	C	C	C	C	C	C	C	C
4	PAYG	C	C	C	C	C	C	LS	LS	LS
6	PAYG	C	C	C	LS	LS	LS	LS	LS	LS
8	PAYG	C	C	LS	LS	LS	LS	LS	LS	LS
10	PAYG	C	LS	LS	LS	LS	LS	LS	LS	LS
20	LS	LS	LS	LS	LS	LS	LS	LS	LS	LS
40	LS	LS	LS	LS	LS	LS	LS	LS	LS	LS

Table 9: The optimal strategy for different values of α and time horizons t.

The probability of default for the state, i.e. the probability that the state will not get the full debt back, equals 50% by definition of the optimal b^*. For the discrete repayment case we sum up the results in Table (10) below for the parameters given above. For each pair (t, α) we calculate the probability of default $\mathbb{P}[\alpha e^{\mu t+\sigma W_t} < 1+\alpha]$, meaning that the state will get less than $C_1 - C_0$ after the PC gets his invested money back.

t \ α	1	2	3	4	5	6	7	8	9	10
1	0.99	0.96	0.89	0.82	0.76	0.72	0.68	0.65	0.63	0.61
2	0.98	0.87	0.77	0.69	0.64	0.60	0.58	0.55	0.54	0.52
4	0.91	0.73	0.63	0.56	0.52	0.49	0.47	0.46	0.45	0.44
6	0.82	0.63	0.54	0.49	0.45	0.43	0.41	0.40	0.39	0.38
8	0.75	0.56	0.48	0.43	0.40	0.38	0.37	0.36	0.35	0.34
10	0.68	0.50	0.43	0.39	0.37	0.35	0.34	0.33	0.32	0.31
20	0.45	0.33	0.28	0.26	0.25	0.24	0.23	0.22	0.21	0.21
40	0.24	0.17	0.15	0.14	0.13	0.13	0.12	0.12	0.12	0.12

Table 10: $\mathbb{P}[\alpha e^{\mu t+\sigma W_t} < 1+\alpha]$ for different values of α and time horizons t.

Again, in Table 10 we see that for short-term periods it is more profitable to use the continuous withdrawal strategy or PAYG.

6 Conclusions

In the last decade, most OECD countries have enacted pension reforms of their traditional defined-benefit pay-as-you-go (PAYG) schemes. PAYG requires the balance between income from contributions and pension expenditure where the current contributions finance the current pensions. The most common reforms have been the changes in the level of benefits (sometimes linked to a longevity index, such as the life expectancy) and increases in the retirement age.

At the same time, there are some countries that combine the PAYG scheme and a defined contribution funding part within the mandatory pension system. These systems have been advocated, particularly by the World Bank, as a practical way to the higher financial market returns with the cost of a scheme with a greater funded component. In this line, the present paper proposes an alternative with the contributors investing an extra amount of money into a fund, so that part of the investment together with the returns can restore the financial sustainability of the PAYG scheme. The contributor faces the investment risk while the state bears the risk that the returns on investment do not cover the deficit in contributions for the period analysed However, in the case of extra returns after debt repayment, the contributor keeps the gains. On the other hand, the loss of the state, in the worst case, will amount to the increase in contribution needed to restore the financial sustainability of the system while the contributor would normally invest and risk a much bigger multiple of this amount.

Two different debt repayment types depending on the amount invested and the timing of the repayment to the state are analysed for a prototypical contributor. The first set of strategies features the repayment of the debt at the end of a pre-specified period as a lump sum. In several examples, we calculate the probability of full payback of the debt, expected loss of the state and the expected value of the gains for the individual. In particular, we compare, for different invested amounts, the case of the repayment after a year versus a repayment after 10 years. As expected, the loss probability decreases when the amount invested and the investment horizon increase.
As an optimisation approach, we study the case when the state requires the individual to transfer any excess above a particular level (barrier) of return – to be optimised – continuously in time as a debt repayment (second type of strategies). Comparing different barriers, we show that the optimal strategy is the biggest possible barrier such that the debt to the state is repaid with a certain pre-specified probability. In an example, we compare the continuous withdrawal and the lump sum repayment strategies with the possibility to pay the increase in contribution directly into the PAYG system. If the period for the repayment is long enough, the optimal strategy tends to be a lump sum debt repayment. Directly paying into the PAYG is the optimal strategy if the investment period is short and the amount invested is relatively small.
The model presented in this paper could be implemented as an alternative to both the PAYG and mixed pension systems as we are not advocating a particular strategy but rather offer possibilities allowing to reflect the actual market and societal situation. Additionally, our model could raise the public awareness of the financial sustainability of the PAYG and promote interest in pension basic principles as the expected gains for the individuals depend on demographic, economic and financial factors.[4]
Finally, based on the methodology presented in this paper, at least three important directions for future research can be identified. First, it would be interesting to explore the smoothed and affordable contributions to be invested by the individual to make the model more applicable in the real world. Another direction would be to study the risk sharing between the government and the individuals under different scenarios by using for instance Value at Risk and Expected Shortfall risk measures. Finally, one can analyse a possible structure and asset allocation problems of the fund that might be used in the real life.

[4]Currently, there is empirical evidence that most individuals have very limited understanding about the core elements of social insurance systems and on the key variables that define the amount of their pensions (Mitchell [16], Lusardi and Mitchell [14], Bucher-Koenen and Lusardi [5] amongst others).

Acknowledgements

1. María del Carmen Boado-Penas is grateful for the financial assistance received from Spanish Government and European Commission [project MINECO-FEDER ECO2015-65826-P].

2. The research of Julia Eisenberg was funded by the Austrian Science Fund (FWF), Project number V 603-N35.

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