Name:

Mat.Nr.: Studium:

Bitte keinen Rotstift verwenden!

105.593 Einführung in Stochastische Prozesse und Zeitreihenanalyse
Vorlesung, 2013S, 2.0h
March 2015
Hubalek/Scherrer

(Dauer 90 Minutes, Permissible materials: one handwritten sheet, format A4, plus a non programmable calculator)

Unless you made a special arrangement for the oral exam you will receive an e-mail with your results of the written exam and the information for registering for the oral exam.

Bsp.	Max.	Punkte
1	5	
2	5	
3	5	
4	5	
$\sum$	20	

1. Consider the ARMA(1,1) process $x_t = ax_{t-1}+\epsilon_t+b\epsilon_{t-1}$, $|a| < 1$, $|b| < 1$, $(\epsilon_t) \sim \mathrm{WN}(\sigma^2)$.

   (a) Show that $x_t = y_t+by_{t-1}$, where (y_t) is an AR(1) process satisfying $y_t = ay_{t-1}+\epsilon_t$.

   (b) Show that (x_t) has the following representation as an MA(∞) process:

$$x_t = \epsilon_t + \sum_{j\geq 0}(a+b)a^j\epsilon_{t-1-j}$$

   (c) Show that the autocovariance function of (x_t) is given by:

$$\gamma(k) = \begin{cases} \dfrac{\sigma^2}{1-a^2}(1+2ab+b^2) & \text{for} k = 0 \\ \\ \dfrac{\sigma^2}{1-a^2}a^{k-1}(a+b)(1+ab) & \text{for} k > 0 \\ \\ \gamma(-k) & \text{for} k < 0 \end{cases}$$

   (d) Show that (x_t) is a white noise Process when $a+b = 0$ or $ab = -1$.

2. Suppose we are given an ARMA(2,1) process $x_t = a_1 x_{t-1} + a_2 x_{t-2} + \epsilon_t + b_1 \epsilon_{t-1}$, $(\epsilon_t) \sim$ WN(σ^2). Suppose the conditions for stability and minimum phase are satisfied and thus we have $\mathbb{H}_x(t) = \mathbb{H}_\epsilon(t)$. We consider the h-step prediction $\hat{x}_{t+h}$ for x_{t+h} from the infinite past $(x_s,\ s \leq t)$ and the corresponding prediction error $\hat{u}_{t+h} = x_{t+h} - \hat{x}_{t+h}$. Proof the following claims:

   (a) 1-step prediction ($h = 1$):

$$\begin{aligned} \hat{x}_{t+1} &= a_1 x_t + a_2 x_{t-1} + b_1 \epsilon_t \\ \hat{u}_{t+1} &= \epsilon_{t+1} \\ \mathbf{E}\hat{u}_{t+1}^2 &= \sigma^2 \end{aligned}$$

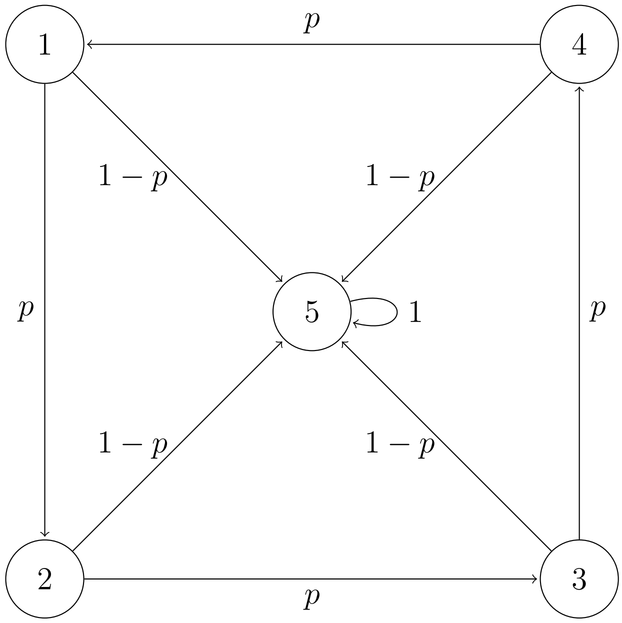
   (b) 2-step prediction ($h = 2$):

$$\begin{aligned} \hat{x}_{t+2} &= a_1 \hat{x}_{t+1} + a_2 x_t \\ \hat{u}_{t+2} &= \epsilon_{t+2} + (a_1 + b_1)\epsilon_{t+1} \\ \mathbf{E}\hat{u}_{t+2}^2 &= \sigma^2(1 + (a_1 + b_1)^2) \end{aligned}$$

   (c) 3-step prediction ($h = 3$):

$$\begin{aligned} \hat{x}_{t+3} &= a_1 \hat{x}_{t+2} + a_2 \hat{x}_{t+1} \\ \hat{u}_{t+3} &= \epsilon_{t+3} + (a_1 + b_1)\epsilon_{t+2} + (a_1^2 + a_1 b_1 + a_2)\epsilon_{t+1} \\ \mathbf{E}\hat{u}_{t+3}^2 &= \sigma^2(1 + (a_1 + b_1)^2 + (a_1^2 + a_1 b_1 + a_2)^2) \end{aligned}$$

3. Suppose you are given a Markov chain $(X_n)_{n\geq 0}$ with state space $I = \{1, 2, 3, 4, 5\}$ and transition probabilities that are specified in the graph below, where p is an arbitrary real number from the interval $(0, 1)$.



(a) Choose (yes, it's up to you!) an initial distribution λ and write it down. Furthermore, write down the transition matrix P.

(b) Let $H = \inf\{n \geq 0 : X_n = 1\}$ and $h_i = \mathbb{P}_i[H < \infty]$ for $i \in I$. Determine $h_1, \ldots, h_5$. No details required, just the result. We use the same notation as in the lecture and our book (Norris).

(c) Let $K = \inf\{n \geq 0 : X_n \in \{3, 5\}\}$. Calculate $\mathbb{E}[K]$ using the initial distribution chosen above. No details required, just the result.

(d) Investigate the convergence respectively calculate the limit of the series

$$\sum_{n=0}^{\infty} p_{ii}^{(n)}, \quad i \in I.$$

Notation as in the lecture and book.

(e) Which states are recurrent, which transient? Provide a short, but mathematically sound argument. Writing down or rewriting the definition of recurrence and transience yields no point!

4. Suppose we are given a probability space $(\Omega, \mathcal{F}, P)$ carrying a Brownian motion $(W(t), t \geq 0)$. Furthermore let $(\mathcal{F}(t), t \geq 0)$ denote the natural filtration of W.

(a) Furthermore, we are given an integer $k \geq 1$ and a process f with

$$f(t) = W(k) \cdot 1_{[k,k+1)}(t), \quad t \geq 0.$$

Show respectively argue, that $f \in M^2_{\text{step}}$.

(b) Compute the stochastic integral $I(f)$ as explicit as possible.

(c) Suppose we are given an integer $n \geq 2$ and a process g with

$$g(t) = \sum_{j=1}^{n-1} W(j) \cdot 1_{[j,j+1)}(t) \quad t \geq 0.$$

Compute expectation and variance of the random variable

$$X = \int_0^\infty g(t) dW(t).$$

(d) Show that the process Y with

$$Y(t) = W(t)^n, \quad t \geq 0,$$

is an Ito-Process and determine the initial value $Y(0)$ and the processes a and b in the corresponding representation

$$Y(t) = Y(0) + \int_0^t a(s) ds + \int_0^t b(s) dW(s), \quad t \geq 0.$$

You need not and should not show the required measurability and integrability conditions for $Y(0)$, a, and b here.

(e) (Continuation) Show that the process Z with

$$Z(t) = \cos(Y(t)), \quad t \geq 0,$$

is an Ito process and compute the initial value $Z(0)$ and the processes A and B from the corresponding representation

$$Z(t) = Z(0) + \int_0^t A(s) ds + \int_0^t B(s) dW(s), \quad t \geq 0.$$

You need not and should not show the required measurability and integrability conditions for $Z(0)$, A, and B here.