Name:

Mat.Nr.:

Bitte keinen Rotstift verwenden!

Finanzmathematik 2: Modelle in stetiger Zeit (Vorlesungsprüfung)
26. Juni 2013
Privatdoz. Dr. Stefan Gerhold

90 Minuten

Unterlagen: ein handbeschriebener A4-Zettel sowie ein nichtprogrammierbarer Taschenrechner sind erlaubt

Anmeldung zur mündlichen Prüfung via TISS möglich. Wenn zu wenig Prüfungstermine online sind, bitte den Vortragenden Stefan Gerhold kontaktieren.

Bsp.	Max.	Punkte
1	10	
2	10	
3	8	
$\sum$	28	

Schriftlich:

AssistentIn:

Mündlich:

Gesamtnote:

1. Fix a time horizon $T \in (0, \infty)$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which there is a Brownian motion $(W_t)_{0 \le t \le T}$. We take as filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ the one generated by W and augmented by the $\mathbb{P}$-nullsets in $\sigma(W_s; s \le T)$. Consider the Black Scholes model, where the bank account and the undiscounted risky asset price are given by (10 Pkt.)

$$dB_t = rB_t dt, \quad B_0 = 1,$$

and

$$dS_t = S_t(\mu dt + \sigma dW_t), \quad S_0 > 0,$$

where $\mu, r \in \mathbb{R}$ and $\sigma > 0$. Moreover, $\mathbb{P}^* \sim \mathbb{P}$ denotes the unique equivalent martingale measure for the discounted price process $\frac{S}{B}$.

   (a) Let H be a nonnegative $\mathcal{F}_T$-measurable (undiscounted) payoff due at time T.

      (i) Construct a probability measure $\widehat{\mathbb{P}} \sim \mathbb{P}^*$ such that

$$E_{\mathbb{P}^*}\left[e^{-rT}H\right] = S_0 E_{\widehat{\mathbb{P}}}\left[\frac{H}{S_T}\right].$$

      Specify in particular the candidate density process $(Z)_{0 \le t \le T}$ and show that it satisfies all necessary properties such that

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*} := Z_T \tag{1}$$

      defines an equivalent probability measure $\widehat{\mathbb{P}} \sim \mathbb{P}^*$.

      (ii) Show that

$$\widehat{W}_t := W_t^* - \sigma t \tag{2}$$

      is a $\widehat{\mathbb{P}}$-Brownian motion, where W^* denotes a $\mathbb{P}^*$-Brownian motion.

      (iii) Show that $\frac{B}{S}$ is a $\widehat{\mathbb{P}}$-martingale.

   (b) The *guarantee option* is given by the payoff

$$H = \max(\alpha S_T, g)$$

   with constants $g \ge 0$ and $0 < \alpha < 1$.

      (i) Show that the unique arbitrage-free price of H can be written as

$$\Pi_0(H) = \alpha S_0 \widehat{\mathbb{P}}\left[\alpha S_T \ge g\right] + g e^{-rT} \mathbb{P}^*\left[\alpha S_T < g\right].$$

      (ii) Derive furthermore the explicit formula given by

$$\Pi_0(H) = \alpha S_0 N\left(\frac{\ln\left(\frac{\alpha S_0}{g}\right) + \left(\frac{1}{2}\sigma^2 + r\right)T}{\sigma\sqrt{T}}\right) + e^{-rT} g N\left(\frac{\ln\left(\frac{g}{\alpha S_0}\right) + \left(\frac{1}{2}\sigma^2 - r\right)T}{\sigma\sqrt{T}}\right),$$

      where N denotes the cumulative distribution function of the standard normal distribution.

(iii) Determine the hedging portfolio V and argue that it replicates $\Pi(H)$, that is show that

$$V_t = \Pi_t(H)$$

for all $t \in [0, T]$. *Hint:* In order to prove the last assertion, you do not need to use the expression for $\Pi_t(H)$ explicitly, but rather write $g(t, S_t)$ for $\Pi_t(H)$ for some sufficiently regular function $g : [0, T] \times \mathbb{R}_+ \to \mathbb{R}$ and apply Itô's formula to $\frac{g(t,S)}{B}$.

2. Consider the setting of the Black Scholes model, as specified in the above example. An American digital call option with maturity $T > 0$ can be exercised at any time $t \in [0, T]$ at the choice of the option holder and yields the payoff $1_{[K,\infty)}(S_t)$ at time $t \in [0, T]$. The option holder wants to find a strategy which maximizes his payoff. (10 Pkt.)

   (a) Consider the following possible situations at time t:

      (i) $S_t \geq K$

      (ii) $S_t < K$

   In each case (i) and (ii), tell whether the option holder would choose to exercise the call option immediately or to wait.

   (b) Show that the price at time 0 of an American digital call option with maturity T, strike K and initial stock price $S_0 = x < K$ is given by

$$C_d^a(0, x) = \mathbb{E}_{\mathbb{P}^*}\left[e^{-r\tau_K} 1_{\{\tau_K \leq T\}} | S_0 = x\right],$$

   where

$$\tau_K = \inf\{t \geq 0 \,|\, S_t = K\}.$$

   (c) It is known that in the Black-Scholes model the price of an American digital call $C_d^a(t, S_t)$ satisfies the PDE

$$rC_d^a(t, x) = \partial_t C_d^a(t, x) + rx\partial_x C_d^a(t, x) + \frac{1}{2}\sigma^2 x^2 \partial_{xx} C_d^a(t, x)$$

   for $t \in [0, T)$ and $0 \leq x \leq K$. Determine the boundary conditions $C_d^a(t, K)$, $0 \leq t < T$ and $C_d^a(T, x)$, $0 \leq x < K$ based on your answers in a).

   (d) In the Black Scholes model the price at time t of an American digital call option with strike K can be computed via the following formula

$$C_d^a(t, x) = \frac{x}{K} N\left(\frac{\ln\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{T - t}}\right) + \left(\frac{x}{K}\right)^{-\frac{2r}{\sigma^2}} N\left(\frac{\ln\left(\frac{x}{K}\right) - \left(r + \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{T - t}}\right), \quad 0 \leq x \leq K,$$

   where N denotes the cumulative distribution function of the standard normal distribution.

   Show that this formula is consistent with the boundary conditions derived in (c).

(e) Suppose that $\mu = r = 0$. What is the probability to exercise the option in the interval $[0, T]$, if the initial stock price S_0 equals $x < K$ (and the holder of the option acts rationally).

3. Fix a time horizon $T \in (0, \infty)$ and a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$ equipped with a standard Brownian motion $(W_t)_{0 \le t \le T}$. Consider the following Itô-process model for the stock price (8 Pkt.)

$$dS_t = S_t(\mu_t dt + \sigma_t dW_t), \quad S_0 > 0,$$

where μ and $\sigma > 0$ are bounded predictable processes. Moreover, consider a continuously monitored variance swap contract with payoff

$$V_T = \frac{1}{T} \int_0^T \sigma_t^2 dt.$$

(a) Prove that

$$V_T = \frac{2}{T} \left(\int_0^T \frac{1}{S_t} dS_t - \ln\left(\frac{S_T}{S_0}\right) \right).$$

(b) Show that for $\kappa \ge 0$

$$-\ln\left(\frac{S_T}{\kappa}\right) = -\frac{S_T - \kappa}{\kappa} + \int_0^\kappa \frac{1}{K^2}(K - S_T)^+ dK + \int_\kappa^\infty \frac{1}{K^2}(S_T - K)^+ dK.$$

Hint: For a twice-differentiable function and $\kappa \ge 0$, the following formula holds:

$$\begin{aligned} f(S_T) &= f(\kappa) + f'(\kappa)(S_T - \kappa) \\ &\quad + \int_0^\kappa f''(K)(K - S_T)^+ dK + \int_\kappa^\infty f''(K)(S_T - K)^+ dK. \end{aligned}$$

(c) Let $r \ge 0$ denote the deterministic constant interest rate and let the call and put prices with maturity T and strike K be given by

$$C(K) = \mathbb{E}_{\mathbb{P}^*}\left[e^{-rT}(S_T - K)^+\right], \quad P(K) = \mathbb{E}_{\mathbb{P}^*}\left[e^{-rT}(K - S_T)^+\right],$$

where $\mathbb{P}^* \sim \mathbb{P}$ denotes some equivalent martingale measure for the discounted stock price $(e^{-rt}S_t)_{0 \le t \le T}$. Show that the price of the variance swap defined via $\mathbb{E}_{\mathbb{P}^*}[e^{-rT}V_T]$ is given by

$$\frac{2}{T} \left(\int_0^{F_T} \frac{1}{K^2} P(K) dK + \int_{F_T}^\infty \frac{1}{K^2} C(K) dK \right),$$

where $F_T = e^{rT} S_0$.