Stochastic Taylor expansion and stochastic viscosity solutions for nonlinear stochastic partial differential equations

Rainer Buckdahn, Ingo Bulla*, Jin Ma

R. Buckdahn and J. Ma (2001) introduced a notion of *stochastic viscosity solution*, inspired by earlier results of P. L. Lions and P. E. Souganidis (1998). By using a Doss-Sussmann-type transformation and the so-called *backward doubly stochastic differential equations* (BDSDEs) introduced by E. Pardoux and S. Peng (1994), they established the existence and uniqueness of stochastic viscosity solution to the stochastic partial differential equation (SPDE)

$$\begin{aligned} u(t,x) &= u(0,x) + \int_0^t \mathcal{A}u(t,x) + f(x,u(t,x),\sigma^*(x)D_x u(t,x))dt \\ &\quad + \sum_{i=1}^k \int_0^t g(t,x,u(t,x))dB_t^i, \quad (t,x) \in (0,T)\times\mathbb{R}^n, \end{aligned} \tag{1}$$

where B is a k-dimensional Brownian motion, f is Lipschitz continuous, g of class C^2, and $\mathcal{A}$ is the generator of a diffusion process with Lipschitz continuous coefficients.

Our contribution extends the notion of stochastic viscosity solutions to the class of SPDEs defined by (1) but with function g not depending on u but on its spatial derivative. The main difficulty here is how to ¨translate¨ the pivotal results from the deterministic theory to the stochastic case. One of the subtle issues is conceivably the measurability with regard to the variable ω, or in particular, the adaptedness of all the devices involved, so that the stochastic calculus can be applied. In fact, it is this simple requirement that causes most of the tedious work.

Our main tool to overcome these problems is a new type of second order ¨Taylor expansion¨ for Itô random fields generalizing the work on stochastic Taylor developement by R. Buckdahn and J. Ma (2002). Buckdahn and Ma proved a stochastic Taylor expansion of Itô random fields at random points (τ,ξ) (τ stopping time and ξ random variable measurable with respect to the σ-field of events occuring before time τ), which holds everywhere on the probability space with the exception of a null set $N_{\tau,\xi}$. We are now able to prove that there exists a universal null set N outside of which we have the stochastic Taylor expansion for any measurable expansion point. The second principal tool is the recently developed theory on BDSDEs (E. Pardoux, S. Peng, 1994). It is mainly used to prove existence of the stochastic viscosity solution. More precisely, we show that the solution of a certain BDSDE provides a stochastic viscosity solution.

*Corresponding author

E-mail addresses: Rainer.Buckdahn@univ-brest.fr (R. Buckdahn), ingobulla@gmx.de (I. Bulla), majin@math.purdue.edu (J. Ma)