Risk-Minimization for Life Insurance Liabilities

Irene Schreiber (LMU Munich)

October 16, 2012

FAM Research Seminar (TU Vienna)

Agenda

Introduction

A review of risk-minimization for payment streams

Risk-minimization for life-insurance liabilities: the single life case

Risk-minimization with basis risk

Risk-minimization with dependent mortality risk

Agenda

Introduction

A review of risk-minimization for payment streams

Risk-minimization for life-insurance liabilities: the single life case

Risk-minimization with basis risk

Risk-minimization with dependent mortality risk

Introduction

- A large number of life insurance and pensions products have mortality and longevity as a primary source of risk
- Inadequate reinsurance capacity on a global basis to effectively address these risks
- Systematic mortality risk cannot be diversified away by pooling
- Securitization as a new form of risk transfer (Blake et al. [10]) $\rightarrow$ creation of a new life market, longevity index based products as hedging instruments
- Risk-minimization: natural hedging method since the market incompleteness is due to the presence of an additional orthogonal source of randomness

- Objective: study the problem of pricing and hedging life insurance liabilities by means of the risk-minimization approach
- Trading in longevity index based products allowed
- 3 scenarios:
  - Single life case: Biagini and Schreiber [4]
  - Homogeneous portfolio with basis risk: Biagini et al. [6]
  - Portfolio consisting of different age cohorts: Biagini et al. [5]
- Main tools:
  - Progressive enlargement of filtration, reduced-form modeling from credit risk $\rightarrow$ Bielecki and Rutkowski [7]
  - Quadratic hedging: risk-minimization $\rightarrow$ Föllmer and Sondermann [15], Møller [18] and Schweizer [19]
  - Affine processes $\rightarrow$ Duffie et al. [13], Duffie et al. [14]
  - Random field theory $\rightarrow$ Adler [1], Goldstein [16], Kennedy [17]

Agenda

Introduction

A review of risk-minimization for payment streams

Risk-minimization for life-insurance liabilities: the single life case

Risk-minimization with basis risk

Risk-minimization with dependent mortality risk

Risk-minimization (RM) for payment streams (Møller [18])

- Finite time horizon $T > 0$, filtered prob. space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{P})$
- Discounted asset price process $\bar{S}$ (local martingale), discounted payment stream $A = (A_t)_{t\in[0,T]}$, $\mathbb{G}$-adapted, square integrable
- An L^2*-strategy* is a pair $\varphi = (\xi, \xi^0)$, such that ξ^0 is a $\mathbb{G}$-adapted process and ξ is a $\mathbb{G}$-predictable process belonging to $L^2(\bar{S})$, with $L^2(\bar{S}) := \left\{\xi \,:\, \xi\ \mathbb{G}\text{-predictable}, \left(\mathbb{E}\left[\int_0^T \xi_s' \,\mathrm{d}[\bar{S}]_s\, \xi_s\right]\right)^{1/2} < \infty\right\}$, such that the discounted value process $V_t(\varphi) = \xi_t \bar{S}_t + \xi_t^0,\ t \in [0, T]$ is right-continuous and square integrable
- *Cumulative cost process* $C(\varphi)$: $C_t(\varphi) = V_t(\varphi) - \int_0^t \xi_s \,\mathrm{d}\bar{S}_s + A_t$
- *Risk process* $R(\varphi)$: $R_t(\varphi) = \mathbb{E}[(C_T(\varphi) - C_t(\varphi))^2 \,|\, \mathcal{G}_t]$, $t \in [0, T]$

- An L^2-strategy $\varphi = (\xi, \xi^0)$ is called *risk-minimizing* (rm), if for any L^2-strategy $\tilde{\varphi} = (\tilde{\xi}, \tilde{\xi}^0)$ such that $V_T(\tilde{\varphi}) = V_T(\varphi) = 0$ $\mathbb{P}$-a.s., $\tilde{\xi}_s = \xi_s$ for $s \leq t$ and $\tilde{\xi}^0_s = \xi^0_s$ for $s < t$, we have

$$R_t(\varphi) \leq R_t(\tilde{\varphi}), \quad t \in [0, T]$$

- *GKW decomposition* (Ansel and Stricker [3]):

$$\mathbb{E}[A_T \,|\, \mathcal{G}_t] = \mathbb{E}[A_T \,|\, \mathcal{G}_0] + \int_{]0,t]} \xi^A_s \,\mathrm{d}\bar{S}_s + L^A_t, \quad t \in [0, T], \tag{1}$$

where $\xi^A \in L^2(\bar{S})$ and L^A is a square integrable martingale null at 0 that is strongly orthogonal to the space of stochastic integrals $\mathcal{I}^2(\bar{S}) := \left\{ \int \xi \,\mathrm{d}\bar{S} \,\middle|\, \xi \in L^2(\bar{S}) \right\}$

Theorem (Møller [18])

The unique risk-minimizing L^2-strategy $\varphi = (\xi, \xi^0)$ for A is given by

$$\xi_t = \xi_t^A,$$
$$\xi_t^0 = \mathbb{E}\left[A_T | \mathcal{G}_t\right] - A_t - \xi_t^A \bar{S}_t = V_t(\varphi) - \xi_t^A \bar{S}_t,$$

for $t \in [0, T]$ with cumulative cost and risk processes

$$C_t(\varphi) = \mathbb{E}\left[A_T | \mathcal{G}_0\right] + L_t^A,$$
$$R_t(\varphi) = \mathbb{E}\left[\left(L_T^A - L_t^A\right)^2 \middle| \mathcal{G}_t\right],$$

where $V_t(\varphi) = \mathbb{E}[A_T \,|\, \mathcal{G}_t] - A_t$ and ξ^A, L^A are given by the GKW decomposition of $\mathbb{E}[A_T \,|\, \mathcal{G}_t]$ in (1)

Agenda

Introduction

A review of risk-minimization for payment streams

Risk-minimization for life-insurance liabilities: the single life case

Risk-minimization with basis risk

Risk-minimization with dependent mortality risk

Introduction

- Joint work with Francesca Biagini
- Objective: study the problem of pricing and hedging life insurance liabilities by means of the risk-minimization approach for the case of one insured person
- Very general setting: general payoff structure, asset prices are local martingales (jumps allowed), existence of intensity not required, no independence assumption
- Main tools:
    - Progressive enlargement of filtration, reduced-form modeling from credit risk $\rightarrow$ Bielecki and Rutkowski [7]
    - Quadratic hedging: risk-minimization $\rightarrow$ Föllmer and Sondermann [15], Møller [18] and Schweizer [19]

The setting

- Finite time horizon $T > 0$, probability space $(\Omega, \mathcal{G}, \mathbb{P})$
- *Background filtration* $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$
- Financial market: bank account + one risky asset with discounted asset price X local $(\mathbb{P}, \mathbb{F})$-martingale $\to$ $\mathbb{P}$ ELMM
- *Remaining lifetime*: $\tau : \Omega \to [0, T] \cup \{\infty\}$
- $\mathbb{H} = (\mathcal{H}_t)_{t\in[0,T]}$, where $\mathcal{H}_t = \sigma\{H_s : 0 \le s \le t\}$ and $H_t := \mathbb{1}_{\{\tau\le t\}}$
- *Progressive enlargement* of filtrations: $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, $\mathcal{G} = \mathcal{G}_T$
- *Hypothesis (H)*: $\mathbb{F}$-(local) martingales are $\mathbb{G}$-(local) martingales
- *Hazard process* Γ of τ under $\mathbb{P}$: $\Gamma_t = -\ln \mathbb{E}[\mathbb{1}_{\{\tau>t\}} \,|\, \mathcal{F}_t]$
- *Survivor/longevity index*: $S_t^\mu = \exp(-\Gamma_t)$, $t \in [0, T]$

- *Systematic mortality risk* component:
  $P_t^T := \mathbb{E}[\mathbb{1}_{\{\tau > T\}} \,|\, \mathcal{F}_t] = \mathbb{E}[S_T^\mu \,|\, \mathcal{F}_t] = P_0^T + \int_{]0,t]} \zeta_s P_{s-}^T \,\mathrm{d}Y_s$,
  $t \in [0, T]$, ζ $\mathbb{F}$-predictable process, $Y \perp X$ $\mathbb{F}$-local martingale
- The martingale $M_t = H_t - \Gamma_{t \wedge \tau}$, $t \in [0, T]$ associated with H is strongly orthogonal to any $\mathbb{F}$-local martingale
- Discounted *life insurance payment process* $A = (A_t)_{t \in [0,T]}$:

$$A_t = \mathbb{1}_{\{\tau \le t\}} \bar{A}_\tau + \mathbb{1}_{\{t = T\}} \mathbb{1}_{\{\tau > T\}} \tilde{A}, \quad t \in [0, T], \tag{2}$$

$\bar{A} = (\bar{A}_t)_{t \in [0,T]}$ $\mathbb{F}$-predictable process, $\mathbb{E}\left[\sup_{t \in [0,T]} \bar{A}_t^2\right] < \infty$, $\tilde{A}$ $\mathcal{G}_T$-measurable random variable, $\mathbb{E}[\tilde{A}^2] < \infty \rightarrow \mathbb{E}[A_t^2] < \infty$

Risk-minimization for life insurance liabilities

Theorem

The payment process A in (2) admits a RM strategy $\varphi = (\xi, \xi^0)$, where $\xi_t = \mathbb{1}_{\{\tau \geq t\}} e^{\Gamma_t} \xi^m_t$ and $\xi^0_t = V_t - \xi_t X_t$ for $t \in [0, T]$, with

$$V_t = \mathbb{E}[A_T \,|\, \mathcal{G}_t] - A_t = \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \xi^m_s \, \mathrm{d}X_s + C_t - A_t,$$

$$C_t = m_0 + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \eta^m_s \, \mathrm{d}Y_s + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \, \mathrm{d}C^m_s + \int_{]0,t]} \psi^M_s \, \mathrm{d}M_s,$$

$$\begin{aligned} m_t &= \mathbb{E}\left[\int_0^T \bar{A}_s e^{-\Gamma_s} \, \mathrm{d}\Gamma_s \,\middle|\, \mathcal{F}_t \right] + \mathbb{E}[\mathbb{1}_{\{\tau > T\}} \tilde{A} \,|\, \mathcal{F}_t], \\ &= m_0 + \int_{]0,t]} \xi^m_s \, \mathrm{d}X_s + \int_{]0,t]} \eta^m_s \, \mathrm{d}Y_s + C^m_t, \quad \textit{and} \end{aligned}$$

$$\psi^M_t = \bar{A}_t - e^{\Gamma_t} \left(\mathbb{E}\left[\int_t^T \bar{A}_s e^{-\Gamma_s} \, \mathrm{d}\Gamma_s \,\middle|\, \mathcal{F}_t \right] + \mathbb{E}[\mathbb{1}_{\{\tau > T\}} \tilde{A} \,|\, \mathcal{F}_t] \right).$$

Proof.

Compute GKW decomposition of A_T - main steps:

- Split up on events $\{\tau \leq t\}$ and $\{\tau > t\}$ for $t \in [0, T]$:
$$\mathbb{E}[A_T \,|\, \mathcal{G}_t] = \underbrace{\mathbb{1}_{\{\tau \leq t\}} \mathbb{E}[\mathbb{1}_{\{\tau \leq T\}} \bar{A}_\tau \,|\, \mathcal{G}_t]}_{a)} + \underbrace{\mathbb{1}_{\{\tau > t\}} \mathbb{E}[\mathbb{1}_{\{\tau > T\}} \tilde{A} \,|\, \mathcal{G}_t]}_{b)}$$
- Compute $a)$ and $b)$ separately
- $\mathcal{G}_t \to \mathcal{F}_t$ (Bielecki and Rutkowski [7]), e.g. for $b)$:
$$\mathbb{E}[\mathbb{1}_{\{\tau > T\}} \tilde{A} \,|\, \mathcal{G}_t] = \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} \underbrace{\mathbb{E}[\mathbb{1}_{\{\tau > T\}} \tilde{A} \,|\, \mathcal{F}_t]}_{\tilde{m}_t}, \quad t \in [0, T]$$
- Find martingale decompositions in $\mathbb{F}$, i.e. for $b)$: $\tilde{m}_t = \ldots$, for $a)$: $\bar{m} = \ldots$, $m = \tilde{m} + \bar{m} = \int \cdots \mathrm{d}X + \int \cdots \mathrm{d}Y + C^m$
- Determine orthogonal structure in $\mathbb{G}$ in terms of X, Y, C^m and M

□

- The cost is generated by the following components:
  - C^m, the orthogonal part due to the predictable decomposition of the $\mathbb{F}$-martingale m
  - Y, the driving process of the conditional survival probability
  - M, the compensated jump process of the time of death
- The integrals with respect to Y and M represent the systematic and unsystematic component of the mortality risk
- Question:
  Can we introduce mortality-linked products into the financial market, that can be used to hedge the cost parts due to Y and M?
- Set $C^m \equiv 0$ and $r \equiv 0$ (constant bank account) for simplicity

Extending the financial market

How to eliminate the systematic risk:

- Zero-coupon *longevity bond* with maturity T $(P_t^T)_{t\in[0,T]}$ pays out the longevity index at time T (Cairns et al. [11]):
$$P_t^T = \mathbb{E}[\,e^{-\Gamma_T}\,|\,\mathcal{G}_t] \overset{(H)}{=} \mathbb{E}[\,e^{-\Gamma_T}\,|\,\mathcal{F}_t] = \mathbb{E}[\mathbb{1}_{\{\tau>T\}}\,|\,\mathcal{F}_t]$$
$$= P_0^T + \int_{]0,t]} \zeta_s P_{s-}^T \,\mathrm{d}Y_s,\ t\in[0,T]$$
- Assume trading in P^T is possible $\rightarrow$ eliminates the cost part associated to the systematic mortality risk:

$$V_t = \int_{]0,t]} \mathbb{1}_{\{\tau\geq s\}} e^{\Gamma_s}\xi_s^m \,\mathrm{d}X_s + \int_{]0,t]} \mathbb{1}_{\{\tau\geq s\}} \frac{e^{\Gamma_s}\eta_s^m}{\zeta_s P_{s-}^T}\,\mathrm{d}P_s^T + C_t - A_t,$$
$$C_t = m_0 + \int_{]0,t]} \psi_s^M \,\mathrm{d}M_s, \quad t\in[0,T]$$

How to eliminate the unsystematic risk:

- *Pure endowment* contract $E = (E_t)_{t \in [0,T]}$ that pays 1 at maturity T if the individual survived: $E_t = \mathbb{E}[\mathbb{1}_{\{\tau > T\}} \,|\, \mathcal{G}_t]$, $t \in [0, T]$
- Here E, P^T and M are closely related to each other: $\mathrm{d}M_t = \frac{\mathbb{1}_{\{\tau \geq t\}}}{P^T_{t-}} \,\mathrm{d}P^T_t - \frac{1}{P^T_{t-} e^{\Gamma_t}} \,\mathrm{d}E_t, \quad t \in [0, T]$
- Assume trading in E is possible $\rightarrow$ additionally eliminates the cost part associated to the unsystematic mortality risk:

$$V_t = m_0 + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \xi^m_s \,\mathrm{d}X_s + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} \left(\frac{e^{\Gamma_s} \eta^m_s}{\zeta_s P^T_{s-}} + \frac{\psi^M_s}{P^T_{s-}} \right) \mathrm{d}P^T_s - \int_{]0,t]} \frac{\psi^M_s}{P^T_{s-} e^{\Gamma_s}} \,\mathrm{d}E_s - A_t$$

Example: unit-linked life insurance

- $\mathbb{F} = \mathbb{F}^W \vee \mathbb{F}^{W^\mu} \vee \mathbb{F}^Q$, W and W^μ independent Brownian motions, Q compound Poisson process
- Jump diffusion model for the discounted asset price

$$\mathrm{d}X_t = \sigma_t X_t \,\mathrm{d}W_t + X_{t-}\,\mathrm{d}\widetilde{Q}_t, \quad X_0 = x, \quad t \in [0, T],$$

  with $\sigma = (\sigma_t)_{t\in[0,T]}$ bounded, $\mathbb{F}$-adapted process, $\widetilde{Q}_t = Q_t - \beta\lambda t$, $Q_t = \sum_{i=1}^{N_t} Y_i$, $t \in [0, T]$, $N = (N_t)_{t\in[0,T]}$ Poisson process with intensity $\lambda > 0$, Y_i are i.i.d. independent of N with $Y_i > -1$ a.s., $i \geq 1$, such that $\mathbb{E}[Y_1] = \beta < \infty$ and $\mathbb{E}[Y_1^2] < \infty$
- *Unit-linked term insurance contract* $\rightarrow$ pays out the discounted asset price in the case of death prior to maturity: $A_T = \mathbb{1}_{\{\tau\leq T\}} X_\tau$

- The hazard process admits the representation $\Gamma_t = \int_0^t \mu_s \, \mathrm{d}s$, $t \in [0, T]$, where the *mortality intensity* μ is a non-negative $\mathbb{F}$-measurable process with

$$\mathrm{d}\mu_t = (a + b\mu_t)\,\mathrm{d}t + c\sqrt{\mu_t}\,\mathrm{d}W_t^{\mu}, \quad \mu_0 = 0, \quad t \in [0, T],$$

  for $b \in \mathbb{R}$ and $a, c \in \mathbb{R}_+$
- Duffie et al. [14]: since μ is an affine process, for $t \in [0, T]$ we have

$$\mathbb{E}[e^{-\Gamma_T} \,|\, \mathcal{G}_t] = e^{-\Gamma_t}\mathbb{E}[e^{-\int_t^T \mu_s \, \mathrm{d}s} \,|\, \mathcal{F}_t^{W^\mu}] = e^{-\Gamma_t} e^{\alpha(t)+\beta(t)\mu_t},$$

  where $\alpha(t) = \frac{2a}{c^2} \ln\left(\frac{2\gamma e^{(\gamma-b)(T-t)/2}}{(\gamma-b)(e^{\gamma(T-t)}-1)+2\gamma}\right)$,
  $\beta(t) = -\frac{2(e^{\gamma(T-t)}-1)}{(\gamma-b)(e^{\gamma(T-t)}-1)+2\gamma}$ and $\gamma := \sqrt{b^2 + 2c^2}$

- $$m_t = \mathbb{E}\left[\int_0^T X_s e^{-\Gamma_s}\,\mathrm{d}\Gamma_s \,\middle|\, \mathcal{F}_t\right]$$
$$= x(1-e^{\alpha(0)}) + \int_{]0,t]} e^{-\Gamma_s}(1-e^{\alpha(s)+\beta(s)\mu_s})\,\mathrm{d}X_s$$
$$- \int_0^t c\sqrt{\mu_s}\beta(s)X_s e^{-\Gamma_s+\alpha(s)+\beta(s)\mu_s}\,\mathrm{d}W_s^\mu, \quad t\in[0,T]$$

- GKW decomposition of A_T:

$$V_t = \mathbb{E}[\mathbb{1}_{\{\tau\le T\}}X_\tau \mid \mathcal{G}_t] - A_t$$
$$= \int_{]0,t]} \mathbb{1}_{\{\tau\ge s\}}(1-e^{\alpha(s)+\beta(s)\mu_s})\,\mathrm{d}X_s + C_t - A_t, \quad t\in[0,T]$$

$$C_t = x(1-e^{\alpha(0)})$$
$$- \int_0^t \mathbb{1}_{\{\tau\ge s\}} c\sqrt{\mu_s}\beta(s)X_s e^{\alpha(s)+\beta(s)\mu_s}\,\mathrm{d}W_s^\mu$$
$$+ \int_{]0,t]} X_s e^{\alpha(s)+\beta(s)\mu_s}\mathrm{d}M_s, \quad t\in[0,T]$$

Agenda

Introduction

A review of risk-minimization for payment streams

Risk-minimization for life-insurance liabilities: the single life case

Risk-minimization with basis risk

Risk-minimization with dependent mortality risk

Introduction

- Joint work with Francesca Biagini and Thorsten Rheinländer
- Objective: study the problem of pricing and hedging life insurance liabilities of a homogeneous insurance portfolio by means of the risk-minimization approach
- Consider a homogeneous insurance portfolio (all individuals are of the same age at time 0), take basis risk into account
- Model the dependency between the index and the insurance portfolio by means of a multidimensional affine mean-reverting diffusion process with stochastic drift
- Additional tool: affine processes
  - Duffie et al. [13], Duffie et al. [14]
  - Biffis [8], Dahl [12]

The setting: insurance portfolio and mortality intensities

- Finite time horizon $T > 0$, probability space $(\Omega, \mathcal{G}, \mathbb{P})$
- Background filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$
- Insurance portfolio: n individuals belonging to the same age cohort
- Remaining lifetimes $\tau^j : \Omega \to [0, T] \cup \{\infty\}$, $j = 1, \dots, n$
- $H_t^j = \mathbb{1}_{\{\tau^j \le t\}}$, $j = 1, \dots, n$, $N_t = \sum_{j=1}^n \mathbb{1}_{\{\tau^j \le t\}}$, $t \in [0, T]$
- $\mathbb{H} = (\mathcal{H}_t)_{t \in [0,T]}$, $\mathcal{H}_t = \mathcal{H}_t^1 \vee \dots \vee \mathcal{H}_t^n$, where $\mathcal{H}_t^j = \sigma\{H_s^j : 0 \le s \le t\}$
- Hazard process Γ^j of τ^j under $\mathbb{P}$: $\Gamma_t^j = -\ln \mathbb{E}[\mathbb{1}_{\{\tau^j > t\}} \,|\, \mathcal{F}_t]$, we set $\Gamma^j = \Gamma$, where $\Gamma_t = \int_0^t \mu_s \,\mathrm{d}s$, $t \in [0, T]$

- Similarly as in Biffis [8] we assume that the mortality intensity μ is given by the following set of stochastic differential equations:

$$\mathrm{d}\mu_t = \gamma_1(\bar{\mu}_t - \mu_t)\mathrm{d}t + \sigma_1\sqrt{\mu_t}\,\mathrm{d}W_t^{\mu},$$
$$\mathrm{d}\bar{\mu}_t = \gamma_2(m(t) - \bar{\mu}_t)\mathrm{d}t + \sigma_2\sqrt{\bar{\mu}_t}\,\mathrm{d}W_t^{\bar{\mu}},$$

for $t \in [0, T]$ where W^{μ} and $W^{\bar{\mu}}$ are independent Brownian motions and $\mu_0 = \bar{\mu}_0 = 0$, where $\gamma_1, \gamma_2, \sigma_1, \sigma_2 > 0$, and $m : [0, T] \to \mathbb{R}_+$ is a continuous deterministic function

- The process $\bar{\mu}$ represents the mortality intensity of the equivalent age cohort of the population
- Survivor/longevity index: $S_t^{\bar{\mu}} = \exp\left(-\int_0^t \bar{\mu}_s\,\mathrm{d}s\right)$, $t \in [0, T]$

The setting: financial market

- Bank account B: $B_t = \exp(rt)$, $t \in [0, T]$, $r > 0$
- Risky asset S with $\mathbb{P}$-dynamics $\mathrm{d}S_t = S_t\,(r\mathrm{d}t + \sigma(t, S_t)\mathrm{d}W_t)$, $S_0 = s$, $t \in [0, T]$, Brownian motion W independent of $(W^{\mu}, W^{\bar{\mu}})$
- Longevity bond P with maturity T (Cairns et al. [11]): pays out the value of the survivor index at T, i.e. $Y_t = \mathbb{E}\left[\frac{S_T^{\bar{\mu}}}{B_T}\,\middle|\,\mathcal{G}_t\right]$, $t \in [0, T]$
- $X = S/B$, $Y = P/B$ are continuous (local) $(\mathbb{P}, \mathbb{F})$-martingales $\rightarrow$ financial market given by X, Y is arbitrage-free

The setting: combined model

- Background filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, where $\mathcal{F}_t = \sigma\{(W_s, W_s^{\mu}, W_s^{\bar{\mu}}) : 0 \leq s \leq t\}$, $t \in [0, T]$
- Enlarged filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$
- Hypothesis (H): all $\mathbb{F}$-(local) martingales are $\mathbb{G}$-(local) martingales
- For $i \neq j$, τ^i, τ^j are conditionally independent given $\mathcal{F}_T$, i.e. $\mathbb{E}[\mathbb{1}_{\{\tau^i>t\}}\mathbb{1}_{\{\tau^j>s\}} \,|\, \mathcal{F}_T] = \mathbb{E}[\mathbb{1}_{\{\tau^i>t\}} \,|\, \mathcal{F}_T]\,\mathbb{E}[\mathbb{1}_{\{\tau^j>s\}} \,|\, \mathcal{F}_T]$, $0 \leq s, t \leq T$
- Fundamental martingales: the compensated process $M_t^j = H_t^j - \Gamma_{t\wedge\tau^j}$, $t \in [0, T]$ follows a $\mathbb{G}$-martingale for each $j = 1, \dots, n$ we define $M_t = \sum_{j=1}^n M_t^j$

- We consider the following life insurance payment streams:
  - Pure endowment: $A_t^{pe} = (n - N_t)\,\frac{C^{pe}}{B_t}\mathbb{1}_{\{t=T\}}$
  - Term insurance: $A_t^{ti} = \int_0^t \frac{C_s^{ti}}{B_s}\,\mathrm{d}N_s = \sum_{j=1}^n \mathbb{1}_{\{\tau^j \le t\}}\frac{C_{\tau^j}^{ti}}{B_{\tau^j}}$
  - Annuity: $A_t^a = \int_0^t (n - N_s)\frac{1}{B_s}\,\mathrm{d}C_s^a = \sum_{j=1}^n \int_0^t \mathbb{1}_{\{\tau^j > s\}}\frac{1}{B_s}\,\mathrm{d}C_s^a$

  where $C^{pe} \in \mathcal{F}_T$, $C^{pe} \ge 0$, $\mathbb{E}[(C^{pe})^2] < \infty$, $C^{ti} \ge 0$ $\mathbb{F}$-predictable, $\mathbb{E}\left[\sup_{t\in[0,T]}(C_t^{ti})^2\right] < \infty$, and $C^a \ge 0$ increasing $\mathbb{F}$-adapted, $\mathbb{E}\left[\sup_{t\in[0,T]}(C_t^a)^2\right] < \infty$
- Unit-linked products:
  $C^{pe} = f(S_T)$, $C_t^{ti} = f(S_t)$ and $C_t^a = \int_0^t f(S_s)\,\mathrm{d}s$ for a function f that satisfies sufficient regularity conditions

RM for term insurance contracts with basis risk

Theorem

The process A^{ti} admits a rm strategy $\varphi = (\xi, \xi^0) = (\xi^X, \xi^Y, \xi^0)$ given by

$$\xi_t = (\xi_t^X, \xi_t^Y) = \left(\frac{(n-N_t)e^{\Gamma_t}\psi_t}{\sigma(t,X_t)X_t}, \frac{(n-N_t)e^{\Gamma_t + rT}\psi_t^{\bar{\mu}}}{Y_t\beta^T(t)\sigma_2\sqrt{\bar{\mu}_t}} \right),$$

$$\xi_t^0 = V_t^{ti}(\varphi) - \xi_t^X X_t - \xi_t^Y Y_t$$

for $t \in [0, T]$, with discounted value process

$$V_t^{ti}(\varphi) = nU_0^{ti} + \int_0^t \xi_s^X \,\mathrm{d}X_s + \int_0^t \xi_s^Y \,\mathrm{d}Y_s + L_t^{ti} - A_t^{ti}, \text{ where}$$

$$L_t^{ti} = \int_0^t (n-N_s)e^{\Gamma_s}\psi_s^{\mu}\,\mathrm{d}W_s^{\mu} + \int_{]0,t]} \left(\frac{C_s^{ti}}{B_s} - \mathbb{E}\left[\int_s^T \frac{C_u^{ti}}{B_u} e^{\Gamma_s - \Gamma_u}\,\mathrm{d}\Gamma_u \,\middle|\, \mathcal{F}_s \right] \right) \mathrm{d}M_s,$$

$$U_t^{ti} = \mathbb{E}\left[\int_0^T \frac{C_s^{ti}}{B_s} e^{-\Gamma_s}\,\mathrm{d}\Gamma_s \,\middle|\, \mathcal{F}_t \right] = U_0^{ti} + \int_0^t \psi_s \mathrm{d}W_s + \int_0^t \psi_s^{\mu}\mathrm{d}W_s^{\mu} + \int_0^t \psi_s^{\bar{\mu}}\mathrm{d}W_s^{\bar{\mu}}.$$

The optimal cost and risk processes are given by

$C_t^{ti}(\varphi) = nU_0^{ti} + L_t^{ti}$ *and* $R_t^{ti}(\varphi) = \mathbb{E}[(L_T^{ti} - L_t^{ti})^2 \,|\, \mathcal{G}_t]$ *for* $t \in [0, T]$.

Proof.

Compute GKW decomposition of A_T^{ti} - main steps:

- $\mathcal{G}_t \to \mathcal{F}_t$:
  $$\mathbb{E}[A_T^{ti} \mid \mathcal{G}_t] = nU_0^{ti} + \int_0^t (n - N_s) e^{\Gamma_s} \,\mathrm{d}U_s^{ti} + \int_{]0,t]} \left(\frac{C_s^{ti}}{B_s} - \mathbb{E}\left[\int_s^T \frac{C_u^{ti}}{B_u} e^{\Gamma_s - \Gamma_u} \,\mathrm{d}\Gamma_u \,\middle|\, \mathcal{F}_s \right] \right) \mathrm{d}M_s, \quad t \in [0, T]$$
- Martingale representation theorem for Brownian filtrations: $U_t^{ti} = U_0^{ti} + \int_0^t \psi_s \,\mathrm{d}W_s + \int_0^t \psi_s^{\mu} \,\mathrm{d}W_s^{\mu} + \int_0^t \psi_s^{\bar{\mu}} \,\mathrm{d}W_s^{\bar{\mu}}$, $t \in [0, T]$, where ψ, ψ^{μ} and $\psi^{\bar{\mu}}$ are $\mathbb{F}$-predictable processes
- Dynamics of X and Y: $\mathrm{d}X_t = \mathrm{d}\left(\frac{S_t}{B_t}\right) = \sigma(t, S_t) X_t \,\mathrm{d}W_t$ and $\mathrm{d}Y_t = Y_t e^{-rT} \beta^T(t) \sigma_2 \sqrt{\bar{\mu}_t} \,\mathrm{d}W_t^{\bar{\mu}}$ for $t \in [0, T]$, where $\partial_t \beta^T(t) = 1 + \gamma_2 \beta^T(t) - \frac{1}{2}\sigma_2^2 (\beta^T(t))^2, \quad \beta^T(T) = 0$
- Orthogonality and integrability

□

Consider a unit-linked term insurance contract:
$A_t^{ti,f} = \int_0^t \frac{f(S_s)}{B_s} \,\mathrm{d}N_s = \sum_{i=1}^n \mathbb{1}_{\{\tau^i \le t\}} \frac{f(S_{\tau^i})}{B_{\tau^i}}$, $t \in [0,T]$, where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a Borel measurable function such that $\mathbb{E}\left[\sup_{t\in[0,T]} f(S_t)^2\right] < \infty$

Corollary

The process $A^{ti,f}$ admits a rm strategy $\varphi = (\xi, \xi^0) = (\xi^X, \xi^Y, \xi^0)$ given by
$\xi_t^X = (n - N_t) e^{\Gamma_t} \int_t^T F_s^u(t, S_t) Z_t^{\mu,u} \,\mathrm{d}u,$
$\xi_t^Y = (n - N_t) e^{\Gamma_t + r(T-t)} Y_t^{-1} \beta^T(t)^{-1} \int_t^T F^u(t, S_t) Z_t^u (\hat{\beta}_2^u(t) + \beta_2^u(t) \hat{Z}_t^u) \,\mathrm{d}u,$
$\xi_t^0 = V_t^{ti}(\varphi) - \xi_t^X X_t - \xi_t^Y Y_t$, *for* $t \in [0,T]$, *with discounted value process*
$V_t^{ti}(\varphi) = nU_0^{ti} + \int_0^t \xi_s^X \,\mathrm{d}X_s + \int_0^t \xi_s^Y \,\mathrm{d}Y_s + L_t^{ti} - A_t^{ti,f}$.
The optimal cost and risk processes are given by
$C_t^{ti}(\varphi) = nU_0^{ti} + L_t^{ti}$ *and* $R_t^{ti}(\varphi) = \mathbb{E}[(L_T^{ti} - L_t^{ti})^2 \,|\, \mathcal{G}_t]$ *for* $t \in [0,T]$.

Proof.

- Fubini's theorem: $U_t^{ti} = \mathbb{E}\left[\int_0^T \frac{f(S_u)}{B_u} e^{-\Gamma_u}\mu_u \,\mathrm{d}u \middle| \mathcal{F}_t\right] = \int_0^T \mathbb{E}\left[\frac{f(S_u)}{B_u}\middle|\mathcal{F}_t\right] \mathbb{E}\left[e^{-\Gamma_u}\mu_u \middle| \mathcal{F}_t\right] \mathrm{d}u,\ t \in [0, T]$
- $\mathbb{E}\left[\frac{f(S_u)}{B_u}\middle|\mathcal{F}_t\right] = F^u(0, S_0) + \int_0^t F_s^u(s, S_s)\sigma(s, S_s)X_s \mathbb{1}_{\{s \le u\}} \,\mathrm{d}W_s$ for $0 \le t,\ u \le T$, where $F^u(u, S_u) = f(S_u)$
- $Z_t^{\mu,u} = \mathbb{E}\left[e^{-\Gamma_u}\mu_u \middle| \mathcal{F}_t\right] = Z_0^{\mu,u} + \int_0^t Z_s^u \left(\hat{\beta}_1^u(s) + \beta_1^u(s)\hat{Z}_s^u\right)\sigma_1\sqrt{\mu_s}\mathbb{1}_{\{s \le u\}} \,\mathrm{d}W_s^{\mu} + \int_0^t Z_s^u \left(\hat{\beta}_2^u(s) + \beta_2^u(s)\hat{Z}_s^u\right)\sigma_2\sqrt{\bar{\mu}_s}\mathbb{1}_{\{s \le u\}} \,\mathrm{d}W_s^{\bar{\mu}},\ 0 \le t,\ u \le T$
- The result follows by the stochastic Fubini theorem

□

Agenda

Introduction

A review of risk-minimization for payment streams

Risk-minimization for life-insurance liabilities: the single life case

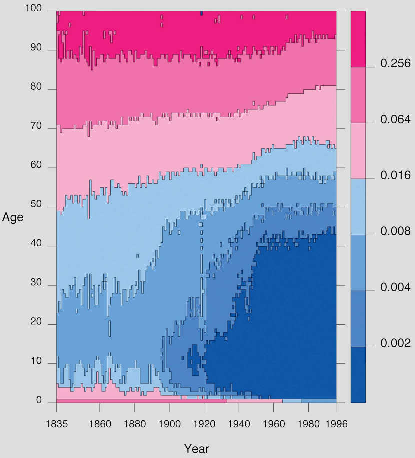
Risk-minimization with basis risk

Risk-minimization with dependent mortality risk

Introduction

- Joint work with Francesca Biagini and Camila Botero
- Objective: study the problem of pricing and hedging life insurance liabilities with *dependent* mortality risk by means of the risk-minimization approach
- Consider a portfolio consisting of individuals of different age cohorts and take into account the cross-generational dependency structure
- Introduce a model for the mortality intensities that is consistent with typical characteristics of historical mortality data
- Additional tool: random field theory
  - Adler [1]
  - Goldstein [16], Kennedy [17]
  - Biffis and Millossovich [9]

Andreev [2]: Danish Female Mortality



Typical characteristics of the Mortality Surface

- For fixed point in time: increasing in age
- For fixed age: decreasing in time
- Downward mortality trend is *not* uniform over age and time

$\rightarrow$ Use random fields to model the mortality intensity

Random Fields

Definition (Adler [1])

A real-valued *random field* is a collection of random variables $(X_t)_{t \in I}$, with index set $I \subseteq \mathbb{R}^N$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with a collection of distribution functions

$$F_{t_1,\dots,t_n}(b_1,\dots,b_n) = \mathbb{P}\left(X_{t_1} \le b_1,\dots,X_{t_n} \le b_n\right),$$

for $n \in \mathbb{N}$, $b_i \in \mathbb{R}$, $t_i \in I$, $i = 1,\dots,n$. Given a square integrable random field $X = (X_t)_{t \in I}$, the *mean function* is defined as $m(t) = \mathbb{E}[X_t]$, $t \in I$ and the *covariance function* is defined as $c(s,t) = \mathrm{Cov}(X_s, X_t)$, $s,t \in I$. A square integrable random field X is *homogeneous* or *stationary* if the mean function is independent of t, i.e. $m(t) = m$, $t \in I$ and the covariance function $c(s,t)$ is a function of $t - s$, $s,t \in I$, only. In this case we write $c(h) = c(0,h)$ for $h \in I$.

Example (Adler [1])

- A *Gaussian random field* is a random field where all finite-dimensional distributions $F_{t_1,\dots,t_n}$, $n \in \mathbb{N}$ are multivariate normal. For $N = 2$ a *Brownian sheet* W is a continuous version of a centered, Gaussian field with covariance function $c((s_1, s_2), (t_1, t_2)) = (s_1 \wedge t_1)(s_2 \wedge t_2)$
- A χ^2*-field* $Y = (Y_t)_{t \in I}$ with parameter $n \in \mathbb{N}$ is defined as

$$Y_t := (Z_t^1)^2 + \cdots + (Z_t^n)^2, \quad t \in I$$

  where $Z^1, \dots, Z^n$ are independent, stationary centered Gaussian random fields with common covariance function $c(h)$, $h \in I$ and variance $c(0) = \sigma^2$

The Setting: insurance portfolio and mortality intensities

- Finite time horizon $T > 0$, probability space $(\Omega, \mathcal{G}, \mathbb{P})$
- Background filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$
- Insurance portfolio B: n individuals belonging to a set of age cohorts $B = \{x_1, \dots, x_m\} \subseteq I$, $I = [0, x^*]$
- Borel function $n^{\cdot} : B \to \mathbb{N}$: # of insureds belonging to each age cohort
- Remaining lifetimes $\tau^{x,j} : \Omega \to [0, T] \cup \{\infty\}$, $x \in B$, $j = 1, \dots, n^x$
- $\mathbb{H} = (\mathcal{H}_t)_{t\in[0,T]}$, $\mathcal{H}_t = \vee_{x\in B}\mathcal{H}_t^x$ with $\mathcal{H}_t^x = \mathcal{H}_t^{x,1} \vee \cdots \vee \mathcal{H}_t^{x,n^x}$, where $\mathcal{H}_t^{x,j} = \sigma\{H_s^{x,j} : 0 \le s \le t\}$ and $H_t^{x,j} = \mathbb{1}_{\{\tau^{x,j}\le t\}}$
- Finite measure ζ on $(B, \mathcal{P}(B))$ allows us to differently weight the subsets of B, i.e. $\int_B n^x \,\zeta(\mathrm{d}x) = \sum_{i=1}^m n^{x_i}\zeta(x_i)$ provides us with the weighted dimension of the portfolio B

- Death counting process belonging to the cohort class x: $N_t^x = \sum_{j=1}^{n^x} \mathbb{1}_{\{\tau^{x,j} \leq t\}}$, $t \in [0, T]$, $x \in B$
- Hazard process $\Gamma^{x,j}$ of $\tau^{x,j}$: $\Gamma_t^{x,j} = -\ln \mathbb{E}[\mathbb{1}_{\{\tau^{x,j} > t\}} \,|\, \mathcal{F}_t]$, $t \in [0, T]$, for $x \in B$: $\Gamma^x := \Gamma^{x,j}$, $j = 1, \ldots, n^x$, Γ^x admits a mortality intensity μ^x, i.e. $\Gamma_t^x = \int_0^t \mu_s^x \,\mathrm{d}s$, $t \in [0, T]$ where $\mu = (\mu_{t,x})_{(t,x) \in [0,T] \times I}$, is a random field generated by a Brownian sheet $W = (W_{t,x})_{(t,x) \in [0,T] \times I}$
- For fixed $x \in I$ we assume that the process $(\mu_t^x)_{t \in [0,T]}$ is an affine diffusion process
- $\mathbb{F}^\mu = (\mathcal{F}_t^\mu)_{t \in [0,T]}$, where $\mathcal{F}_t^\mu = \{\mathcal{F}_{t,x}^\mu : 0 \leq x \leq x^*\} = \vee_{x \in I} \mathcal{F}_{t,x}^\mu$ and $\mathcal{F}_{t,x}^\mu = \sigma\{W_{s,v} : 0 \leq s \leq t,\ 0 \leq v \leq x\}$
- Survivor indices: $S_t^{\mu^x} = \exp\left(-\int_0^t \mu_s^x \,\mathrm{d}s\right)$, $t \in [0, T]$, $x \in I$

The setting: financial market

- Bank account B: $B_t = \exp(rt)$, $t \in [0, T]$, $r > 0$
- Risky asset S with $\mathbb{P}$-dynamics $\mathrm{d}S_t = S_t \left(r\mathrm{d}t + \sigma(t, S_t)\mathrm{d}W_t^X \right)$, $S_0 = s$, $t \in [0, T]$, Brownian motion W^X independent of the Brownian sheet W
- $\mathbb{F}^X = \left(\mathcal{F}_t^X\right)_{t \in [0,T]}$, $\mathcal{F}_t^X = \sigma\{W_s^X : 0 \leq s \leq t\}$
- Family of longevity bonds with maturity T (Cairns et al. [11]): $Y_t^x = \mathbb{E}\left[\frac{S_T^{\mu^x}}{B_T} \,\middle|\, \mathcal{G}_t \right]$, $t \in [0, T]$, $x \in I$
- $X := \frac{S}{B}$, Y^x, $x \in I$ are continuous (local) $(\mathbb{P}, \mathbb{F})$-martingales $\rightarrow$ financial market is arbitrage-free

The setting: combined model

- Background filtration $\mathbb{F} = \mathbb{F}^X \vee \mathbb{F}^\mu$, enlarged filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$
- Hypothesis (H): all $\mathbb{F}$-(local) martingales are $\mathbb{G}$-(local) martingales
- Remaining lifetimes are conditionally independent given $\mathcal{F}_T$
- Fundamental martingales: the compensated process $M_t^{x,j} = H_t^{x,j} - \Gamma^x_{t \wedge \tau^{x,j}}$, $t \in [0, T]$ follows a $\mathbb{G}$-martingale for each $x \in B$, $j = 1, \ldots, n^x$ we define $M_t^x = \sum_{j=1}^{n^x} M_t^{x,j}$
- Unit-linked life insurance liabilities:
  - Pure endowment: $A_t^{pe} = \frac{f(S_t)}{B_t} \sum_{i=1}^m \zeta(x_i) \sum_{j=1}^{n^{x_i}} \mathbb{1}_{\{\tau^{x_i,j} > t\}} \mathbb{1}_{\{t=T\}}$
  - Term insurance: $A_t^{ti} = \sum_{i=1}^m \zeta(x_i) \sum_{j=1}^{n^{x_i}} \frac{f(S_{\tau^{x_i,j}})}{B_{\tau^{x_i,j}}} \mathbb{1}_{\{\tau^{x_i,j} \leq t\}}$
  - Annuity: $A_t^a = \sum_{i=1}^m \zeta(x_i) \sum_{j=1}^{n^{x_i}} \int_0^t \mathbb{1}_{\{\tau^{x_i,j} > s\}} \frac{f(S_s)}{B_s} \mathrm{d}s$

RM for unit-linked term insurance (dependent mortality risk)

Theorem

The payment process A^{ti} admits a rm strategy $\varphi = (\xi^X, \xi^Y, \xi^0)$ with discounted value process
$V_t^{ti}(\varphi) = \mathbb{E}[A_T^{ti} \,|\, \mathcal{G}_0] + \int_0^t \xi_s^X \,\mathrm{d}X_s + \int_0^t \xi_s^Y \,\mathrm{d}Y_s + L_t^{ti} - A_t^{ti}$, *and*
$\xi_t^0 = V_t^{ti}(\varphi) - \xi_t^X X_t - \xi_t^Y Y_t$,
$\xi_t^X = \sum_{i=1}^m \zeta(x_i)(n^{x_i} - N_t^{x_i}) e^{\Gamma_t^{x_i}} \int_t^T \bar{Z}_t^{x_i,u} F_s^u(t, S_t) \,\mathrm{d}u$, *and the investment in the family of (discounted) longevity bonds $Y = (Y^{x_1}, \dots, Y^{x_m})$ is given by $\xi_t^Y = (\xi_t^{Y^{x_1}}, \dots, \xi_t^{Y^{x_m}})$ with $\xi_t^{Y^{x_i}} =$*
$\zeta(x_i)(n^{x_i} - N_t^{x_i}) \frac{e^{\Gamma_t^{x_i}} e^{r(T-t)}}{Z_t^{x_i,T} \beta^{x_i,T}(t)} \int_t^T F^u(t, S_t) Z_t^{x_i,u} (\hat{\beta}^{x_i,u}(t) + \beta^{x_i,u}(t) \hat{Z}_t^{x_i,u}) \,\mathrm{d}u$
and $L_t^{ti} = \sum_{i=1}^m \zeta(x_i) \int_{]0,t]} \left(\frac{f(S_s)}{B_s} - \mathbb{E}\left[\int_s^T \frac{f(S_u)}{B_u} e^{\Gamma_s^{x_i} - \Gamma_u^{x_i}} \,\mathrm{d}\Gamma_u^{x_i} \,\middle|\, \mathcal{F}_s \right] \right) \mathrm{d}M_s^{x_i}$,
$t \in [0, T]$.

Example: Gaussian intensity field model

- $\mu_{t,x} = \bar{\mu}(t,x) + O_{t,x}$, $t \in [0,T]$, $x \in I$, where $\bar{\mu}$ is a deterministic function and $O_{t,x} = \frac{\sigma}{\sqrt{2\theta\alpha}} e^{-\theta t} e^{-\alpha x} W_{\nu_1(t),\nu_2(x)}$, $t \in [0,T]$, $x \in I$, with $\nu_1(t) = e^{2\theta t}$, $\nu_2(x) = e^{2\alpha x}$ for α, $\theta > 0$
- $\bar{\mu}(t,x) = \exp\left(a(x) + b(x)k(t)\right)$, $t \in [0,T]$, $x \in I$ (Lee-Carter model)
- $\mathbb{E}[\mu_{t,x}] = \bar{\mu}(t,x)$ and $\mathrm{Cov}(\mu_{t,x}, \mu_{s,y}) = \frac{\sigma^2}{2\alpha\theta} e^{-\theta|t-s|} e^{-\alpha|x-y|}$, as well as $\mathrm{Corr}(\mu_{t,x}, \mu_{s,y}) = e^{-\theta|t-s|} e^{-\alpha|x-y|}$, $t \in [0,T]$, $x, y \in I$
- $\mathrm{d}\mu_t^x = \theta\left[\left(\bar{\mu}(t,x) + \frac{\partial_t \bar{\mu}(t,x)}{\theta}\right) - \mu_t^x\right]\mathrm{d}t + \frac{\sigma}{\sqrt{\alpha}}\mathrm{d}\tilde{W}_t^{\nu_2(x)}$, $t \in [0,T]$, where $\tilde{W}_t^{\nu_2(x)} := \frac{W_{t,\nu_2(x)}}{\sqrt{\nu_2(x)}}$, $t \in [0,T]$, is a standard Brownian motion
  $\rightarrow \mu^x$ affine
- Sharp bracket: $\langle \mu^x, \mu^y \rangle_t = \frac{\sigma^2}{\alpha} e^{-\alpha|x-y|} t$, $t \in [0,T]$, $x, y \in I$

Thank you very much for your attention → Questions?

References I

[1] R. J. Adler. *The Geometry of Random Fields*. John Wiley, Chichester, 1981.

[2] K. Andreev. *Demographic surfaces: estimation, assessment and presentation, with application to Danish mortality, 1835-1995*. PhD thesis, University of southern Denmark, Odense, 1999.

[3] J. P. Ansel and C. Stricker. Décomposition de Kunita-Watanabe. In *Séminaire de Probabilités XXVII*, Lecture Notes in Mathematics Volume 1557. Springer, 1993.

[4] F. Biagini and I. Schreiber. Risk-minimization for life insurance liabilities. Preprint, University of Munich, 2012.

References II

[5] F. Biagini, C. Botero, and I. Schreiber. Risk-minimization for life insurance liabilities with dependent mortality risk. Preprint, University of Munich, 2012.

[6] F. Biagini, T. Rheinländer, and I. Schreiber. Risk-minimization for life insurance liabilities with basis risk. Preprint, University of Munich and London School of Economics and Political Science, 2012.

[7] T. R. Bielecki and M. Rutkowski. *Credit Risk: Modeling, Valuation and Hedging*. Springer Finance, 2004.

[8] E. Biffis. Affine processes for dynamic mortality and actuarial valuations. *Insurance: Mathematics & Economics*, 37(3):443–468, 2005.

[9] E. Biffis and P. Millossovich. A bidimensional approach to mortality risk. *Decisions in Economics an Finance*, 29(2):71–94, 2006.

References III

[10] D. Blake, A. J. G. Cairns, and K. Dowd. The birth of the life market. *Asia-Pacific Journal of Risk and Insurance*, 3(1):6–36, 2008.

[11] A. J. G. Cairns, D. Blake, and K. Dowd. Pricing death: Frameworks for the valuation and securitization of mortality risk. *ASTIN Bulletin*, 36(1):79–120, 2006.

[12] M. Dahl. Stochastic mortality in life insurance: Market reserves and mortality-linked insurance contracts. *Insurance: Mathematics & Economics*, 35(1):113–136, 2004.

[13] D. Duffie, J. Pan, and K. Singleton. Transform analysis and asset pricing for affine jump-diffusions. *Econometrica*, 68:1343–1376, 2000.

[14] D. Duffie, D. Filipović, and W. Schachermayer. Affine processes and applications in finance. *Annals of Applied Probability*, 13(3): 984–1053, 2003.

References IV

[15] H. Föllmer and D. Sondermann. Hedging of non-redundant contingent claims. In A. Mas-Colell and W. Hildenbrand, editors, *Contributions to Mathematical Economics*, pages 205–223. North Holland, 1986.

[16] R. S. Goldstein. The term structure of interest rates as a random field. *Review of Financial Studies*, 13(2):365–384, 2000.

[17] D. Kennedy. The term structure of interest rates as a Gaussian random field. *Mathematical Finance*, 4:247–258, 1994.

[18] T. Møller. Risk-minimizing hedging strategies for insurance payment processes. *Finance and Stochastics*, 4(5):419–446, 2001.

[19] M. Schweizer. A guided tour through quadratic hedging approaches. In E. Jouini, J. Cvitanic, and M. Musiela, editors, *Option Pricing, Interest Rates and Risk Management*, pages 538–574. Cambridge University Press, 2001.

Affine Diffusion Processes (Duffie et al. [14])

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space, $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ satisfying the usual conditions and let X be an n-dimensional $\mathbb{F}$-Markov process specified as the strong solution to the following SDE:

$$\mathrm{d}X_t = \delta(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t,$$

where W is an $\mathbb{F}$-standard Brownian motion in $\mathbb{R}^n$,
$\delta(t,x) = d_0(t) + d_1(t)x$ with $d_0 : [0, T] \to \mathbb{R}^n$ and $d_1 : [0, T] \to \mathbb{R}^{n \times n}$ continuous and $(\sigma(t,x)\sigma(t,x)')_{ij} = (v_0(t))_{ij} + (v_1(t))'_{ij} x$ with $v_0 : [0, T] \to \mathbb{R}^{n \times n}$ and $v_1 : [0, T] \to \mathbb{R}^{n \times n \times n}$ also continuous. Let $c \in \mathbb{C}$, $a, b \in \mathbb{C}^n$ and $\Lambda(t,x) = \lambda_0(t) + \lambda_1(t)'x$ for $\lambda_0 : [0, T] \to \mathbb{R}$ and $\lambda_1 : [0, T] \to \mathbb{R}^n$ continuous. Under certain technical conditions for $0 \le t \le u \le T$ the following expression holds:

$$\mathbb{E}\left[e^{-\int_t^u \Lambda(s,X_s)\mathrm{d}s} e^{a'X_u}\left(b'X_u + c\right) \,\middle|\, \mathcal{F}_t\right] = e^{\alpha^u(t)+\beta^u(t)'X_t}\left[\hat{\alpha}^u(t) + \hat{\beta}^u(t)'X_t\right]$$

where α^u and β^u are functions solving the following ODEs:

$$\partial_t\beta^u(t) = \lambda_1(t) - d_1(t)'\beta^u(t) - \frac{1}{2}\beta^u(t)'v_1(t)\beta^u(t),$$
$$\partial_t\alpha^u(t) = \lambda_0(t) - d_0(t)'\beta^u(t) - \frac{1}{2}\beta^u(t)'v_0(t)\beta^u(t),$$

and $\hat{\alpha}^u$ and $\hat{\beta}^u$ are functions solving the following ODEs:

$$\partial_t\hat{\beta}^u(t) = -d_1(t)'\hat{\beta}^u(t) - \beta^u(t)'v_1(t)\hat{\beta}^u(t),$$
$$\partial_t\hat{\alpha}^u(t) = -d_0(t)'\hat{\beta}^u(t) - \beta^u(t)'v_0(t)\hat{\beta}^u(t),$$

for $t \in [0, u]$ with boundary conditions $\alpha^u(u) = 0$, $\beta^u(u) = a$ and $\hat{\beta}^u(u) = b$, $\hat{\alpha}^u(u) = c$.

Proof.

Compute GKW decomposition of A_T^{ti} - main steps:

- $\mathbb{E}[A_T^{ti} \mid \mathcal{G}_t] = \sum_{i=1}^m \zeta(x_i) \sum_{j=1}^{n^{x_i}} \mathbb{E}\left[\frac{f(S_{\tau^{x_i,j}})}{B_{\tau^{x_i,j}}} \mathbb{1}_{\{\tau^{x_i,j} \leq T\}} \,\middle|\, \mathcal{G}_t \right]$, $t \in [0, T]$
- $\mathcal{G}_t \to \mathcal{F}_t$:
$\mathbb{E}\left[\frac{f(S_{\tau^{x_i,j}})}{B_{\tau^{x_i,j}}} \mathbb{1}_{\{\tau^{x_i,j} \leq T\}} \,\middle|\, \mathcal{G}_t \right] = U_0^{x_i} + \int_0^t L_s^{x_i,j} \,\mathrm{d}U_s^{x_i} + \int_{]0,t]} \cdots \mathrm{d}M_s^{x_i,j}$
- $U_t^{x_i} = \mathbb{E}\left[\int_0^T \frac{f(S_u)}{B_u} e^{-\Gamma_u^{x_i}} \mathrm{d}\Gamma_u^{x_i} \middle| \mathcal{F}_t \right] =$
$\int_0^T \mathbb{E}\left[\frac{f(S_u)}{B_u} \middle| \mathcal{F}_t^X \right] \mathbb{E}\left[e^{-\Gamma_u^{x_i}} \mu_u^{x_i} \middle| \mathcal{F}_t^\mu \right] \mathrm{d}u$, $t \in [0, T]$, $i = 1, \ldots, m$
- For $0 \leq t, u \leq T$, $i = 1, \ldots, m$: $\bar{Z}_t^{x_i,u} = \mathbb{E}\left[e^{-\Gamma_u^{x_i}} \mu_u^{x_i} \,\middle|\, \mathcal{F}_t^\mu \right] =$
$\bar{Z}_0^{x_i,u} + \int_0^t Z_s^{x_i,u} \sigma_s^{x_i} \left[\hat{\beta}^{x_i,u}(s) + \beta^{x_i,u}(s) \hat{Z}_s^{x_i,u} \right] \mathbb{1}_{\{s \leq u\}} \mathrm{d}\tilde{W}_s^{\nu_2(x_i)}$
- $\mathrm{d}Y_t^{x_i} = \frac{Z_t^{x_i,T}}{B_T} \sigma_t^{x_i} \beta^{x_i,T}(t) \mathrm{d}\tilde{W}_t^{\nu_2(x_i)}$, $t \in [0, T]$, $i = 1, \ldots, m$

□

Example: χ^2-intensity field model

- $\mu_{t,x} = (c(t,x)O_{t,x})^2$, $t \in [0,T]$, $x \in I$ where $c(t,x)$ is a continuously differentiable function
- $\mathbb{E}[\mu_{t,x}] = c^2(t,x)\mathrm{Cov}(O_{t,x}, O_{t,x}) = c^2(t,x)\frac{\sigma^2}{2\theta\alpha}$ for $t \in [0,T]$ and $x \in I$ and $\mathrm{Cov}(\mu_{t,x}, \mu_{s,y}) = 2c^2(t,x)c^2(s,y)\mathrm{Cov}(O_{t,x}, O_{s,y})^2 = \frac{\sigma^4}{2\theta^2\alpha^2}c^2(t,x)c^2(s,y)e^{-2\theta|t-s|}e^{-2\alpha|x-y|}$ i.e. $\mathrm{Corr}(\mu_{t,x}, \mu_{s,y}) = e^{-2\theta|t-s|}e^{-2\alpha|x-y|}$ for $s,t \in [0,T]$ and $x,y \in I$
- $\mathrm{d}\mu_t^x = 2\left(\theta - \frac{\partial_t c(t,x)}{c(t,x)}\right)\left(\frac{\sigma^2}{2\alpha}\bar{c}(t,x) - \mu_t^x\right)\mathrm{d}t + \sqrt{\frac{4}{\alpha}\sigma^2 c^2(t,x)\mu_t^x}\,\mathrm{d}\tilde{W}_t^{\nu_2(x)}$ for $t \in [0,T]$, where $\bar{c}(t,x) = \frac{c^3(t,x)}{(\theta c(t,x) - \partial_t c(t,x))}$ $\rightarrow$ μ^x affine
- $\langle \mu^x, \mu^y \rangle_t = \frac{4\sigma^2}{\alpha}e^{-\alpha|x-y|}\int_0^t \mu_s^x \mu_s^y c(s,x)c(s,y)\,\mathrm{d}s$, $t \in [0,T]$, $x,y \in I$