Optimal risk sharing for

law invariant monetary utility functions

Walter Schachermayer, TU Wien

Optimal Risk Sharing is a classical topic in Mathematical Economics

Actuarial literature on Re-Insurance:

- Borch (1962)
- Arrow (1963)
- Bühlmann (1979, 1984)
- Gerber (1979)

  ...

- Barrieu, El Karoui (2002-2005)
- Dana, Scarsini (2005)

A new ingredient is the increasing use of *Risk Measures* in the finance industry (Basel II).

Definition

A functional $\rho : L^\infty(\Omega, \mathcal{F}, \mathbf{P}) \to \mathbb{R}$ is a *convex risk measure* if it is

- *monotone*, i.e., $X_1 \leq X_2 \Rightarrow \rho(X_1) \geq \rho(X_2)$
- *convex*, i.e., $\rho(\lambda X_1 + (1-\lambda)X_2) \leq \lambda\rho(X_1) + (1-\lambda)\rho(X_2)$ for $0 \leq \lambda \leq 1$
- *cash invariant*, i.e., $\rho(X + const) = \rho(X) - const$

Example 1

Value at Risk $\mathsf{V@R}_\alpha$:

$$\mathsf{V@R}_\alpha(X) = -\sup\{x \mid \mathbf{P}[X \le x] \le \alpha\},$$

where $\alpha \in]0,1[$, e.g., $\alpha = 5\%$.

This “risk measure” fails to be convex!

Example 2

Average Value at Risk AV@R$_\alpha$:

$$\text{AV@R}_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha \text{V@R}_\gamma(X)d\gamma.$$

Artzner, Delbaen, Eber, Heath (1997-2005): coherent measures of risk.

Example 3

Standard Deviation Principle:

$$\rho(X) = -\mathbf{E}[X] + \beta\mathbf{E}\left[\Big(X - \mathbf{E}[X]\Big)^2\right]^{\frac{1}{2}}$$

where $\beta \geq 0$.

This “risk measure” fails to be monotone.

Example 4

Semi-Deviation Principle:

$$\rho(X) = -\mathbf{E}[X] + \beta \mathbf{E}\left[\left(X - \mathbf{E}[X]\right)_{-}^{2}\right]^{\frac{1}{2}}$$

Example 5

Entropic Risk Measure:

$$\rho(X) = -\frac{1}{\gamma} \ln \mathbf{E}\left[e^{-\gamma X}\right],$$

for $\gamma > 0$.

Basic Question

Two (or n) economic agents, $i = 1, 2$, are endowed with risky portfolios $X_1, X_2 \in L^\infty(\Omega, \mathcal{F}, \mathbf{P})$ and use risk measures ρ_1, ρ_2. What kind of risk exchange do we expect to happen?

More Formally

We look for $\xi_1, \xi_2 \in L^\infty(\Omega, \mathcal{F}, \mathbf{P})$ sucht that $X_1 + X_2 = \xi_1 + \xi_2$ and such that the agents $i = 1, 2$ are "happier" with ξ_i then with X_i (where "happiness" is measured by the risk measures ρ_i).

Definition (to relate convex risk measures with classical utility theory)

A function $U : L^\infty(\Omega, \mathcal{F}, \mathbf{P}) \to \mathbb{R}$ is a *monetary utility function* if U is

- *monotone*, i.e., $X_1 \leq X_2 \Rightarrow U(X_1) \leq U(X_2)$
- *concave*,
- *cash invariant*, i.e., $U(X + const) = U(X) + const$

Obvious

U is a monetary utility function $\Leftrightarrow$ $\rho := -U$ is a convex risk measure.

What does it mean to be “happier” after the risk exchange for agents $i = 1, 2$ endowed with risky position $X_i \in L^\infty(\Omega, \mathcal{F}, \mathbf{P})$ and a monetary utility function)?

We call a pair $\xi_1, \xi_2 \in L^\infty \times L^\infty$ *an admissible allocation* if $\xi_1 + \xi_2 = X_1 + X_2$.

Definition

An admissible allocation (ξ_1, ξ_2) is *Pareto optimal* if, for each admissible allocation (η_1, η_2) with

$$U_1(\eta_1) \geq U_1(\xi_1), \quad U_2(\eta_2) \geq U_2(\xi_2)$$

implies that

$$U_1(\eta_1) = U_1(\xi_1), \quad U_2(\eta_2) = U_2(\xi_2).$$

Observation

For *monetary* utility functions U_1, U_2 and a Pareto optimal allocation (ξ_1, ξ_2) we have that $(\xi_1 + const, \xi_2 - const)$ is Pareto optimal too, for each $const \in \mathbb{R}$.

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Definition

An admissible allocation (ξ_1, ξ_2) is *individually rational* if

$$U_1(\xi_1) \geq U_1(X_1), \quad U_2(\xi_2) \geq U_2(X_2).$$

Remark (speaking economically)

The search of an optimal (i.e. Pareto optimal and individually rational) risk exchange has two aspects: firstly the two agents have a *common interest* to find a Pareto optimal allocation (ξ_1, ξ_2); secondly they have an *adverse interest* in fixing the price c.

Proposition

Given X_1, X_2 and U_1, U_2 as above, there is $r \geq 0$, called the *rent of risk exchange*, such that, for every Pareto optimal admissible allocation (ξ_1, ξ_2) we have

$$U_1(\xi_1) + U_2(\xi_2) = U_1(X_1) + U_2(X_2) + r$$

For each admissible, Pareto optimal allocation ξ_1, ξ_2,... there is an interval $[c_1, c_2] \subseteq \mathbb{R}$ with $c_2 - c_1 = r$ and such that $(\xi_1 + c, \xi_2 - c)$ is individually rational iff $c \in [c_1, c_2]$.

Example

(one of the results of the paper [Jouini, Schachermayer, Touzi])

Suppose that $U_1(X) = -\rho_{\mathsf{AV@R}_\alpha}(X)$ and $U_2(\,.\,)$ is in a rather general class of monetary utility functions (including the "semi-deviation" as well as the "entropic" utility).

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Then there is a **unique** (up to a constant) Pareto optimal admissible allocation, namely

$$\xi_1 = -(X-k)_-, \quad \xi_2 = X + (X-k)_-$$

for some $k \in \mathbb{R}$ (which can, in principle, be computed).

Mathematical Tools

Legendre-Fenchel transform:

$$U_i^*(f) = \sup_{X \in L^\infty(\Omega, \mathcal{F}, \mathbf{P})} [U_i(X) - \langle X, f \rangle],$$

$f \in L^1(\Omega, \mathcal{F}, \mathbf{P})$ (or, maybe, $f \in L^\infty(\Omega, \mathcal{F}, \mathbf{P})^*$).

Remark: The cash invariance and monotonicity of U_i implies that U_i is Lipschitz w.r. to $\| . \|_\infty$. Hence one is sure that the duality theory works for the dual pair $\langle L^\infty, (L^\infty)^* \rangle$:

$$U_i(X) = \inf_{f \in (L^\infty)^*} [U_i^*(f) + \langle X, f \rangle]. \qquad \text{(D*)}$$

If U_i is $\sigma(L^\infty, L^1)$ lower semi-continous then we have the stronger assertion

$$U_i(X) = \inf_{f \in (L^1)} [U_i^*(f) + \langle X, f \rangle]. \qquad \text{(D)}$$

Wellknown facts:

$$f \in supergrad\,(U_i(X)) \Leftrightarrow -X \in subgrad\,(U_i^*(f))$$

Sup-Convolution:

$$U_1 \Box U_2(X) = \sup_{\xi_1+\xi_2=X} U_1(\xi_1) + U_2(\xi_2).$$

$$(U_1 \Box U_2)^* = U_1^* + U_2^*$$

Back to the problem of finding the admissible allocation (ξ_1, ξ_2) which is Pareto-optimal. Let U_1, U_2, and the total risk $X = X_1 + X_2$ be fixed.

Necessary and sufficient condition for Pareto-optimality (first order condition)

$$supergrad\,(U_1(\xi_1)) \cap supergrad\,(U_2(\xi_2)) \neq \emptyset \qquad \text{(FO)}$$

Admitting the formula

$$subgrad\,(U_1^* + U_2^*) = subgrad\,(U_1^*) + subgrad\,(U_2^*) \qquad (*)$$

we have the following recipe to find a Pareto optimal allocation for a given total risk $X \in L^\infty(\Omega, \mathcal{F}, \mathbf{P})$.

Find $f \in supergrad(U_1 \Box U_2)(X)$ then we have

$$
\begin{aligned}
-X \in subgrad\left((U_1 \Box U_2)^*(f)\right) &= subgrad\left(U_1^*(f) + U_2^*(f)\right) \\
\text{Supposing (*):} \qquad &= subgrad\left(U_1^*(f)\right) + subgrad\left(U_2^*(f)\right).
\end{aligned}
$$

Now choose $-\xi_i \in subgrad\left(U_i^*(X)\right)$ such that $\xi_1 + \xi_2 = X$.

As $f \in supergrad\left(U_1^*(\xi_1)\right) \cap supergrad\left(U_2^*(\xi_2)\right)$ the first order condition (FO) is satisfied so that (ξ_1, ξ_2) is Pareto optimal.

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Definition

A function $U : L^{\infty}(\Omega, \mathcal{F}, \mathbf{P}) \to \mathbb{R}$ is *law invariant* if $law(X) = law(Y)$ implies $U(X) = U(Y)$.

Theorem 1

A law invariant monetary utility function $U : L^\infty \to \mathbb{R}$ is weak star lower semi-continuous. Hence the Legendre-Fenchel duality works for the dual pair $\langle L^\infty, L^1 \rangle$.

Theorem 2

Let U_1, U_2 be law invariant monetary utility functions such that

$$U_1 \Box U_2(0) = \sup \{U_1(\xi) + U_2(-\xi) \mid \xi \in L^\infty\} < \infty.$$

Then, for $X \in L^\infty$, there exists a Pareto optimal allocation $(\xi_1, \xi_2) \in L^\infty \times L^\infty$.

Thank you

for your attention!