Name:

Mat.Nr.:

Kennz.:

Bitte keinen Rotstift verwenden!

# 105.057 Finanzmathematik II: zeitstetige Modelle (Vorlesungsprüfung)
## 15. Juni 2009
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(Dauer 90 Minuten, alle Unterlagen sind erlaubt)

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| Bsp. | Max. | Punkte |
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| $\sum$ | 15 | |

Schriftlich:

AssistentIn:

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**Gesamtnote:**

1. Sei $(\Omega, \mathcal{F}, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$ ein filtrierter Wahrscheinlichkeitsraum, $(M_t)_{t\in[0,\infty)}$ ein Martingal auf $(\Omega, \mathcal{F}, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$ mit $\mathbb{E}M_0 = m \in \mathbb{R}$ sowie $\tau : \Omega \to [0, \infty)$ eine Stoppzeit die nur endlich viele Werte annimmt. $M_\tau$ ist eine Zufallsvariable wie gewohnt definiert durch $M_\tau : \Omega \to \mathbb{R} \quad \omega \mapsto M_{\tau(\omega)}(\omega)$.

   Berechne $\mathbb{E}M_\tau$.

2. Take the Vasicek model from interest theory, i. e. assume that the short rate $r_t$ follows

$$dr_t = (a - br_t)dt + \sigma dW_t, \tag{1}$$

with $a, b, \sigma > 0$ and $W_t$ a one dimensional standard Brownian motion w. r. t. the risk neutral probability measure $\mathbb{P}$.

Calculate the price $P(0, T)$ of the zero-coupon bond with maturity $T > 0$.

**Hints:**

(a) Apply the "stochastic Fubini theorem": $\int_0^T \int_0^u f(u,s)dW_s du = \int_0^T \int_s^T f(u,s)du dW_s$.

(b) The solution of (1) is given by

$$r_t = r_0 e^{-bt} + \frac{a}{b}(1 - e^{-bt}) + \sigma e^{-bt} \int_0^t e^{bs} dW_s,$$

hence $r_t$ is Gaussian.

3. Consider a stock price process $S$ modeled as a geometric Brownian motion with volatility $\sigma > 0$ constant. For simplicity we assume that the interest rate $r$ and the dividend yield $q$ are equal to zero. Hence, the dynamics of the stock price under the risk-neutral measure $\mathbb{P}$ are given by

$$S_t = S_0 \exp\left(\sigma W_t - \frac{1}{2}\sigma^2 t\right) \qquad (0 \le t \le T), \tag{2}$$

where $W = (W_t)_{0\le t\le T}$ denotes a standard Brownian motion on the filtered probability space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$. Here $T$ is a fixed, finite time horizon and $S_0 > 0$.

There exist options with a payoff structure similar to a call option, which depend on the *maximum* of the stock price process over the lifetime of the option. These are typically called *lookback* options.

There are two types of lookback options: *floating* strike options with payoff

$$(\max_{0\le t\le T} S_t - S_T)^+,$$

and *fixed* strike options with payoff

$$(\max_{0\le t\le T} S_t - K)^+,$$

where $K > 0$ is constant.

Now, the pricing of floating strike lookback options requires the knowledge of the joint law of the bivariate random variable $(\max_{0\le t\le T} S_t, S_T)$, while the pricing of fixed strike lookback options requires only the knowledge of the law of the univariate random variable $\max_{0\le t\le T} S_t$.

In the stock price model described by (2), show that we can simplify the pricing of the floating strike option, such that it requires only the knowledge of a univariate random variable; that is, find a probability measure $\widetilde{\mathbb{P}}$ (equivalent to $\mathbb{P}$) and a price process $\hat{S}_t$ derived from $S_t$ such that

$$\mathbb{E}(\max_{0\le t\le T} S_t - S_T)^+ = S_0\, \widetilde{\mathbb{E}}(\max_{0\le t\le T} \hat{S}_t - 1)^+,$$

where $\widetilde{\mathbb{E}}$ denotes the expectation with respect to the probability measure $\widetilde{\mathbb{P}}$.

Further, calculate the dynamics of $\hat{S}$ under $\widetilde{\mathbb{P}}$ and show that $Law(\hat{S}|\widetilde{\mathbb{P}}) = Law(S|\mathbb{P})$.

**Hints:**

(a) You should use the following *reflection principle* for a Brownian motion with drift $H_t = bt + \sigma W_t$:

$$Law(\max_{0\le t\le T} H_t - H_T|\mathbb{P}) = Law(-\min_{0\le t\le T} H_t|\mathbb{P}).$$

(b) What is the process of the exponent in (2)? How is the maximum of $S$ related to the maximum of the exponent in (2)?