A Variant of Strassen’s Theorem with an Application to the Consistency of Option Prices

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Let \((M_n)_{n \in \mathbb{N}}\) be a martingale and \(\phi : \mathbb{R} \to \mathbb{R}\) convex. Then by Jensen’s inequality we have that

\[
\mathbb{E}[\phi(M_s)] \leq \mathbb{E}[\phi(M_t)], \quad s \leq t,
\]

\[
\int_{\mathbb{R}} \phi(x) \, d\mu_s(x) \leq \int_{\mathbb{R}} \phi(x) \, d\mu_t(x), \quad s \leq t.
\]

Let \(\mu_1\) and \(\mu_2\) be two probability measures on \(\mathbb{R}\) with finite mean \((\mathcal{M})\). Then \(\mu_1\) is smaller in convex order than \(\mu_2\) \((\mu_1 \leq_c \mu_2)\) if

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\int_{\mathbb{R}} \phi(x) \, d\mu_1(x) \leq \int_{\mathbb{R}} \phi(x) \, d\mu_2(x),
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for all convex functions \(\phi : \mathbb{R} \to \mathbb{R}\).
Let \((M_n)_{n \in \mathbb{N}}\) be a martingale and \(\phi : \mathbb{R} \to \mathbb{R}\) convex. Then by Jensen’s inequality we have that

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Strassen’s Theorem

Strassen’s Theorem, 1965

Let \((\mu_n)_{n \in \mathbb{N}}\) be a sequence in \(\mathcal{M}\). Then there exists a martingale \((M_n)_{n \in \mathbb{N}}\) such that \(M_n \sim \mu_n\) if and only if \(\mu_s \leq_c \mu_t\) for all \(s \leq t\).

Lemma

Let \((\mu_n)_{n \in \mathbb{N}}\) be a sequence in \(\mathcal{M}\) and define the call function of \(\mu_n\) as

\[
R_{\mu_n}(x) = \int_{\mathbb{R}} (y - x)^+ \mu_n(dy), \quad x \in \mathbb{R}.
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Then \(\mu_s \leq_c \mu_t\) for all \(s \leq t\) if and only if \((\mu_n)_{n \in \mathbb{N}}\) has constant mean and

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Application - Classical Problem

- Given a finite set of European call option prices $r_{t,i}$, with maturity $t \in \{1, \ldots, T\}$ and strike $K_i \in \{K_1, \ldots, K_N\}$ and given the price of the underlying asset $S_0$, when does there exist an arbitrage-free model which generates these prices?

- A model is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a non-negative martingale $S$ such that

$$\mathbb{E}[(S_t - K_i)^+] = r_{t,i}.$$  

Literature:

- Carr and Madan (2005) → necessary and sufficient conditions
- Davis and Hobson (2007) → arbitrage strategies
- Cousot (2007) → positive bid-ask spread on options (but not on the underlying).
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Conditions for single maturities

- For each maturity $t$ the linear interpolation $L_t$ of the points $(K_i, r_{t,i})$ has to be convex, decreasing and all slopes of $L_t$ have to be in $[-1, 0]$.
- Intuition: for every random variable $S_t$ the function $K \mapsto \mathbb{E}[(S_t - K)^+]$ has these properties.

![Graph showing stock price, option prices, and linear interpolation $L_t$ with slopes in $[-1, 0]$]
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Intertemporal conditions

- For all strikes $K_i$ we have that $r_{t,i} \leq r_{t+1,i}$.
- Intuition: for every martingale $S = (S_t)_{t \in \{0,\ldots,T\}}$ the function $t \mapsto \mathbb{E}[(S_t - K)^+]$ is increasing by Strassen's theorem.
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Necessary and Sufficient Conditions

- For all maturities $t$

\[
0 \geq \frac{r_{t,i+1} - r_{t,i}}{K_{i+1} - K_i} \geq \frac{r_{t,i} - r_{t,i-1}}{K_i - K_{i-1}} \geq -1, \quad \text{for } i \in \{1, \ldots, N - 1\},
\]

and

\[
r_{t,i} = r_{t,i-1} \text{ implies } r_{t,i} = 0, \quad \text{for } i \in \{1, \ldots, N\}.
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- Note that we set $K_0 = 0$ and $r_{t,0} = S_0$ for all $t \in \{1, \ldots, T - 1\}$.

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- It is possible to state arbitrage strategies if any of these conditions fails.
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- It is possible to state arbitrage strategies if any of these conditions fails.
**Application - New Problem**

- Additional to the classical Problem assume that there is a **positive bid-ask spread** on the underlying \((S_t \leq \bar{S}_t)\).

- What is the payoff of a European call option at maturity \(t\)?

\[
\text{Is it } (\bar{S}_t - K)^+ \text{? or } (S_t - K)^+ ?
\]

- We assume that there is a third process \((S^C_t)_{t\in\{0,...,T\}}\) such that \(S_t \leq S^C_t \leq \bar{S}_t\) and such that the payoff is given by

\[
(S^C_t - K)^+.
\]

Options are cash-settled.

- An arbitrage-free model is a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and four non-negative processes:

\[
\underline{S}, \bar{S}, S^C, S^*.
\]

- \(S^*\) is a martingale which evolves in the bid-ask spread: \(\underline{S}_t \leq S^*_t \leq \bar{S}_t\).

- \(S^C\) is not a traded asset, hence \(S^C\) does not have to be a martingale.
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- Additional to the classical Problem assume that there is a positive bid-ask spread on the underlying ($S_t \leq S$).
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Unbounded Bid-Ask Spread

- If we allow models where the bid ask can get arbitrarily large than there are no intertemporal conditions.
- For all maturities $t$ the following conditions are then necessary and sufficient for the existence of arbitrage-free models:
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  0 \geq \frac{r_{t,i+1} - r_{t,i}}{K_{i+1} - K_i} \geq \frac{r_{t,i} - r_{t,i-1}}{K_i - K_{i-1}} \geq -1, \quad \text{for } i \in \{2, \ldots, N-1\},
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Bounded Bid-Ask Spreads

- We focus on models where the bid-ask spread is bounded: there has to exist $\epsilon \geq 0$ and $p \in [0, 1]$ such that

$$\mathbb{P}(\overline{S}_t - \underline{S}_t > \epsilon) \leq p.$$ 

- In particular, $\mathbb{P}(|S^C_t - S^*_t| > \epsilon) \leq p$.
- The option prices allow us to construct measures which correspond to the law of $S^C$ (temporal conditions).
- Strassen’s theorem is not applicable anymore since $S^C$ does not have to be a martingale.
But, $S^C$ has to be close to a martingale.
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Problem Formulation

Let $d$ be a metric on $\mathcal{M}$ and $\epsilon > 0$.

Formulation 1

Given a sequence $(\mu_n)_{n \in \mathbb{N}}$ in $\mathcal{M}$, when does there exist a martingale $(M_n)_{n \in \mathbb{N}}$ such that

$$d(\mu_n, \mathcal{L}M_n) \leq \epsilon, \quad \text{for all } n \in \mathbb{N}?$$

Formulation 2

Given a sequence $(\mu_n)_{n \in \mathbb{N}}$ in $\mathcal{M}$, when does there exist a sequence $(\nu_n)_{n \in \mathbb{N}}$ which is increasing in convex order (peacock) such that

$$d(\mu_n, \nu_n) \leq \epsilon, \quad \text{for all } n \in \mathbb{N}?$$

We want to solve this problem for different $d$:
- Infinity Wasserstein distance
- Modified Prokhorov distance
- Prokhorov distance, Lévy distance, modified Lévy distance, Stop-Loss distance, ...
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Infinity Wasserstein distance

- The modified Prokhorov distance with parameter $p \in [0, 1]$ is the mapping $d_p^P : \mathcal{M} \times \mathcal{M} \to [0, \infty]$, defined by

$$d_p^P(\mu, \nu) := \inf \left\{ h > 0 : \nu(A) \leq \mu(A^h) + p, \text{ for all closed sets } A \subseteq \mathbb{R} \right\}$$

where $A^h = \left\{ x \in S : \inf_{a \in A} |x - a| \leq h \right\}$.

- The modified Prokhorov distance is not a metric in general!

- The infinity Wasserstein distance $W^\infty$ is defined by

$$W^\infty(\mu, \nu) = d_0^P(\mu, \nu).$$
Infinity Wasserstein distance

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  $$
Minimal distance coupling

**Theorem (Strassen 1965, Dudley 1968)**

Given measures $\mu, \nu$ on $\mathbb{R}$, $p \in [0, 1]$, and $\epsilon > 0$ there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with random variables $X \sim \mu$ and $Y \sim \nu$ such that

$$\mathbb{P}(|X - Y| > \epsilon) \leq p,$$

if and only if

$$d_p^\mathbb{P}(\mu, \nu) \leq \epsilon.$$

This is exactly what we need: we are interested in models where

$$\mathbb{P}(|S_t^C - S_t^*| > \epsilon) \leq p.$$
Minimal distance coupling

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First answer

Question

Given a sequence \((\mu_n)_{n \in \mathbb{N}}\) in \(\mathcal{M}\), \(p \in [0, 1]\) and \(\epsilon > 0\) when does there exist a peacock \((\nu_n)_{n \in \mathbb{N}}\) such that

\[ d_p^P(\mu_n, \nu_n) \leq \epsilon, \quad \text{for all } n \in \mathbb{N}? \]

- **Answer:** If \(p > 0\), then there always exists such a peacock!
- **Conclusion:** if we allow models where \(P(\bar{S}_t - S_t > \epsilon) \leq p\), for \(p \in (0, 1]\).

Then for all maturities \(t\) the following conditions are necessary and sufficient for the existence of arbitrage-free models:

\[
0 \geq \frac{r_{t,i+1} - r_{t,i}}{K_{i+1} - K_i} \geq \frac{r_{t,i} - r_{t,i-1}}{K_i - K_{i-1}} \geq -1, \quad \text{for } i \in \{2, \ldots, N - 1\},
\]

and

\[ r_{t,i} = r_{t,i-1} \text{ implies } r_{t,i} = 0, \quad \text{for } i \in \{2, \ldots, N\}. \]
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- **Conclusion:** if we allow models where \(\mathbb{P}(S^*_t - S_t > \epsilon) \leq p\), for \(p \in (0, 1]\). Then for all maturities \(t\) the following conditions are necessary and sufficient for the existence of arbitrage-free models:

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Solution for $W^\infty (\rho = 0)$, Part 1

- Let $B^\infty (\mu , \epsilon )$ be the closed ball wrt. $W^\infty$ with center $\mu$ and radius $\epsilon$. Let $\mathcal{M}_m$ be the set of all probability measures on $\mathbb{R}$ with mean $m \in \mathbb{R}$.

- Given $\epsilon > 0$, a measure $\mu \in \mathcal{M}$ and $m \in \mathbb{R}$ such that $B^\infty (\mu , \epsilon ) \cap \mathcal{M}_m \neq \emptyset$ there exist unique measures $S(\mu ) , T(\mu ) \in B^\infty (\mu , \epsilon ) \cap \mathcal{M}_m$ such that

$$S(\mu ) \leq c \nu \leq c T(\mu ) \quad \text{for all } \nu \in B^\infty (\mu , \epsilon ) \cap \mathcal{M}_m.$$ 

- The call functions of $S(\mu )$ and $T(\mu )$ are given by

$$R^\min_{\mu}(x; m, \epsilon ) = R_{S(\mu )}(x) = \left( m + R_{\mu}(x - \epsilon ) - (\mathbb{E}_{\mu} + \epsilon ) \right) \vee R_{\mu}(x + \epsilon ),$$

$$R^\max_{\mu}(x; m, \epsilon ) = R_{T(\mu )}(x) = \conv \left( m + R_{\mu}(\cdot + \epsilon ) - (\mathbb{E}_{\mu} - \epsilon ), R_{\mu}(\cdot - \epsilon ) \right)(x).$$
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$$R^\min_{\mu}(x; m, \varepsilon) = R_{S(\mu)}(x) = \left( m + R_{\mu}(x - \varepsilon) - (\mathbb{E} \mu + \varepsilon) \right) \lor R_{\mu}(x + \varepsilon),$$

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Solution for $W^\infty (p = 0)$, Part 2

Question

Given a sequence $(\mu_n)_{n \in \mathbb{N}}$ in $\mathcal{M}$ and $\epsilon > 0$ when does there exist a peacock $(\nu_n)_{n \in \mathbb{N}}$ such that

$$W^\infty (\mu_n, \nu_n) = d^P_0 (\mu, \nu) \leq \epsilon,$$

for all $n \in \mathbb{N}$?

Answer: if and only if

$$I := \bigcap_{n \in \mathbb{N}} \left[ \mathbb{E} \mu_n - \epsilon, \mathbb{E} \mu_n + \epsilon \right] \neq \emptyset,$$

and there exists $m \in I$ such that for all $N \in \mathbb{N}$, $x_1, \ldots, x_N \in \mathbb{R}$, we have

$$R_{\mu_1}^{\text{min}} (x_1; m, \epsilon) + \sum_{n=2}^{N} \left( R_{\mu_n} (x_n + \epsilon \sigma_n) - R_{\mu_n} (x_{n-1} + \epsilon \sigma_n) \right) \leq R_{\mu_{N+1}}^{\text{max}} (x_N; m, \epsilon),$$

where $\sigma_n = \text{sgn}(x_{n-1} - x_n)$.

If $\epsilon = 0$ this simplifies to

$$R_{\mu_1} (x) \leq R_{\mu_2} (x) \leq \cdots \leq R_{\mu_{N+1}} (x) \leq \cdots.$$
Necessary and Sufficient Conditions for single maturities

- If we restrict ourselves to models where \( \mathbb{P}(S_t - S_t > \epsilon) = 0 \) we get the following temporal conditions:

\[
0 \geq \frac{r_{t,i+1} - r_{t,i}}{K_{i+1} - K_i} \geq \frac{r_{t,i} - r_{t,i-1}}{K_i - K_{i-1}} \geq -1, \quad \text{for } i \in \{2, \ldots, N - 1\},
\]

and

\[
r_{t,i} = r_{t,i-1} \text{ implies } r_{t,i} = 0, \quad \text{for } i \in \{2, \ldots, N\}.
\]

\[
\frac{r_{t,2} - r_{t,1}}{K_2 - K_1} \geq \frac{r_{t,1} - S_0}{K_1 - \epsilon} \quad \text{and} \quad \frac{r_{t,1} - S_0}{K_1 + \epsilon} \geq -1.
\]
If we restrict ourselves to models where \( \mathbb{P}(S_t - S_t^+ > \epsilon) = 0 \) we get the following temporal conditions:

\[
0 \geq \frac{r_{t,i+1} - r_{t,i}}{K_{i+1} - K_i} \geq \frac{r_{t,i} - r_{t,i-1}}{K_i - K_{i-1}} \geq -1, \quad \text{for } i \in \{2, \ldots, N-1\},
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and

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r_{t,i} = r_{t,i-1} \text{ implies } r_{t,i} = 0, \quad \text{for } i \in \{2, \ldots, N\}.
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\frac{r_{t,2} - r_{t,1}}{K_2 - K_1} \geq \frac{r_{t,1} - S_0}{K_1 - \epsilon} \quad \text{and} \quad \frac{r_{t,1} - S_0}{K_1 + \epsilon} \geq -1.
\]
Necessary Conditions for multiple maturities

- If we restrict ourselves to models where $\mathbb{P}(\overline{S}_t - \underline{S}_t > \epsilon) = 0$ then we get the following intertemporal conditions:
  - If $K_i + \epsilon < K_j - \epsilon \sigma_s < K_l + \epsilon$, $s \leq t$ and $s \leq u$ then the following conditions are necessary:

    \[
    \frac{r_{s}^{CVB}(\sigma_s, K_j) - r_{t,i}}{(K_j - \epsilon \sigma_s) - (K_i + \epsilon)} \leq \frac{r_{u,l} - r_{s}^{CVB}(\sigma_s, K_j)}{K_l + \epsilon - (K_s - \epsilon \sigma_s)},
    \]
    \[
    \frac{r_{s}^{CVB}(\sigma_s, K_j) - r_{t,i}}{(K_j - \epsilon \sigma_s) - (K_i + \epsilon)} \leq 0, \quad \text{and}
    \]
    \[
    \frac{r_{u,l} - r_{s}^{CVB}(\sigma_s, K_j)}{K_l + \epsilon - (K_s - \epsilon \sigma_s)} \geq -1
    \]

    where

    \[
    r_{s}^{CVB} = r_{1,j_1} + \sum_{t=2}^{s} (r_{t,j_t} - r_{t,i_{t-1}}) + 2 \epsilon 1_{\{\sigma_1 = -1\}}.
    \]
Necessary Conditions for multiple maturities

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\frac{r_s^{CVB}(\sigma_s, K_j) - r_{t,i}}{(K_j - \epsilon \sigma_s) - (K_i + \epsilon)} \leq 0, \quad \text{and}
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\]
Conclusion

- If there are no transaction costs on the underlying then necessary sufficient conditions can be derived from *Strassen's theorem*.
- If there is no bound on the bid-ask spread on the underlying there are no intertemporal conditions.
- If the bid-ask spread is bounded by a constant we need a generalization of Strassen’s theorem. It can be used to derive consistency conditions.