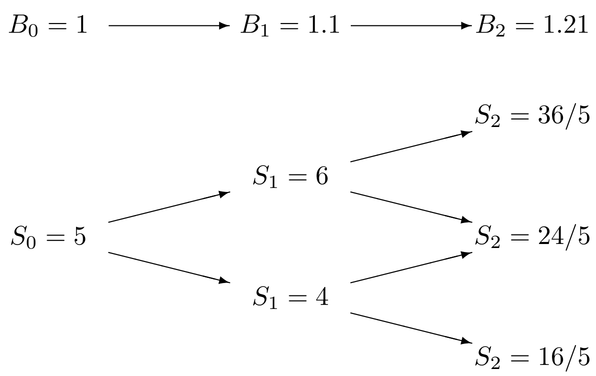
Consider the following two-step Cox-Ross-Rubinstein model ($a = -1/5$, $r = 1/10$, $b = 1/5$) for a financial market with bond B and stock S :



Assume furthermore that $P(\{\omega_i\}) = 0.2$, $1 \leq i \leq 3$ and $P(\{\omega_4\}) = 0.4$, where the 4-th trajectory is the one going twice up.

(a) Compute the set of equivalent martingale measures $\mathcal{M}^e(\tilde{S})$ and the Radon-Nikodym derivative of the martingale measure Q with respect to P. (1)

(b) Compute the arbitrage-free price at time $N = 0$ of a claim with payoff

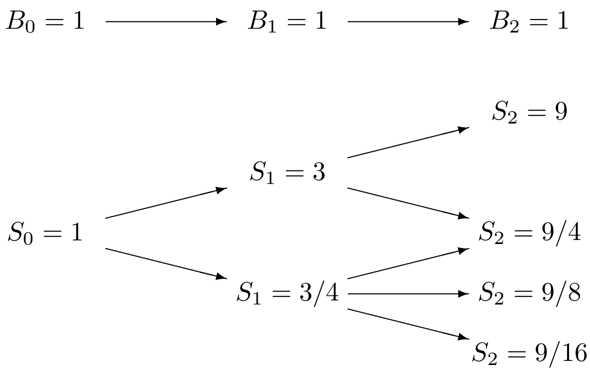
$$\big(9 \, , \, 9 \, , \, 4 \, , \, 0 \,\big)$$

and maturity $N = 2$. (1)

(c) Compute the replicating strategy of the claim introduced in part (b). (1)

(d) Compute your expected total gain (or loss) if you buy the claim of (b) at time $N = 0$, i.e. compute the difference of the expected outcome of the claim (with respect to the measure P) and the invested premium. Is this in contradiction to no-arbitrage arguments? (1)

Consider a two-step model for a financial market with bond B and stock S. The bond B is assumed to be constant, i.e. $B_n = 1$ for $n = 0, 1, 2$, the stock S is assumed to move according to the following tree:



Assume furthermore that $P(\{\omega_i\}) = 0.2$, $1 \leq i \leq 5$.

(d) Compute the set of equivalent martingale measures $\mathcal{M}^e(\tilde{S})$ as well as the set of absolutely continuous martingale measures $\mathcal{M}^a(\tilde{S})$. (1)

(e) Compute the arbitrage-free price(s) at times $n = 0, 1$ of a European digital option $1_{S_2 \geq K}$ with strike $K = 9/2$ and maturity $N = 2$. (1)

(f) How to replicate the European digital option from (e). (1)

TASK 2

Fix $T > 0$. Consider a Bachelier Model with constant interest rate $r = 0$ and stock price S, i.e. under the martingale measure Q the stock price process is given by

$$S_t = S_0 + \sigma B_t, \quad S_0, \sigma > 0, \quad 0 \leq t \leq T.$$

Here, B denotes a standard Brownian motion under the martingale measure Q.

(a) Compute the arbitrage-free price v of an option with payoff

$$(S_T - K)^2.$$

Hint: consider $(S_T - K) = (S_T - S_0) + (S_0 - K)$ to make the computation easier. (2)

(b) Compute the Delta of the price v, i.e. the derivative of the price with respect to S_0 and argue how one can construct the Hedging strategy of $(S_T - K)^2$ with the Delta. (2)

Task 3

Consider a Black-Scholes Model with constant interest rate $r = 0$ and stock price S, i.e. under the martingale measure Q the stock price process is given by

$$S_t = S_0 \exp(-\frac{\sigma^2 t}{2} + \sigma B_t), \quad S_0, \sigma > 0, \quad 0 \leq t \leq T.$$

Here, B denotes a standard Brownian motion under the martingale measure Q.

(a) Compute the arbitrage-free price $p(S_0, T)$ of a European digital option

$$1_{(S_T \geq K)}$$

and give an interpretation of the price in terms of Q-probabilities. (2)

(b) Write down a put-call-parity for the European digital Option and prove it. Give an argument why there are several ones. (2)

TASK 1

Consider the following two-step Cox-Ross-Rubinstein model ($a = -1/5$, $r = 1$, $b = 3/2$) for a financial market with bond B and stock S. Assume furthermore that $P(\{\omega_i\}) = 0.25$, $1 \leq i \leq 4$.

(a) Compute the set of equivalent martingale measures $\mathcal{M}^e(\tilde{S})$ and the Radon-Nikodym derivative of the martingale measure Q with respect to P. (1)

(b) Compute the arbitrage-free price at time $N = 0$ of a claim with payoff

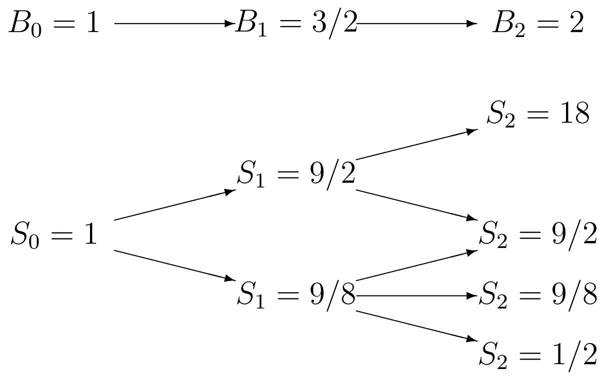
$$\left(4 \, , \, 3 \, , \, 0 \, , \, 0 \right)$$

and maturity $N = 2$. It this claim of European type, i.e. does it only depend on the price at time $N = 2$? (1)

(c) Compute the replicating strategy of the claim introduced in part (b). (1)

(d) Compute your expected total gain (or loss) if you buy the claim of (b) at time $N = 0$, i.e. compute the difference of the expected outcome of the claim (with respect to the measure P) and the invested premium. Is this in contradiction to no-arbitrage arguments? (1)

Consider a two-step model for a financial market with bond B and stock S. The bond B and the stock S are assumed to move according to the following tree:



Assume furthermore that $P(\{\omega_i\}) = 0.2$, $1 \leq i \leq 5$.

(e) Compute the set of equivalent martingale measures $\mathcal{M}^e(\tilde{S})$ as well as the set of absolutely continuous martingale measures $\mathcal{M}^a(\tilde{S})$. (1)

(f) Compute the arbitrage-free price(s) at times $n = 0, 1$ of a European put option with strike $K = 9/4$ and maturity $N = 2$. Apply the put-call parity to calculate the prices of a European Call option. (1)

TASK 2

Consider a Bachelier Model with constant interest rate $r = 0$ and stock price S, i.e. under the martingale measure Q the stock price process is given by

$$S_t = S_0 + \sigma B_t, \quad S_0, \sigma > 0, \quad 0 \leq t \leq T.$$

Here, B denotes a standard Brownian motion under the martingale measure Q.

(a) Compute the arbitrage-free price v of an option with payoff

$$1_{S_T \geq S_0},$$

i.e. which pays one if $S_T \geq S_0$ and 0 otherwise. (2)

(b) Find a replicating strategy $(\phi_t)_{0 \leq t \leq T}$ for the option, i.e. solve the equation

$$1_{S_T \geq S_0} = v + \int_0^T \phi_s dS_s.$$

Write down the Hedging portfolio for the option and calculate the amount of money at each time in the Bank account. Recall that the derivative of the price v for time to maturity $T - t$ at the actual price S_t yields the hedging portfolio. (2)

TASK 3

Consider a Black-Scholes Model with constant interest rate $r = 0$ and stock price S, i.e. under the martingale measure Q the stock price process is given by

$$S_t = S_0 \exp(-\frac{\sigma^2 t}{2} + \sigma B_t), \quad S_0, \sigma > 0, \quad 0 \leq t \leq T.$$

Here, B denotes a standard Brownian motion under the martingale measure Q.

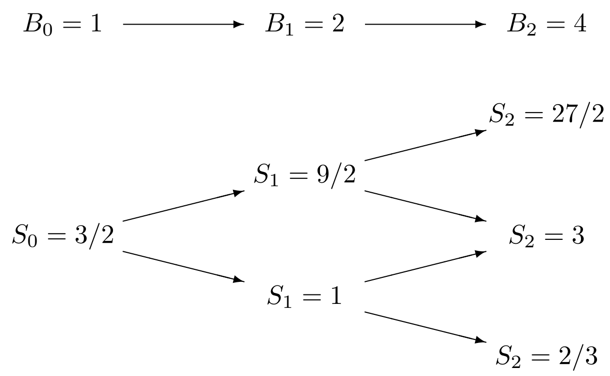
(a) Compute the arbitrage-free price p of the scaled European put option with payoff

$$(0.01 S_T - K)_-.$$

How does the put-call parity look for this type of option? (2)

(b) Calculate actual prices for $S_0 = 100, K = 1, \sigma = 0.2, T = 0,5$ and the actual hedging portfolio in this case. (2)

Consider the following two-step Cox-Ross-Rubinstein model ($a = -1/3$, $r = 1$, $b = 2$) for a financial market with bond B and stock S :



Assume furthermore that $P(\{\omega_i\}) = 0.1$, $1 \leq i \leq 3$ and $P(\{\omega_4\}) = 0.7$, where the 4-th trajectory is the one going twice up.

(a) Compute the set of equivalent martingale measures $\mathcal{M}^e(\tilde{S})$ and the Radon-Nikodym derivative of the martingale measure Q with respect to P. (1)

(b) Compute the arbitrage-free price at time $N = 0$ of a claim with payoff

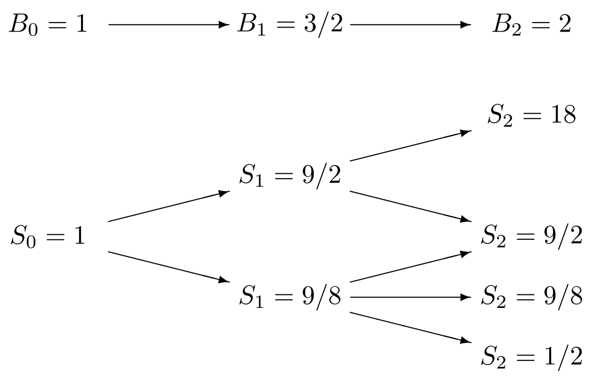
$$\big(9\ ,\ 9\ ,\ 4\ ,\ -2/3\ \big)$$

and maturity $N = 2$. (1)

(c) Compute the replicating strategy of the claim introduced in part (b). (1)

(d) Compute your expected total gain (or loss) if you buy the claim of (b) at time $N = 0$, i.e. compute the difference of the expected outcome of the claim (with respect to the measure P) and the invested premium. Is this in contradiction to no-arbitrage arguments? (1)

Consider a two-step model for a financial market with bond B and stock S. The bond B and the stock S are assumed to move according to the following tree:



Assume furthermore that $P(\{\omega_i\}) = 0.2$, $1 \leq i \leq 5$.

(d) Compute the set of equivalent martingale measures $\mathcal{M}^e(\tilde{S})$ as well as the set of absolutely continuous martingale measures $\mathcal{M}^a(\tilde{S})$. (1)

(e) Compute the arbitrage-free price(s) at times $n = 0, 1$ of a European put option with strike $K = 9/4$ and maturity $N = 2$. (1)

(f) Apply the put-call parity to calculate the prices of a European Call option. (1)

TASK 2

Consider a Bachelier Model with constant interest rate $r = 0$ and stock price S, i.e. under the martingale measure Q the stock price process is given by

$$S_t = S_0 + \sigma B_t, \quad S_0, \sigma > 0, \quad 0 \leq t \leq T.$$

Here, B denotes a standard Brownian motion under the martingale measure Q.

(a) Compute the arbitrage-free price v of an option with payoff

$$(S_T - S_0)^3.$$

(2)

(b) Find a replicating strategy $(\phi_t)_{0 \leq t \leq T}$ for the option, i.e. solve the equation

$$(S_T - S_0)^3 = v + \int_0^T \phi_s dS_s.$$

Write down the Hedging portfolio for the option and calculate the amount of money at each time in the Bank account. (2)

TASK 3

Consider a Black-Scholes Model with constant interest rate $r = 0$ and stock price S, i.e. under the martingale measure Q the stock price process is given by

$$S_t = S_0 \exp(-\frac{\sigma^2 t}{2} + \sigma B_t), \quad S_0, \sigma > 0, \quad 0 \leq t \leq T.$$

Here, B denotes a standard Brownian motion under the martingale measure Q.

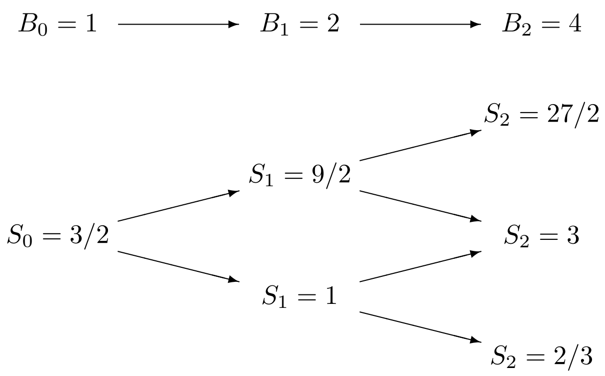
(a) Compute the arbitrage-free price $p(S_0, T)$ of the digital option, i.e. an option with payoff

$$1_{S_T \geq K}.$$

How would a put-call parity look for this type of option? (2)

(b) Argue if replication is possible for this option and how many stocks should be hold at $0 \leq t \leq T$ in a replicating portfolio. Recall the formula for Delta-Hedging. (2)

Consider the following two-step Cox-Ross-Rubinstein model ($a = -1/3$, $r = 1$, $b = 2$) for a financial market with bond B and stock S :



Assume furthermore that $P(\{\omega_i\}) = 0.1$, $1 \leq i \leq 3$ and $P(\{\omega_4\}) = 0.7$, where the 4-th trajectory is the one going twice up.

(a) Compute the set of equivalent martingale measures $\mathcal{M}^e(\tilde{S})$ and the Radon-Nikodym derivative of the martingale measure Q with respect to P. (1)

(b) Compute the arbitrage-free price at time $N = 0$ of a claim with payoff

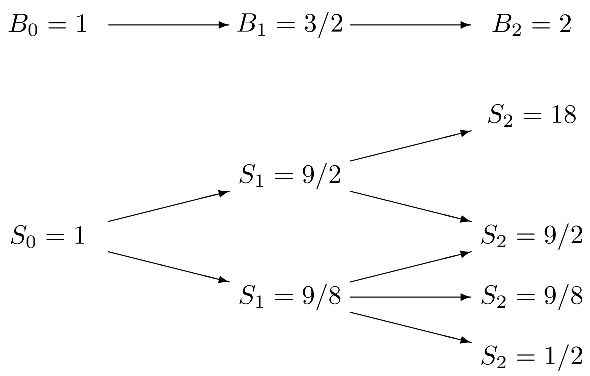
$$\big(\, 9\,,\; 9\,,\; 4\,,\; -2/3\,\big)$$

and maturity $N = 2$. (1)

(c) Compute the replicating strategy of the claim introduced in part (b). (1)

(d) Compute your expected total gain (or loss) if you buy the claim of (b) at time $N = 0$, i.e. compute the difference of the expected outcome of the claim (with respect to the measure P) and the invested premium. Is this in contradiction to no-arbitrage arguments? (1)

Consider a two-step model for a financial market with bond B and stock S. The bond B is assumed to be constant, i.e. $B_n = 1$ for $n = 0, 1, 2$, the stock S is assumed to move according to the following tree:



Assume furthermore that $P(\{\omega_i\}) = 0.2$, $1 \leq i \leq 5$.

(d) Compute the set of equivalent martingale measures $\mathcal{M}^e(\tilde{S})$ as well as the set of absolutely continuous martingale measures $\mathcal{M}^a(\tilde{S})$. (1)

(e) Compute the arbitrage-free price(s) at times $n = 0, 1$ of a European put option with strike $K = 9/4$ and maturity $N = 2$. (1)

(f) Apply the put-call parity to calculate the prices of a European Call option. (1)

TASK 2

Consider a Bachelier Model with constant interest rate $r = 0$ and stock price S, i.e. under the martingale measure Q the stock price process is given by

$$S_t = S_0 + \sigma B_t, \quad S_0, \sigma > 0, \quad 0 \leq t \leq T.$$

Here, B denotes a standard Brownian motion under the martingale measure Q.

(a) Compute the arbitrage-free price v of an option with payoff

$$(S_T - S_0)^2.$$

(2)

(b) Find a replicating strategy $(\phi_t)_{0 \leq t \leq T}$ for the option, i.e. solve the equation

$$(S_T - S_0)^2 = v + \int_0^T \phi_s dS_s.$$

Write down the Hedging portfolio for the option and calculate the amount of money at each time in the Bank account. Recall the equation

$$2 \int_0^T B_s dB_s = B_T^2 - T$$

for the solution of this problem and note that

$$E(B_T^2 | \mathcal{F}_s) = B_s^2 + (T - s)$$

for $0 \leq s \leq T$. (2)

TASK 3

Consider a Black-Scholes Model with constant interest rate $r = 0$ and stock price S, i.e. under the martingale measure Q the stock price process is given by

$$S_t = S_0 \exp(-\frac{\sigma^2 t}{2} + \sigma B_t), \quad S_0, \sigma > 0, \quad 0 \leq t \leq T.$$

Here, B denotes a standard Brownian motion under the martingale measure Q.

(a) Compute the arbitrage-free price $p(S_0, T)$ of the ”famous” square option, i.e. an option with payoff

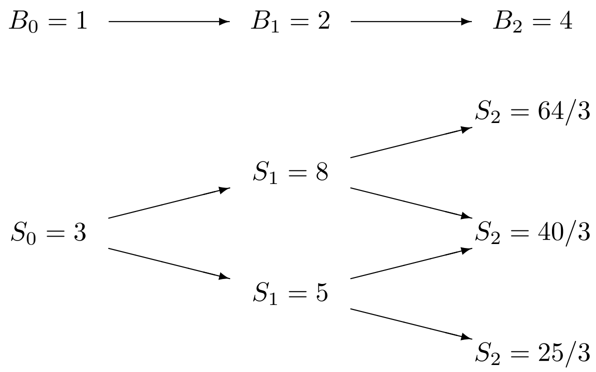
$$(S_T^2 - K)_+.$$

How would a put-call parity look for this type of option? (2)

(b) Argue if replication is possible for this option and how many stocks should be hold at $0 \leq t \leq T$ in a replicating portfolio. Recall the formula for Delta-Hedging. (2)

TASK 1

Consider the following two-step Cox-Ross-Rubinstein model ($a = 2/3$, $r = 1$, $b = 5/3$) for a financial market with bond B and stock S :



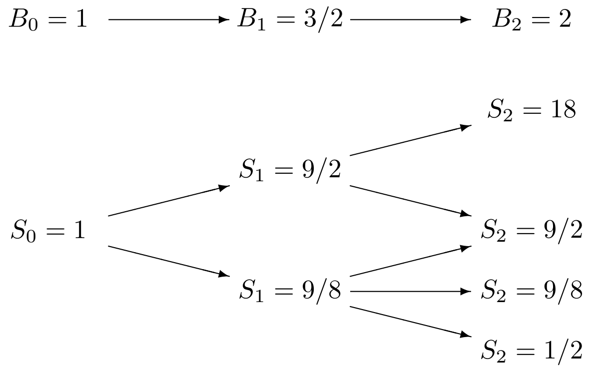
Assume furthermore that $P(\{\omega_i\}) = 0.25$, $1 \leq i \leq 4$.

(a) Compute the set of equivalent martingale measures $\mathcal{M}^e(\tilde{S})$ and the Radon-Nikodym derivative of the martingale measure Q with respect to the historical measure P. The prices of the stock S seem increasing. Why is there no simple arbitrage by going long in the stock? (1)

(b) Compute the arbitrage-free prices at time $n = 0, 1$ of a European Call with strike price $K = 40/3$ and maturity $N = 2$. (1)

(c) Compute the replicating strategy of the claim introduced in part (b). Explain the replicating strategy in words (1)

Consider a two-step model for a financial market with bond B and stock S. The bond B is assumed to be constant, i.e. $B_n = 1$ for $n = 0, 1, 2$, the stock S is assumed to move according to the following tree:



Assume furthermore that $P(\{\omega_i\}) = 0.2$, $1 \leq i \leq 5$.

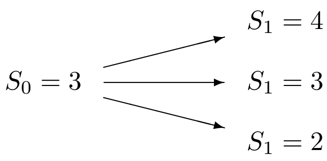
(d) Compute the set of equivalent martingale measures $\mathcal{M}^e(\tilde{S})$ as well as the set of absolutely continuous martingale measures $\mathcal{M}^a(\tilde{S})$. (1)

(e) Compute the arbitrage-free price(s) at times $n = 0, 1$ of a European put option with strike $K = 9/4$ and maturity $N = 2$. (1)

(f) Apply the put-call parity to calculate the prices of a European Call option. (1)

TASK 2

Consider a one-step model for a financial market with bond B and stock S. The bond B is assumed to be constant, i.e. $B = 1$, the stock S is assumed to move according to the following tree:



Assume furthermore that $P(\{\omega_1\}) = 1/10$, $P(\{\omega_2\}) = 5/10$ and $P(\{\omega_3\}) = 4/10$.
Define a utility function u by

$$u(x) := \log(x), \quad x \in \mathbb{R}_{>0}.$$

Consider the utility maximization problem

$$U(x) := \sup_{Y \in K} E\big(u(x+Y)\big), \qquad (*)$$

where $K := \big\{ Y = (\phi \cdot S)_T : \phi \text{ predictable } \big\}$.

Solve the utility maximization problem $(*)$ using the "**pedestrian method**", i.e.:

(a) Compute the set K. (1)

(b) Compute the optimizer $\hat{Y}$ in $(*)$ and compute the value function U. (1)

Solve the utility maximization problem $(*)$ using the "**duality approach**", i.e.:

(c) Compute the set of absolutely continuous martingale measures $\mathcal{M}^a(\tilde{S})$. (1)

(d) Prove that the dual optimization problem

$$V(y) := \inf_{Q \in \mathcal{M}^a(\tilde{S})} E\Big(v\big(y\frac{dQ}{dP}\big)\Big) \qquad (**)$$

obtains its minimum at $Q = (1/4, 1/2, 1/4)$. Using this, compute the dual value function V. (1)

(e) Verify that the primal value function U, computed in part (a), and the dual value function V, computed in part (d), are indeed conjugate. (1)

Task 3

Consider a Black-Scholes Model with constant interest rate $r = 0$ and stock price S, i.e. under the martingale measure P the stock price process is given by

$$S_t = S_0 \exp(-\frac{\sigma^2 t}{2} + \sigma B_t), \quad S_0, \sigma > 0, \quad 0 \leq t \leq T.$$

Here, B denotes a standard Brownian motion under the martingale measure P.

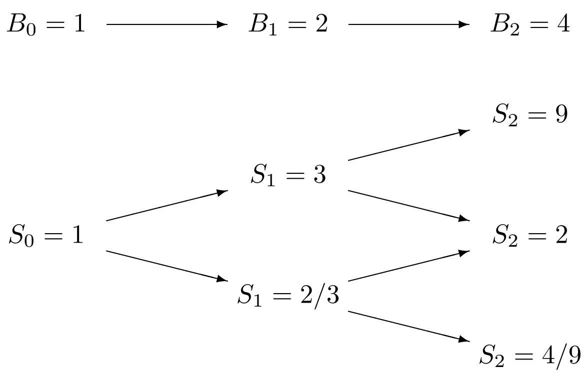
Compute the arbitrage-free price v of a straddle option, i.e. an option with payoff

$$|S_T - K|.$$

Argue if replication is possible for the straddle. Apply then the put-call parity and the replication strategy for the European call to obtain a strategy of replication for the straddle. (3)

TASK 1

Consider the following two-step Cox-Ross-Rubinstein model ($a = -1/3$, $r = 1$, $b = 2$) for a financial market with bond B and stock S :



Assume furthermore that $P(\{\omega_i\}) = 0.25$, $1 \leq i \leq 4$.

(a) Compute the set of equivalent martingale measures $\mathcal{M}^e(\tilde{S})$. (1)

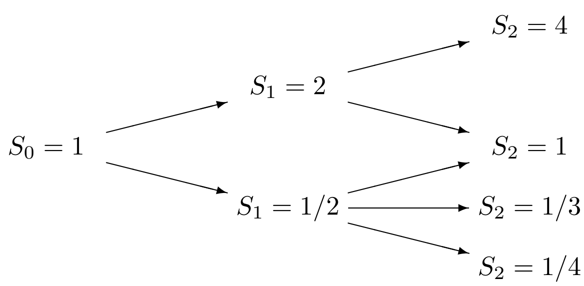
(b) Compute the arbitrage-free price at time $N = 0$ of a claim with payoff

$$\left(6 \,,\; 6 \,,\; 8/3 \,,\; -4/9 \right)$$

and maturity $N = 2$. (1)

(c) Compute the replicating strategy of the claim introduced in part (b). (2)

Consider a two-step model for a financial market with bond B and stock S. The bond B is assumed to be constant, i.e. $B \equiv 1$, the stock S is assumed to move according to the following tree:



Assume furthermore that $P(\{\omega_i\}) = 0.2$, $1 \le i \le 5$.

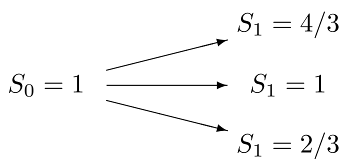
(a) Compute the set of equivalent martingale measures $\mathcal{M}^e(\tilde{S})$ as well as the set of absolutely continuous martingale measures $\mathcal{M}^a(\tilde{S})$. (1)

(b) Compute the arbitrage-free price(s) of a European put option with strike $K = 1$ and maturity $N = 2$ at times $N = 1, 2$. (1)

(c) Interpret the set $\mathcal{M}^a(\tilde{S})$ as a closed convex set and describe its extreme points. (1)

TASK 2

Consider a one-step model for a financial market with bond B and stock S. The bond B is assumed to be constant, i.e. $B \equiv 1$, the stock S is assumed to move according to the following tree:



Assume furthermore that $P(\{\omega_1\}) = 1/10$, $P(\{\omega_2\}) = 5/10$ and $P(\{\omega_3\}) = 4/10$.
Define a utility function u by

$$u(x) := -\exp(-2x), \quad x \in \mathbb{R}.$$

Recall that the convex conjugate v of u is given by

$$v(y) := \frac{y}{2}\big(\ln \frac{y}{2} - 1\big), \quad y \geq 0.$$

Consider the utility maximization problem

$$U(x) := \sup_{Y \in K} E\big(u(x+Y)\big), \tag{$*$}$$

where $K := \big\{Y = (\phi \cdot S)_T : \phi \text{ predictable } \big\}$.

Solve the utility maximization problem $(*)$ using the "**pedestrian method**", i.e.:

(a) Compute the set K. (1)

(b) Compute the optimizer $\hat{Y}$ in $(*)$ and compute the value function U. (1)

Solve the utility maximization problem $(*)$ using the "**duality approach**", i.e.:

(c) Compute the set of absolutely continuous martingale measures $\mathcal{M}^a(\tilde{S})$. (1)

(d) The solution to the dual optimization problem

$$V(y) := \inf_{Q \in \mathcal{M}^a(\tilde{S})} E\Big(v\big(y\frac{dQ}{dP}\big)\Big) \tag{$**$}$$

is given by

$$\hat{Q} = \Big(\frac{2}{9}, \frac{5}{9}, \frac{2}{9}\Big).$$

Using this, compute the dual value function V. (1)

(e) Verify that the primal value function U, computed in part (a), and the dual value function V, computed in part (d), are indeed conjugate. (1)

Task 3

Consider the Black-Scholes Model with constant bond and stock S, i.e. under the martingale measure Q_T, the bond price process is identically one and the stock price process is given by

$$S_t = S_0 \exp\Big(\sigma \tilde{B}_t - \frac{\sigma^2}{2} t\Big), \quad S_0, \sigma > 0, \quad 0 \leq t \leq T.$$

Here, $\tilde{B}$ denotes a standard Brownian motion under the martingale measure Q_T.

(a) Compute the arbitrage-free price $C(T, K, S_0)$ of a European option with the payoff

$$\begin{array}{cc} K_2 - K_1 & \ln S_T \leq K_1 \\ K_2 - \ln S_T & K_1 \leq \ln S_T \leq K_2 \\ 0 & \ln S_T \geq K_2 \end{array}$$

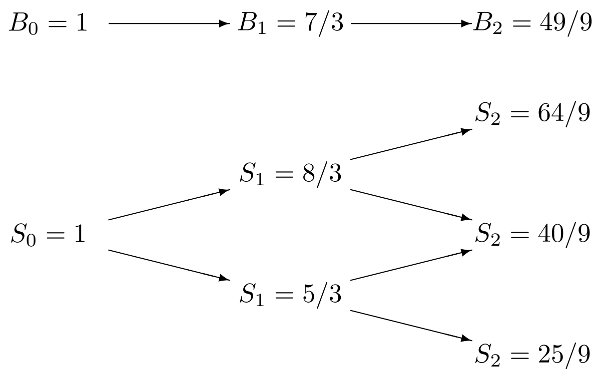
Here, $K_1 \leq K_2 \in \mathbb{R}$ denote strike values and T denotes the maturity. (1)
Hint: Recall that

$$\int_a^\infty x\phi(x)dx = \frac{1}{\sqrt{2\pi}} \int_a^\infty x \exp\big(\frac{-x^2}{2}\big)dx = \frac{1}{\sqrt{2\pi}} \exp\big(\frac{-a^2}{2}\big) = \phi(a).$$

(b) Compute the limit $\lim_{K_1 \to -\infty} C(T, K_1, K_2, S_0)$ and interpret the result. (1)

TASK 1

Consider the following two-step Cox-Ross-Rubinstein model ($a = 2/3$, $r = 4/3$, $b = 5/3$) for a financial market with bond B and stock S :



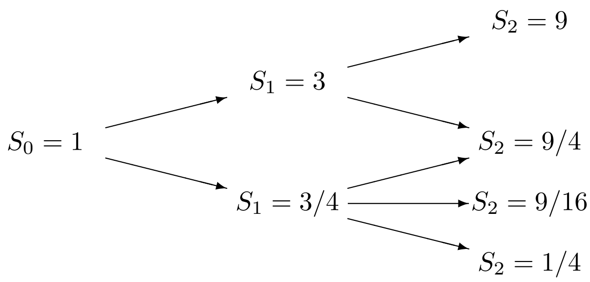
Assume furthermore that $P(\{\omega_i\}) = 0.25$, $1 \leq i \leq 4$.

(a) Compute the set of equivalent martingale measures $\mathcal{M}^e(\tilde{S})$ and the Radon-Nikodym derivative of the martingale measure(s) with respect to P. The prices of the stock S are surely increasing. Why is the model nevertheless arbitrage-free? Why does one not make a certain gain by simply investing into the stock? (1)

(b) Compute the arbitrage-free prices at time $n = 0, 1$ of a European Call with strike price $K = 40/9$ and maturity $N = 2$. (1)

(c) Compute the replicating strategy of the claim introduced in part (b). (1)

Consider a two-step model for a financial market with bond B and stock S. The bond B is assumed to be constant, i.e. $B_n = 1$ for $n = 0, 1, 2$, the stock S is assumed to move according to the following tree:



Assume furthermore that $P(\{\omega_i\}) = 0.2$, $1 \leq i \leq 5$.

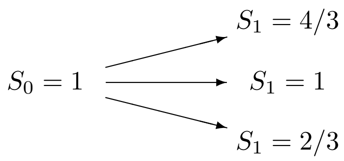
(d) Compute the set of equivalent martingale measures $\mathcal{M}^e(\tilde{S})$ as well as the set of absolutely continuous martingale measures $\mathcal{M}^a(\tilde{S})$. (1)

(e) Compute the arbitrage-free price(s) at times $n = 0, 1$ of a European put option with strike $K = 9/4$ and maturity $N = 2$. (1)

(f) Apply the put-call parity to calculate the prices of a European Call option. (1)

TASK 2

Consider a one-step model for a financial market with bond B and stock S. The bond B is assumed to be constant, i.e. $B = 1$, the stock S is assumed to move according to the following tree:



Assume furthermore that $P(\{\omega_1\}) = 1/10$, $P(\{\omega_2\}) = 5/10$ and $P(\{\omega_3\}) = 4/10$.
Define a utility function u by

$$u(x) := \log(x), \quad x \in \mathbb{R}_{>0}.$$

Consider the utility maximization problem

$$U(x) := \sup_{Y \in K} E\big(u(x+Y)\big), \tag{$*$}$$

where $K := \big\{ Y = (\phi \cdot S)_T : \phi \text{ predictable } \big\}$.

Solve the utility maximization problem $(*)$ using the "**pedestrian method**", i.e.:

(a) Compute the set K. (1)

(b) Compute the optimizer $\hat{Y}$ in $(*)$ and compute the value function U. (1)

Solve the utility maximization problem $(*)$ using the "**duality approach**", i.e.:

(c) Compute the set of absolutely continuous martingale measures $\mathcal{M}^a(\tilde{S})$. (1)

(d) Prove that the dual optimization problem

$$V(y) := \inf_{Q \in \mathcal{M}^a(\tilde{S})} E\Big(v\big(y\frac{dQ}{dP}\big)\Big) \tag{$**$}$$

obtains its minimum at $Q = (1/4, 1/2, 1/4)$. Using this, compute the dual value function V. (1)

(e) Verify that the primal value function U, computed in part (a), and the dual value function V, computed in part (d), are indeed conjugate. (1)

TASK 3

Consider a Black-Scholes Model with constant interest rate $r = 0$ and stock price S, i.e. under the martingale measure P the stock price process is given by

$$S_t = S_0 \exp(-\frac{\sigma^2 t}{2} + \sigma B_t), \quad S_0, \sigma > 0, \quad 0 \leq t \leq T.$$

Here, B denotes a standard Brownian motion under the martingale measure P.

Compute the arbitrage-free prices $v_{+/-}$ of options with payoff

$$(S_T - K)_+$$

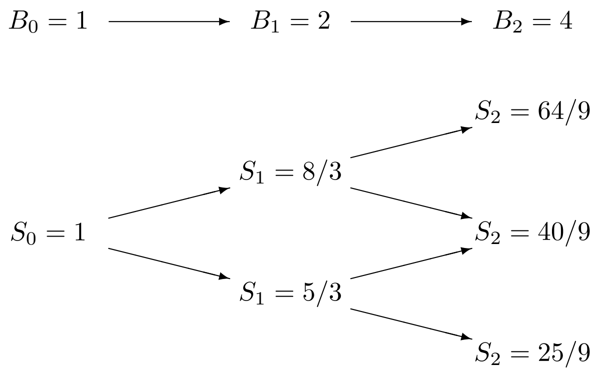
and

$$(S_T - K)_-$$

for a strike value $K > 0$. Calculate the difference $v_+ - v_-$ and show that it is independent of σ. Provide an argument for this result. (3)

TASK 1

Consider the following two-step Cox-Ross-Rubinstein model ($a = 2/3$, $r = 1$, $b = 5/3$) for a financial market with bond B and stock S :



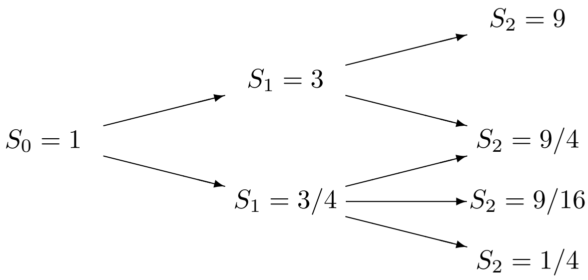
Assume furthermore that $P(\{\omega_i\}) = 0.25$, $1 \le i \le 4$.

(a) Compute the set of equivalent martingale measures $\mathcal{M}^e(\tilde{S})$ and the Radon-Nikodym derivative of the martingale measure Q with respect to P. The prices of the stock S are surely increasing. Why is the model nevertheless arbitrage-free? Why does one not make a certain gain by simply investing in the stock? (1)

(b) Compute the arbitrage-free prices at time $n = 0, 1$ of a European Call with strike price $K = 40/9$ and maturity $N = 2$. (1)

(c) Compute the replicating strategy of the claim introduced in part (b). (1)

Consider a two-step model for a financial market with bond B and stock S. The bond B is assumed to be constant, i.e. $B_n = 1$ for $n = 0, 1, 2$, the stock S is assumed to move according to the following tree:



Assume furthermore that $P(\{\omega_i\}) = 0.2$, $1 \leq i \leq 5$.

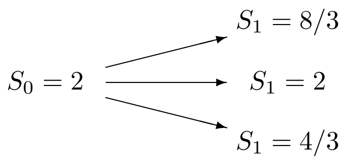
(d) Compute the set of equivalent martingale measures $\mathcal{M}^e(\tilde{S})$ as well as the set of absolutely continuous martingale measures $\mathcal{M}^a(\tilde{S})$. (1)

(e) Compute the arbitrage-free price(s) at times $n = 0, 1$ of a European put option with strike $K = 9/4$ and maturity $N = 2$. (1)

(f) Apply the put-call parity to calculate the prices of a European Call option. (1)

TASK 2

Consider a one-step model for a financial market with bond B and stock S. The bond B is assumed to be constant, i.e. $B = 1$, the stock S is assumed to move according to the following tree:



Assume furthermore that $P(\{\omega_1\}) = 1/10$, $P(\{\omega_2\}) = 5/10$ and $P(\{\omega_3\}) = 4/10$.
Define a utility function u by

$$u(x) := \log(x), \quad x \in \mathbb{R}_{>0}.$$

Consider the utility maximization problem

$$U(x) := \sup_{Y \in K} E\big(u(x+Y)\big), \tag{$*$}$$

where $K := \big\{Y = (\phi \cdot S)_T : \phi \text{ predictable } \big\}$.

Solve the utility maximization problem $(*)$ using the "**pedestrian method**", i.e.:

(a) Compute the set K. (1)

(b) Compute the optimizer $\hat{Y}$ in $(*)$ and compute the value function U. (1)

Solve the utility maximization problem $(*)$ using the "**duality approach**", i.e.:

(c) Compute the set of absolutely continuous martingale measures $\mathcal{M}^a(\tilde{S})$. (1)

(d) Prove that the dual optimization problem

$$V(y) := \inf_{Q \in \mathcal{M}^a(\tilde{S})} E\Big(v\big(y\frac{dQ}{dP}\big)\Big) \tag{$**$}$$

obtains its minimum at $Q = (1/4, 1/2, 1/4)$. Using this, compute the dual value function V. (1)

(e) Verify that the primal value function U, computed in part (a), and the dual value function V, computed in part (d), are indeed conjugate. (1)

TASK 3

Consider a Black-Scholes Model with constant interest rate $r = 0$ and stock price S, i.e. under the martingale measure P the stock price process is given by

$$S_t = S_0 \exp(-\frac{\sigma^2 t}{2} + \sigma B_t), \quad S_0, \sigma > 0, \quad 0 \leq t \leq T.$$

Here, B denotes a standard Brownian motion under the martingale measure P.

Compute the arbitrage-free price v of an option with payoff

$$(\log S_T - K)^2.$$

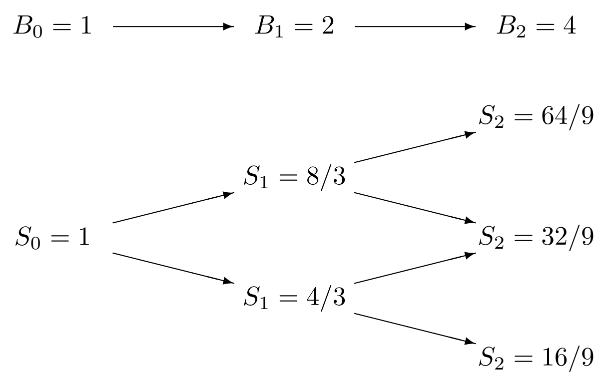
and argue if replication is possible. Show first that there is a strategy $(\phi_s)_{0 \leq s \leq T}$ such that

$$(\log S_T - K)^2 = \int_0^T \phi_s dB_s + v$$

and apply then the formula $dS_t = \sigma S_t dB_t$. (3)

TASK 1

Consider the following two-step Cox-Ross-Rubinstein model ($a = 1/3$, $r = 1$, $b = 5/3$) for a financial market with bond B and stock S :



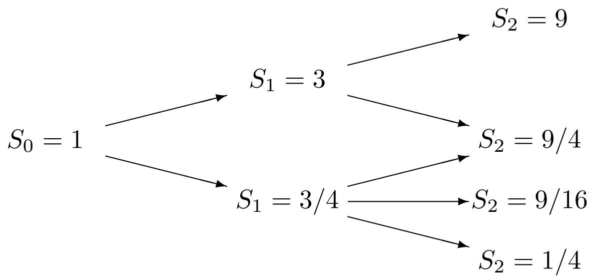
Assume furthermore that $P(\{\omega_i\}) = 0.25$, $1 \leq i \leq 4$.

(a) Compute the set of equivalent martingale measures $\mathcal{M}^e(\tilde{S})$ and the Radon-Nikodym derivative of the martingale measure Q with respect to P. The prices of the stock S are surely increasing. Why is the model nevertheless arbitrage-free? Why does one not make a certain gain by simply investing in the stock? (1)

(b) Compute the arbitrage-free prices at time $n = 0, 1$ of a European Call with strike price $K = 32/9$ and maturity $N = 2$. (1)

(c) Compute the replicating strategy of the claim introduced in part (b). (1)

Consider a two-step model for a financial market with bond B and stock S. The bond B is assumed to be constant, i.e. $B_n = 1$ for $n = 0, 1, 2$, the stock S is assumed to move according to the following tree:



Assume furthermore that $P(\{\omega_i\}) = 0.2$, $1 \leq i \leq 5$.

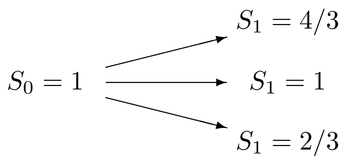
(d) Compute the set of equivalent martingale measures $\mathcal{M}^e(\tilde{S})$ as well as the set of absolutely continuous martingale measures $\mathcal{M}^a(\tilde{S})$. (1)

(e) Compute the arbitrage-free price(s) at times $n = 0, 1$ of a European put option with strike $K = 9/4$ and maturity $N = 2$. (1)

(f) Apply the put-call parity to calculate the prices of a European Call option. (1)

TASK 2

Consider a one-step model for a financial market with bond B and stock S. The bond B is assumed to be constant, i.e. $B = 1$, the stock S is assumed to move according to the following tree:



Assume furthermore that $P(\{\omega_1\}) = 1/10$, $P(\{\omega_2\}) = 5/10$ and $P(\{\omega_3\}) = 4/10$.
Define a utility function u by

$$u(x) := \log(x), \quad x \in \mathbb{R}_{>0}.$$

Consider the utility maximization problem

$$U(x) := \sup_{Y \in K} E\big(u(x+Y)\big), \qquad (*)$$

where $K := \big\{ Y = (\phi \cdot S)_T : \phi \text{ predictable } \big\}$.

Solve the utility maximization problem $(*)$ using the "**pedestrian method**", i.e.:

(a) Compute the set K. (1)

(b) Compute the optimizer $\hat{Y}$ in $(*)$ and compute the value function U. (1)

Solve the utility maximization problem $(*)$ using the "**duality approach**", i.e.:

(c) Compute the set of absolutely continuous martingale measures $\mathcal{M}^a(\tilde{S})$. (1)

(d) Prove that the dual optimization problem

$$V(y) := \inf_{Q \in \mathcal{M}^a(\tilde{S})} E\Big(v\big(y \frac{dQ}{dP}\big)\Big) \qquad (**)$$

obtains its minimum at $Q = (1/4, 1/2, 1/4)$. Using this, compute the dual value function V. (1)

(e) Verify that the primal value function U, computed in part (a), and the dual value function V, computed in part (d), are indeed conjugate. (1)

TASK 3

Consider a Bachelier Model with constant interest rate $r = 0$ and stock price S, i.e. under the martingale measure P the stock price process is given by

$$S_t = S_0 + \sigma B_t, \quad S_0, \sigma > 0, \quad 0 \leq t \leq T.$$

Here, B denotes a standard Brownian motion under the martingale measure P.

(a) Compute the arbitrage-free price v of an option with payoff

$$(S_T - S_0)^2. \tag{1}$$

(b) Find a replicating strategy $(\phi_t)_{0 \leq t \leq T}$ for the option, i.e. solve the equation

$$(S_T - S_0)^2 = v + \int_0^T \phi_s dS_s.$$

Write down the Hedging portfolio for the option and calculate the amount of money at each time in the Bank account. Recall the equation

$$2 \int_0^T B_s dB_s = B_T^2 - T$$

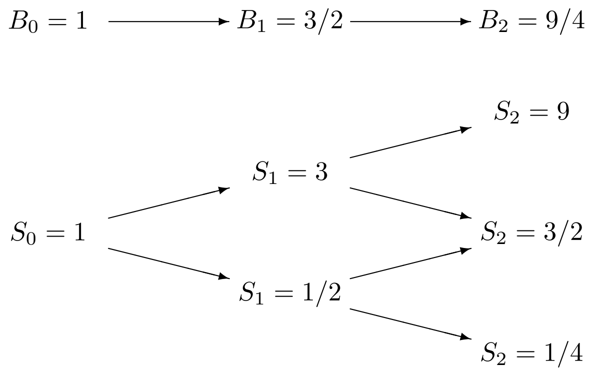
for the solution of this problem and note that

$$E(B_T^2 | \mathcal{F}_s) = B_s^2 + (T - s)$$

for $0 \leq s \leq T$. (2)

EXAMPLE 1

Consider the following two-step Cox-Ross-Rubinstein model ($a = -1/2$, $r = 1/2$, $b = 2$) for a financial market with bond B and stock S :



Assume furthermore that $P(\{\omega_i\}) = 0.25$, $1 \leq i \leq 4$.

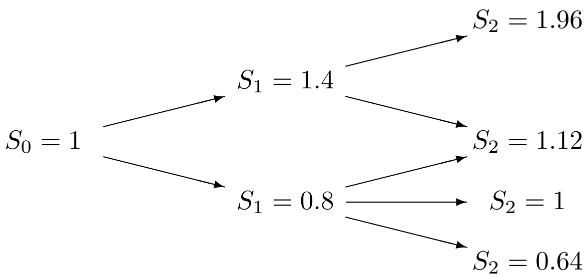
(a) Compute the set of equivalent martingale measures $\mathcal{M}^e(\tilde{S})$ and the Radon-Nikodym derivatives of those martingale measures with respect to the measure P. (1)

(b) Compute the arbitrage-free prices at time $N = 0$ and at time $N = 1$ for a European call option with strike $K = 7$ and maturity $N = 2$. (2)

(c) Compute the set K of discounted values of portfolios with zero initial wealth at time $N = 2$ and show – by the concrete numbers – the following fundamental characterization: for all random variables X we have that $X \in K$ if and only if $E_Q(X) = 0$ for all equivalent martingale measures Q. (1)

EXAMPLE 2

Consider a two-step model for a financial market with bond B and stock S. The bond B is assumed to be constant, i.e. $B \equiv 1$, the stock S is assumed to move according to the following tree:



Assume furthermore that $P(\{\omega_i\}) = 0.2$, $1 \leq i \leq 5$.

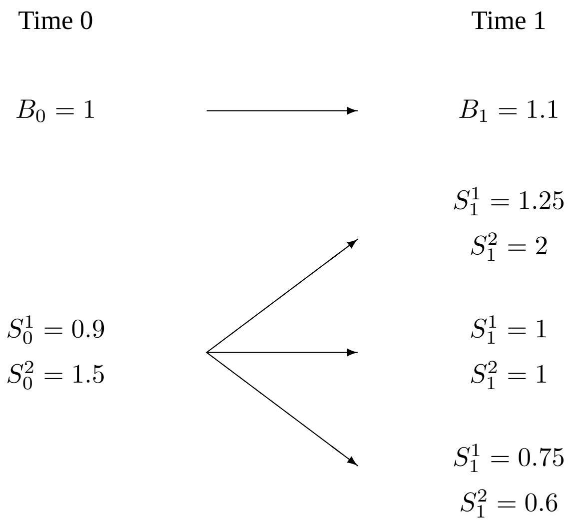
(a) Compute the set of equivalent martingale measures $\mathcal{M}^e(\tilde{S})$ as well as the set of absolutely continuous martingale measures $\mathcal{M}^a(\tilde{S})$. (1)

(b) Compute the arbitrage-free price(s) at $N = 0$ and $N = 1$ of a European put option with strike $K = 1$ and maturity $N = 2$. (2)

(c) Using the put-call-parity, compute the arbitrage-free price(s) of a European call option with strike $K = 1$ and maturity $N = 2$. (1)

EXAMPLE 3: COMPLETE MARKETS

Consider a one-period model with bond B and two stocks S^1 and S^2. Suppose that bond and stock prices evolve as follows:



(a) Show that every claim with maturity $T = 1$, that depends on S^1 and S^2, is attainable by a (B, S^1, S^2)-portfolio. (2)

(b) Calculate the set of equivalent martingale measures $\mathcal{M}^e(B, S^1, S^2)$. (1)

(c) Show that each model (B, S^1) and (B, S^2) is incomplete and compute the respective sets of equivalent martingale measures. Show that the intersection of those sets gives exactly the equivalent martingale measure of the market (B, S^1, S^2). (1)

(d) Show that the random variable S_1^2 cannot be replicated in with a portfolio in (B, S^1) and that the random variable S_1^1 cannot be replicated with a portfolio in (B, S^2). (2)