Credit Risk$^+$ - Correlations and Extensions

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Basics About Credit Portfolios

Let m be the number of individual obligors resp. credit risks and $(N_1, \ldots, N_m)$ be a random vector of Bernoulli default indicators, i.e. binary values

$$N_i = \begin{cases} 1 & \text{if obligor } i \text{ defaults (within the observed period),} \\ 0 & \text{otherwise,} \end{cases}$$

giving the number of defaults.
Furthermore, let

$$p_i := \mathbb{P}[N_i = 1] \in [0, 1]$$

denote the probability of default (within the observed period) of obligor i and the total number of defaults

$$N := \sum_{i=1}^{m} N_i. \tag{1}$$

Problems with Bernoulli Random Variables

The probability of exactly $n \in \{0, 1, \dots, m\}$ defaults is the sum over the probabilities of all the possible subsets of n obligors defaulting together, i.e.

$$\mathbb{P}[N = n] = \sum_{\substack{I \subset \{1,\dots,m\} \\ |I| = n}} \mathbb{P}[N_i = 1 \text{ for } i \in I,\, N_i = 0 \text{ for } i \in \{1, \dots, m\} \setminus I] . \tag{2}$$

Moreover, if the $N_1, \dots, N_m$ are independent, then

$$\mathbb{P}[N = n] = \sum_{\substack{I \subset \{1,\dots,m\} \\ |I| = n}} \left[\left(\prod_{i \in I} p_i \right) \prod_{i \in \{1,\dots,m\} \setminus I} (1 - p_i) \right] . \tag{3}$$

The Total Variation Metric

We will use Poisson distributed random variables instead of Bernoulli random variables. Therefore we are interested in bounds for this model error.
We define the following metric on the set of measures with mass 1 on $\mathbb{N}_0$ ($\mathcal{M}_1(\mathbb{N}_0)$):

$$d_{\mathrm{TV}}(\mu,\nu) = \sup_{A\subset\mathbb{N}_0} |\mu(A) - \nu(A)|, \qquad \mu,\nu \in \mathcal{M}_1(\mathbb{N}_0),$$

It can be shown that this defines a metric and the supremum is attained on the set

$$M := \{\, k \in \mathbb{N}_0 \mid \mu(\{k\}) > \nu(\{k\}) \,\}.$$

Poisson Approximation of the Sum of Bernoulli RV's

With the Stein-Chen method we get the following result for a Poisson approximation (see Eichelsbacher [Eic03]):

Theorem (Barbour und Hall, 1984)

Let $X_1, \ldots, X_m$ be independent Bernoulli random variables, i.e., they are $\{0, 1\}$-valued, with $p_i := \mathbb{P}[X_i = 1]$. Then $W := X_1 + \cdots + X_m$ is the random variable counting the number of ones. Define $\lambda = \mathbb{E}[W] = p_1 + \cdots + p_m$ and let $N \sim \text{Poisson}(\lambda)$. Then

$$\mathbb{P}[W \in M] - \mathbb{P}[N \in M] \leq \frac{1 - e^{-\lambda}}{\lambda} \sum_{i=1}^{m} p_i^2. \tag{4}$$

Credit Risk+

The classical CreditRisk$^+$ model was introduced by Credit Suisse First Boston ([Bos97]) and various extensions and applications are presented in the book edited by Gundlach and Lehrbass ([GE03]). The portfolio loss of a portfolio of m obligors modelled by K sources of systematic risk and idiosyncratic risk is expressed by

$$L = \sum_{i=1}^{m} \sum_{k=0}^{K} \sum_{n=1}^{N_{i,k}} L_{i,k,n} \tag{5}$$

where $N_{i,k}$ denotes the number of defaults of obligor i due to risk k. The following assumptions are made:

Default Intensities - 1

We assume that

- we derive the default intensity λ_i for obligor i from the default probability p_i coming from some a priori rating. Possible choices are $\lambda_i = p_i$ (matching of expectation) or $\lambda_i = -\ln(1-p_i)$ (matching of the probability of no default).
- there are K risk factors $\Lambda_k, k = 1, \ldots, K$ which are independently Gamma-distributed with $\mathbb{E}[\Lambda_k] = 1$ and $\mathrm{Var}(\Lambda_k) = \sigma_k^2$. These factors will scale the a priori loss intensity λ_i and model stochastic changes.
- Remark 1: The independence assumption on these factors limits the means to model dependence of the default intensities.
- Remark 2: The Gamma distribution assumption is relatively arbitrary but it makes mixing of Poisson intensities easy.

Default Intensities - 2

Finally we assume

- that the affiliation of the risk of obligor i to these factors or to idiosyncratic risk (non-systematical risk) is given by nonnegative weights $w_{i,k}$ which sum up to one (i.e. $\sum_{k=0}^{K} w_{i,k} = 1$).
- Thus the intensity conditioned on $\Lambda_1, \ldots, \Lambda_K$ is

$$\lambda_i' = \lambda_i (w_{i,0} + \sum_{k=1}^{K} \Lambda_k w_{i,k}).$$

- Due to the normalization of the risk factors we have

$$\mathbb{E}[\lambda_i'] = \lambda_i (w_{i,0} + \sum_{k=1}^{K} \mathbb{E}[\Lambda_k] w_{i,k}) = \lambda_i.$$

Default Numbers

1. The default numbers due to idiosyncratic risk $N_{1,0}, N_{2,0}, \ldots, N_{m,0}$ are independent from everything else.
2. Every non-idiosyncratic default number $N_{i,k}$ is conditionally Poisson distributed with parameter given as product of the individual default intensity λ_i, the risk contribution factor $w_{i,k}$, and the sector risk Λ_k. This is

$$\mathcal{L}\left(N_{i,k} \middle| \Lambda_1, \ldots, \Lambda_m \right) \overset{\text{a.s.}}{=} \mathcal{L}\left(N_{i,k} \middle| \Lambda_k \right) \overset{\text{a.s.}}{=} \text{Poisson}(\lambda_i w_{i,k} \Lambda_k)\,. \tag{6}$$

3. Conditionally on $\Lambda_1, \ldots, \Lambda_K$, the family of sector risk based defaults $N_{i,k}$ is independent.

Loss Granularity

- Losses are integer multiples of the basic loss unit E.
- Thus we have to round real (stochastic or deterministic) exposures to multiples of this loss unit E. There are various methods to reduce this rounding error (stochastic rounding, interpolation of the quantile).
- Losses are non-negative. Thus we don't have any profit scenario.

Stochastic Loss Given Default

- Let $(L_{i,k,n})_{n\in\mathbb{N}}$ denote the loss of obligor i due to a default event caused by risk factor k given the n^{th} default. This sequence is iid and independent of all other random variables.
- We denote by $(q_{i,k,\nu})_{\nu\in\mathbb{N}_0}$ the distribution $(L_{i,k,n})_{n\in\mathbb{N}}$. This is

$$\mathbb{P}[L_{i,k,n} = \nu] = \mathbb{P}[L_{i,k,1} = \nu] = q_{i,k,\nu}.$$

- Moreover let $\varphi_{L_{i,k,1}}(s)$ denote the probability generating function of $L_{i,k,1}$ (i.e. $\mathbb{E}[s^{L_{i,k,1}}]$)- where we take the first loss given default as a kind of prototype of the iid sequence.

Weighted Probability Function - Definition

Fix $\gamma = (\gamma_1, \ldots, \gamma_K) \in [0, \infty)^K$. Then we calculate the weighted probability generating function

$$\varphi_{L,\gamma}(s) := \mathbb{E}\Big[\Lambda_1^{\gamma_1} \ldots \Lambda_K^{\gamma_K} s^L\Big]$$

of the total loss $L = \sum_{i=1}^m \sum_{k=0}^K \sum_{n=1}^{N_{i,k}} L_{i,k,n}$.
For the vector $\gamma = 0$, we will obtain the usual generating function φ_L of L. Vectors γ of the form $(0, \ldots, 1, \ldots, 0)$ will be useful later on for calculating risk contributions.

Weighted Probability Function - Properties

Consider the X-weighted pgf

$$\varphi_{L,X}(s) := \mathbb{E}[Xs^L] = \sum_{n=0}^{\infty} \mathbb{E}[X1_{\{L=n\}}]s^n$$

for some integrable random variable X which is meaningful at least for $|s| < 1$. Similarly to ordinary pgfs we see that

$$\varphi_{L,X}^{(n)}(0) = n!\mathbb{E}[X1_{\{L=n\}}]$$

which are exactly the quantities needed in order to calculate risk contributions (see the section later).

Weighted Probability Function - Calculation

Together with the assumptions on conditional independence of the obligors and independence of the risk factors we arrive at the following form of the weighted pgf:

$$\varphi_{L,\gamma}(s) = C_\gamma \exp\left(\psi_0(s) - \sum_{k=1}^{K}(\alpha_k + \gamma_k)\ln\left(1 - \frac{\psi_k(s)}{\beta_k}\right)\right)$$

where the expressions $\psi_k(s)$ are of the form

$$\psi_k(s) = \sum_{i=1}^{m} \lambda_i w_{i,k}(\varphi_{L_{i,k,1}}(s) - 1).$$

and the constant C_γ equals 1 in the 'interesting' cases where the vector γ_k has entries 0 and 1 only.

Calculating the Distribution

- Rearrangements and considerations about the expansion of logarithms and exponentials of power series provide algorithms to calculate the distribution.
- Haaf, Reiß and Schoenmakers ([HRS03]) proved that these algorithms are numerically stable.
- Rather basic algorithms like the Cauchy product of series, expansions of a logarithm and Leibnitz' rule for higher derivatives are used.

VaR - 1

Definition

For a real-valued random variable X and a level $\delta \in (0,1)$, define the δ-quantile of X by

$$q_\delta(X) = \min\{x \in \mathbb{R} \mid \mathbb{P}[X \leq x] \geq \delta\}. \tag{7}$$

- By the right-continuity of the distribution function this minimum exists.
- This definition of a quantile is often denoted by $q_\delta(X)^-$ and referred to as lower quantile [Del00, AT02].

The quantile $q_\delta(X)$ of a loss variable X is also called Value-at-Risk (VaR) at level $1 - \delta$.

VaR - 2

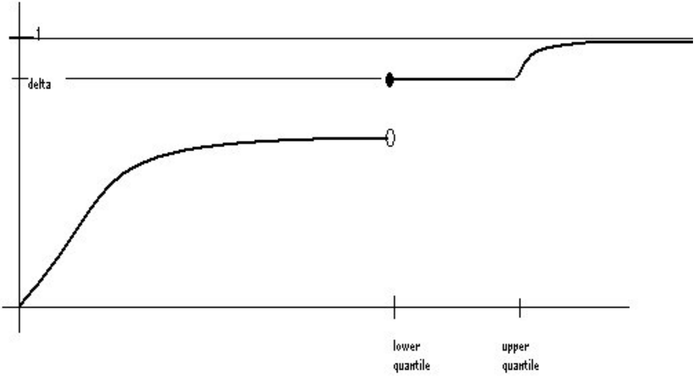
Rewriting (7) as

$$q_\delta(X) = \min\{x \in \mathbb{R} \mid \mathbb{P}[X > x] \leq 1 - \delta\},$$

we see that $q_\delta(X)$ is the smallest threshold which is exceeded by the loss X with probability at most $1 - \delta$.

- Attention: VaR lacks the theoretically desirable property of subadditivity which has the economical meaning of positive effects of diversification.
- It just measures the 'starting point' of a certain tail and not its shape.
- But it is wide spread and often mentioned in documents on regulatory capital.
- Can be calculated in the Credit Risk$^+$ framework by adding up the probabilities.

Lower and Upper Quantile



ES - Definition

Definition (Expected Shortfall)

The expected shortfall of the loss variable X at level $\delta \in (0,1)$ is defined as

$$\mathrm{ES}_\delta[X] = \frac{\mathbb{E}\left[X 1_{\{X > q_\delta(X)\}}\right] + q_\delta(X)(\mathbb{P}[X \leq q_\delta(X)] - \delta)}{1-\delta}.$$

If $\mathbb{P}[X \leq q_\delta(X)] = \delta$, in particular if the distribution function is also left-continuous at $x = q_\delta(X)$, the definition simplifies to

$$\mathrm{ES}_\delta[X] = \mathbb{E}[X \,|\, X > q_\delta(X)]\,.$$

ES - Properties

Expected shortfall fulfills a long list of desirable theoretical properties:

- monotonicity: $X \leq 0 \Rightarrow \mathrm{ES}_\delta[X] \leq 0$.
- positive homogeneity: $\mathrm{ES}_\delta[hX] = h\,\mathrm{ES}_\delta[X], h > 0$.
- translation invariance: $a \in \mathbb{R}$ then $\mathrm{ES}_\delta[X + a] = \mathrm{ES}_\delta[X] + a$. The latter three together are summarized by coherence.
- convexity: If $\alpha \in [0, 1]$, then

$$\mathrm{ES}_\delta[\alpha X + (1 - \alpha)Y] \leq \alpha\,\mathrm{ES}_\delta[X] + (1 - \alpha)\,\mathrm{ES}_\delta[Y].$$

ES - Calculation

- The quantities derived from the distribution (probabilities, quantiles) needed to calculate expected shortfall can already be calculated.
- The quantity $\mathbb{E}[L1_{\{L>q_\delta(L)\}}] = \mathbb{E}[L] - \mathbb{E}[L1_{\{L\leq q_\delta(X)\}}]$ can be calculated by

$$\mathbb{E}[L1_{\{L\leq q_\delta(L)\}}] = \sum_{l=1}^{q_\delta(L)} l\mathbb{P}[L = l].$$

Default Correlations in CRP

We calculate the covariance of N_i and N_j representing the number of defaults of obligors i and j, respectively. Since their conditional distribution given the risk factor vector Λ is Poisson with parameter $\lambda_i' = \lambda_i(w_{i,0} + \sum_{k=1}^{K} w_{ik}\Lambda_k)$ we obtain

$$\begin{aligned} \mathrm{Cov}(N_i, N_j) &= \mathbb{E}[\mathrm{Cov}(N_i, N_j|\Lambda)] + \mathrm{Cov}(\mathbb{E}[N_i|\Lambda], \mathbb{E}[N_j|\Lambda]) \\ &= \mathbb{E}[\mathbf{1}_i(j)\lambda_i'] + \mathrm{Cov}\left(\lambda_i', \lambda_j'\right) \\ &= \mathbf{1}_i(j)\lambda_i + \lambda_i\lambda_j \sum_{k,l=1}^{K} w_{ik}w_{jl}\,\mathrm{Cov}(\Lambda_k, \Lambda_l)\,. \end{aligned} \tag{8}$$

Default Correlations in CRP

We see that the covariance between default counters depends on the covariance structure of the sectors. One can write the default correlation keeping only terms of first order in the default intensities as

$$\rho_{i,j} = \frac{\mathrm{Cov}(N_i, N_j)}{\sqrt{\mathrm{Var}(N_i)}\sqrt{\mathrm{Var}(N_j)}} \approx \sqrt{\lambda_i \lambda_j} \sum_{k,l=1}^{K} w_{ik} w_{jl} \sigma_{kl} \quad \text{for } i \neq j \quad (9)$$

where σ_{kl} denotes the covariance between risk sector k and l.

- Keep in mind that the approximation in (9) is only valid if the intensities and the covariances are sufficiently small.

Sectors equal Industries

This is the original approach introduced by Credit Suisse First Boston [Bos97].

- Each obligor belongs to the risk sector of his/her industry sector.
- Only defaults in the same industry sector are dependent.

Tractable first approach - but does not reflect empirically observed high correlations (up to 0.8).

One Risk Sector

Due to high correlations observed one could assign all obligors to only 1 risk sector. This sector can be interpreted as overall country risk.

- This assumes linear correlation of $\sqrt{\lambda_i \lambda_j}\,\mathrm{Var}(\Lambda)$ between defaults.
- This will probably overestimate the risk.
- Advantage: Data is accessible from public sources.

Variance Matching - 1

Bürgisser et al. [BKWW99] propose to use one sector whose variance equals the variance of the loss modelled using industries.

- For 1 sector we have $\sigma_L^2 = \sigma_\Lambda^2 \mathbb{E}[L]^2 + \sum_i \lambda_i \nu_i^2$, where ν_i denotes the exposure of obligor i.
- For K risk sectors we get

$$\begin{aligned}\sigma_L^2 &= \sum_{k=1}^{K} \sigma_{\Lambda_k}^2 \mathbb{E}[L_k]^2 + \sum_{\substack{k,l \\ k \neq l}} \mathrm{corr}(\Lambda_k, \Lambda_l)\sigma_k \sigma_l \mathbb{E}[L_k]\mathbb{E}[L_l] \\ &+ \sum_{k=1}^{K} \sum_{i \in \Lambda_k} \lambda_i \nu_i^2.\end{aligned}$$

Variance Matching - 2

Thus we match the systematic variance terms and use this variance in the 1 sector model:

$$\sigma_\Lambda^2 \mathbb{E}[L]^2 = \sum_{k=1}^{K} \sigma_{\Lambda_k}^2 \mathbb{E}[L_k]^2 + \sum_{\substack{k,l \\ k \neq l}} \operatorname{corr}(\Lambda_k, \Lambda_l) \sigma_k \sigma_l \mathbb{E}[L_k] \mathbb{E}[L_l].$$

PGF of the loss by the MGF of the Factors

We can write the conditional pgf of the loss as

$$\varphi_L(s|\Lambda) = \exp(\sum_{k=0}^{K} \Lambda_k \psi_k(s))$$

where $\psi_k(s) = \sum_{i=1}^{m} \lambda_i w_{ik}(\varphi_{L_{i,k,1}}(s) - 1)$ and $\Lambda_0 = 1$.
By taking expectation with respect to the risk factor vector Λ:

$$\varphi_L(s) = \mathbb{E}[\varphi_L(s|\Lambda)] = \mathbb{E}[\exp(\sum_{k=0}^{K} \Lambda_k \psi_k(s))]$$

which corresponds to the valuation of the mgf of the sector variables at the vector $P = (\psi_0(s), \dots, \psi_K(s))$.

Hidden Gamma Model

Götz Giese [Gie] proposes to set

$$\Lambda_k = \sigma_k^2 (Y_k + Y)$$

with independent standard Gamma rv's Y_k and Y $(Y_K \sim \Gamma(\alpha_k, 1), Y \sim \Gamma(\alpha, 1))$. Covariances among sectors are

$$\mathrm{Cov}(\Lambda_k, \Lambda_l) = \sigma_k^2 \sigma_l^2 (\mathbf{1}_k(l)\alpha_k + \alpha).$$

Hidden Gamma Model - Default Covariances

We see that the covariance of 2 different default counting rv's in this model is given by

$$\mathrm{Cov}(N_i, N_j) = \lambda_i \lambda_j \sum_{k,l=1}^{K} w_{ik} w_{jl} \sigma_{ll} \sigma_{kk} (1_{\{k=l\}} \alpha_k + \alpha). \qquad (10)$$

Thus we can control the covariance by choosing α (in some bounds due to the normalization of the risk factors).

Hidden Gamma Model - PGF

The pgf is given by

$$\varphi_L(s) = \exp(\psi_0(s) + \sum_{k=1}^{K}(\alpha - \frac{1}{\sigma_{kk}}) \ln(1 - \sigma_{kk}\psi_k(s)) - \alpha \ln(1 - \sum_{k=1}^{K}\sigma_{kk}\psi_k(s)))$$

which corresponds to the classical model if α equals 0. This expression can be expanded in a numerically stable way.

Hidden Gamma Model - Weighted PGF

For a less general form of the weighted pgf namely $\mathbb{E}[\Lambda_l \exp(s^L)] = \varphi_{L,\Lambda_l}(s)$ one can find the following closed form which can be implemented:

$$\begin{aligned}\varphi_{L,\Lambda_l}(s) &= \sigma_l^2(\exp(\psi_0(s) - \sum_{k=1}^{K} \alpha_k \ln(1 - \sigma_k^2 \psi_k(s)) - \\ &\quad \alpha \ln(1 - \sum_{k=1}^{K} \sigma_k^2 \psi_k(s))) \\ &\quad (\alpha_l/(1 - \sigma_l^2 \psi_l(s)) + \alpha/(1 - \sum_{k=1}^{K} \sigma_k^2 \psi_k(s))). \quad (11)\end{aligned}$$

We see that this expression is consistent with the case of no sector correlations which means setting $\alpha = 0$.

Compound Gamma Model

In this model risk factor variables are considered which are independently gamma distributed conditioned on a positive random variable T.

- For each Λ_k we have the conditional shape parameter $\alpha_k T$ with $\alpha_k > 0$ and the scale parameter β_k is assumed to be constant.
- T itself is assumed to be $\Gamma(\alpha, \beta)$ with $\alpha = \beta = \frac{1}{\sigma^2}$.
- Thus we have $\mathbb{E}[T] = 1$ and $\mathrm{Var}(T) = \sigma^2$.

Compound Gamma Model - Sector Covariance

For the covariance among sectors we get

$$\begin{aligned}\mathrm{Cov}[\Lambda_k, \Lambda_l] &= \mathbb{E}[\mathrm{Cov}[\Lambda_k, \Lambda_l | T]] + \mathrm{Cov}(\mathbb{E}[\Lambda_k | T]], \mathbb{E}[\Lambda_l | T]) \\ &= \mathbf{1}_k(l)\frac{1}{\beta_k} + \sigma^2.\end{aligned}$$

Which leads to default covariations of the form

$$\mathrm{Cov}(N_i, N_j) = \lambda_i \lambda_j \sum_{k,l=1}^{K} w_{ik} w_{jl} (\mathbf{1}_k(l)\frac{1}{\beta}_k + \sigma^2).$$

Compound Gamma Model - PGF

The pgf in this model can be calculated by conditioning on T and then taking the expectation. The pgf is given by

$$\varphi_L(s) = \exp(\psi_0(s) - \frac{1}{\sigma^2}\ln(1+\sigma^2\sum_{k=1}^{K}\beta_k \ln(1-\frac{\psi_k(s)}{\beta_k})))$$

an expression which can be expanded using the expansions of logarithms and exponentials. One can show that the structure of the coefficients again allows numerically stable computations. Using L'Hopital's rule to see that

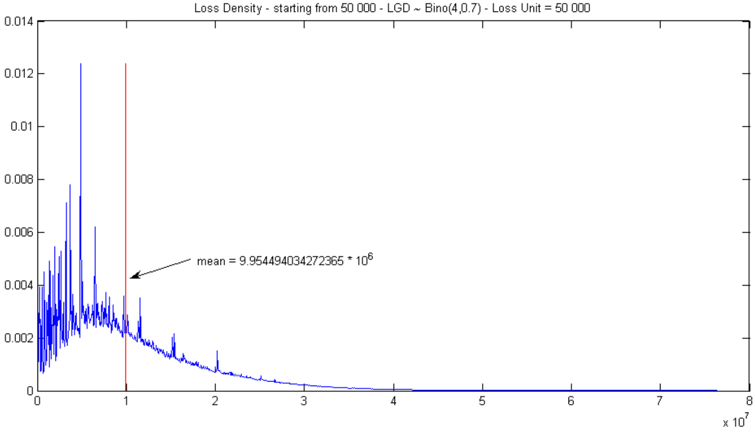
$$\lim_{x\to 0}\frac{\ln(1+ax)}{x} = a \quad \text{for } a \in \mathbb{R}$$

we recognize that we get the result from independent sectors back if we let σ^2 tend to 0.

Example - 1

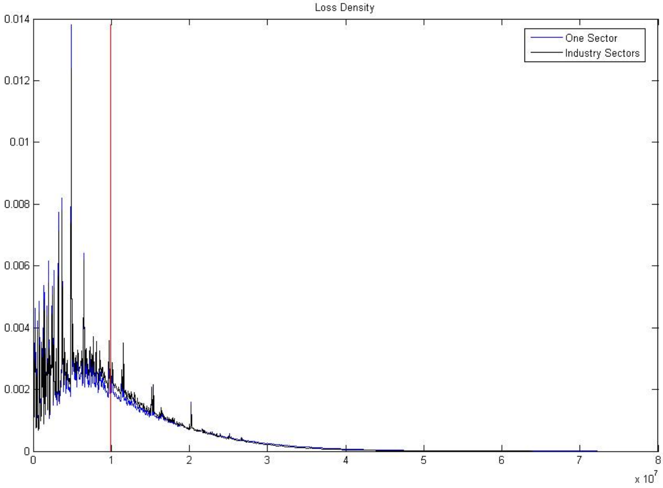
The following plot shows the discrete probability function of a portfolio with the following characteristics:

- 25 obligors, 3 industry sectors, risk affiliation to the specific sector
- pd: 0.05 - 0.3, $std(pd) \sim 0.5 * pd$
- LGD: each exposure split unto 4 fractions, LGD stochastic with $\mathbb{P}[lgd = \nu \frac{k}{4}]$ is the probability of a Bino(4,0.7) rv.

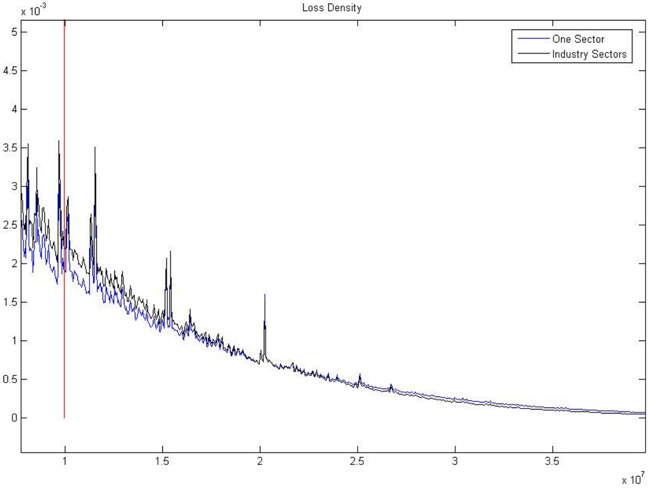

Loss Density - starting from 50 000 - LGD ~ Bino(4,0.7) - Loss Unit = 50 000
mean = 9.954494034272365 * 10^6
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
0
1
2
3
4
5
6
7
8
x 10^7

Example - 2

Now we have the same data as before, but we use only one sector in order to show the influence of implicit modelling of correlations by the weights.


Loss Density
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
0
1
2
3
4
5
6
7
8
x 10^7
One Sector
Industry Sectors

The following plot shows the region around the mean value in more detail.



Capital Allocation

Kalkbrener [Kal05] introduced an axiomatic approach to capital allocation consistent with coherent risk measures.

- Let $L_1(\mathbb{P})$ denote the space of all integrable random variables $X\colon \Omega \to \mathbb{R}$ on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.
- Then, if $Y \in L_1(\mathbb{P})$ denotes the portfolio loss and $X_1, \dots, X_n \in L_1(\mathbb{P})$ with $X_1 + \cdots + X_n = Y$ denote the losses of the n subportfolios, we can ask how to allocate the risk capital $\mathrm{ES}_\delta[Y]$ to the n subportfolios in an risk-adequate way.

Allocation with ES - definition

Definition (Allocation of risk capital by expected shortfall)

For a subportfolio loss $X \in L_1(\mathbb{P})$ within a portfolio loss $Y \in L_1(\mathbb{P})$ define the expected shortfall contribution at level $\delta \in (0,1)$ of X to Y by

$$\mathrm{ES}_\delta[X, Y] = \frac{\mathbb{E}[X 1_{\{Y > q_\delta(Y)\}}] + \beta_Y \, \mathbb{E}[X 1_{\{Y = q_\delta(Y)\}}]}{1 - \delta} \tag{12}$$

with β_Y defined by

$$\beta_Y := \begin{cases} \frac{\mathbb{P}[Y \le q_\delta(Y)] - \delta}{\mathbb{P}[Y = q_\delta(Y)]} & \text{if } \mathbb{P}[Y = q_\delta(Y)] > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Properties

Expected shortfall contribution at level $\delta \in (0,1)$ has the following properties:

1. Consistency with expected shortfall: $\mathrm{ES}_\delta[X,X] = \mathrm{ES}_\delta[X]$ for all $X \in L_1(\mathbb{P})$.
2. Diversification: $\mathrm{ES}_\delta[X,Y] \leq \mathrm{ES}_\delta[X,X]$ for all $X,Y \in L_1(\mathbb{P})$.
3. Linearity: For all $\alpha, \beta \in \mathbb{R}$ and $X,Y,Z \in L_1(\mathbb{P})$,

$$\mathrm{ES}_\delta[\alpha X + \beta Y, Z] = \alpha\,\mathrm{ES}_\delta[X,Z] + \beta\,\mathrm{ES}_\delta[Y,Z].$$

Calculation in CRP

Lemma

For every obligor $i \in \{1, \dots, m\}$ *and total loss* $l \in \mathbb{N}_0$,

$$\mathbb{E}[L_{i,0} 1_{\{L=l\}}] = \lambda_i w_{i,0} \sum_{\nu=1}^{l} \nu \, \mathbb{P}[L = l - \nu] q_{i,0,\nu} \tag{14}$$

and, for every risk $k \in \{1, \dots, K\}$,

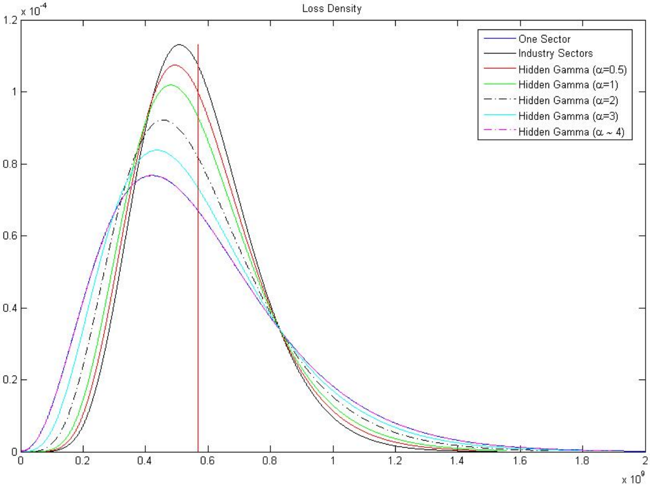
$$\mathbb{E}[L_{i,k} 1_{\{L=l\}}] = \lambda_i w_{i,k} \sum_{\nu=1}^{l} \nu \, \mathbb{E}[\Lambda_k 1_{\{L=l-\nu\}}] q_{i,k,\nu}. \tag{15}$$

Influence on Risk Contributions

Sector	1	Sum	Ratio
Obligors			
1	149027.51	149027.51	134%
2	467968.57	467968.57	130%
3	1377074.98	1377074.98	128%
4	150211.47	150211.47	123%
5	616668.72	616668.72	121%
6	378579.27	378579.27	121%
7	1012675.84	1012675.84	121%
8	414248.76	414248.76	120%
9	4865814.81	4865814.81	119%
10	14026490.8	14026490.8	94%
11	486203.29	486203.29	169%
12	286701.07	286701.07	166%
13	791006.63	791006.63	161%
14	3194806.87	3194806.87	153%
15	350413.86	350413.86	142%
16	582353.52	582353.52	133%
17	611403.08	611403.08	133%
18	284683.57	284683.57	131%
19	292549.12	292549.12	131%
20	516825.09	516825.09	126%
21	529775.88	529775.88	126%
22	1101510.67	1101510.67	125%
23	895410.34	895410.34	125%
24	4882783.85	4882783.85	122%
25	9117951.81	9117951.81	101%
	47383139.4		
Sum over all		47383139.4	
ES 99%		47383139.4	
Quantile		40250000.00	
Ratio of ES		110.82%	
Ratio of Quantile		109.82%	

	1	2	3	Sum
Obligors				
1	111023.92	0	0	111023.92
2	359986.41	0	0	359986.41
3	1074084.9	0	0	1074084.9
4	121984.45	0	0	121984.45
5	508239.27	0	0	508239.27
6	312502.41	0	0	312502.41
7	839897.8	0	0	839897.8
8	344140.29	0	0	344140.29
9	4094618.05	0	0	4094618.05
10	14936659.4	0	0	14936659.4
11	0	287449.24	0	287449.24
12	0	172827.09	0	172827.09
13	0	490811.56	0	490811.56
14	0	2085314.93	0	2085314.93
15	0	246146.84	0	246146.84
16	0	0	437599.29	437599.29
17	0	0	460337.63	460337.63
18	0	0	217657.77	217657.77
19	0	0	223980.18	223980.18
20	0	0	409717.11	409717.11
21	0	0	420881.21	420881.21
22	0	0	877715.01	877715.01
23	0	0	718002.94	718002.94
24	0	0	4014915.65	4014915.65
25	0	0	8990019.63	8990019.63
	22703136.9	3282549.66	16770826.4	
	Sum over all			42756513
ES 99%		42756513		
Quantile			36650000.00	

In this plot you see loss distribution of a portfolio consisting of 1000 obligors calculated with 7 models.

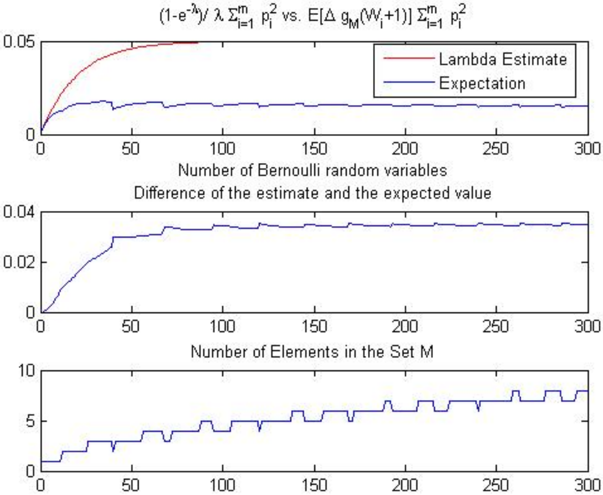


Example for a Stein-Chen Estimate - Binomial Case

In the case of $p_i = p$ for $i = 1, \ldots, m$ the situation simplifies because we have a binomial distribution. $\lambda = mp$ and

$$\frac{1 - e^{-\lambda}}{\lambda} \sum_{i=1}^{m} p_i^2 = (1 - \exp(-mp))p \to p \text{ for } m \to \infty.$$

This situation is illustrated in the following plot where p equals 0.05 and m, the number of Bernoulli rv's, ranges from 1 to 300:


(1-e^-λ)/ λ Σ^m_i=1 p_i^2 vs. E[Δ g_M(W_i+1)] Σ^m_i=1 p_i^2
Lambda Estimate
Expectation
0.05
0
0
50
100
150
200
250
300
Number of Bernoulli random variables
Difference of the estimate and the expected value
0.04
0.02
0
Number of Elements in the Set M
10
5

The Poisson Distribution

A rv N has distribution Poisson(λ) if

$$\mathbb{P}[N = k] = \frac{\lambda^k}{k!} e^{-\lambda}.$$

Thus $\mathbb{E}[X] = \lambda$ and $\mathrm{Var}(X) = \lambda$. For the probability generating function the following holds

$$\mathbb{E}[s^N] = \varphi(s) = \exp(\lambda(s-1)) \quad , s \in \mathbb{R}.$$

The Gamma Distribution

A rv X has distribution $\Gamma(\alpha, \beta,)$ if its density for $x > 0$ is

$$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}.$$

Thus $\mathbb{E}[X] = \frac{\alpha}{\beta}$ and $\mathrm{Var}(X) = \frac{\alpha}{\beta^2}$.
The following identity is useful calculating general moments:

$$\mathbb{E}[X^\gamma e^{Xz}] = \frac{\Gamma(\alpha+\gamma)}{\beta^\gamma \Gamma(\alpha)} (1 - z/\beta)^{-(\alpha+\gamma)}$$

for $\gamma \in [0, \infty)$ and $z \in (-\infty, \beta)$.

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