Problem of random summation and its role in risk aggregation models

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Problem of random summation and its role in risk aggregation models

$$\sum_{j=1}^{N(t)} Y_j$$

- Y_j are assumed to be *iid* random variables
- $N(t)$ is a *counting process*

Initial approach

PhD topic : Insurance models with stochastic premium

$$S(t) = u + ct - \sum_{j=1}^{N(t)} Y_j$$

$$\Psi(u) = \mathbf{P}\left(\inf\{\, S(t)\,\} < 0 \mid S(0) = u\,\right)$$

Proposed model:

$$S(t) = u + \sum_{i=1}^{\widetilde{N}(t)} X_i - \sum_{j=1}^{N(t)} Y_j$$

- G. Temnov. (2004). Risk process with random income. Journal of Mathematical Sciences. 121 (2), 236 - 244.

Solution to ruin problems

Solution to classical ruin problem : Pollaczek – Khinchine Formula

$$\Psi_{cl}(z) = (1 - \frac{\lambda_1 a_X}{c}) \sum_{k=0}^{\infty} \left(\frac{\lambda_1 a_X}{c} \right)^k (1 - \widetilde{F}_X^{*k}(z)), \tag{1}$$

where

$$\widetilde{F}_X(x) = \frac{1}{a_X} \int_0^x (1 - F_X(y))dy, \quad x \geq 0, \tag{2}$$

Case with stochastic income :

$$\Psi(z) = q \sum_{k=0}^{\infty} (1-q)^k (1 - F_h^{*k}(z));$$

$$\ln \frac{1}{1 - (1-q)\widehat{f}_h(s)} = \sum_{n=1}^{\infty} \frac{1}{n} \int_{0+}^{\infty} e^{isx} dW^{n*}(x);$$

$$W(x) = \frac{1}{1 + \lambda_2/\lambda_1} \sum_{k=0}^{\infty} \left(\frac{\lambda_2/\lambda_1}{1 + \lambda_2/\lambda_1} \right)^k F_X(x) * \overline{G}_Y^{*k}(x), \ \overline{G}_Y(x) = 1 - G_Y(-x-0).$$

Work with PRiSMa-Lab

Initial task :

Operational risk measurement

The common scheme: for each BL L_i $(i = 1, \ldots, m)$:

- Modelling loss severity (single–loss df) $F(x) = \mathbf{P}\left(X_1^L < x\right)$
- Modelling loss frequency $\mathcal{L} = \mathbf{P}\left(N^L = n\right)$
- Loss aggregation $S_L = \sum_{j=1}^{N^L} X_j^L \Rightarrow F^{S_L} = ?$

- Basic measure in the capital allocation problem

$$\mathrm{VaR}_\alpha(S_L) = \inf\{s \in \mathbb{R} : \mathbf{P}(S_L > s) \leq 1 - \alpha\}, \text{ OpRisk: } \alpha = 0.999$$

Operational risk measurement - from basic to advanced

- Methodology; numerical techniques : analyzing accuracy, speed ...
- Finding an optimal scheme

Taking into account factors that affect regularity of data:

- Peculiarity of severity distributions (e.g., presence of outlying data points)
- Inflation, trends and other scaling factors
- Dependence between different types of risks

Loss aggregation and ch.f.

$$S_{N(t)} = \sum_{k=1}^{N(t)} X_k, \quad \mathbf{P}(X_k < x) =: F_X(x), \quad \mathbf{P}(N(t) = k) =: \alpha_k \tag{3}$$

$h(x) = \sum_k \alpha_k x^k,$

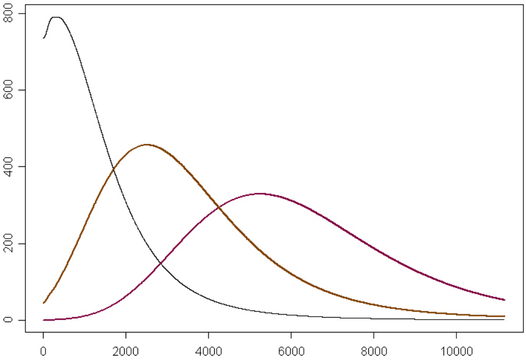
Characteristic function (ch.f.)

$$\widehat{f}_X(u) = \int_{-\infty}^{\infty} e^{iux} dF_X(x); \quad \boxed{\widehat{g}_S(u) = h\left(\widehat{f}_X(u)\right) = \sum_k a_k \widehat{f}_X^k(u)}$$

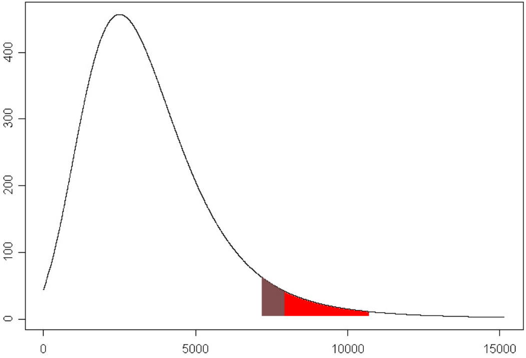
$$g_S(x) = \sum_k a_k f_X^{*k}(x). \tag{4}$$

$$\text{Poisson: } g_S(u) = \sum_k \frac{(\lambda t \widehat{f}_X(u))^k e^{\lambda t}}{k!} = \exp\left[\lambda t(\widehat{f}_X(u) - 1)\right] \tag{5}$$

Aggregate loss distribution



Aggregate loss – error bounds



Quantile as a function of the model parameters

If $Y = H(X)$, where contin. rv X and Y, cdf F_Y; pdf f_X

$$F_Y(y) = \mathbf{P}(Y \leq y) = \mathbf{P}[H(X) \leq y] = \int_{x:\, H(x) \leq y} f_X(x) dx.$$

$$F_Y(y) = \mathbf{P}(Y \leq y) = \mathbf{P}[H(X_1, \ldots, X_n) \leq y] = \int \ldots \int_{\mathbf{x}:\, H(\mathbf{x}) \leq y} f_{\mathbf{X}}(x_1, \ldots x_n) dx_1, \ldots x_n$$

$Q_{\boldsymbol{\theta}} \equiv Q(\boldsymbol{\theta})$: (α, σ); $Q_{\boldsymbol{\theta}}$: $\Omega \in \mathbb{R}^2 \longrightarrow \mathbb{R}$.

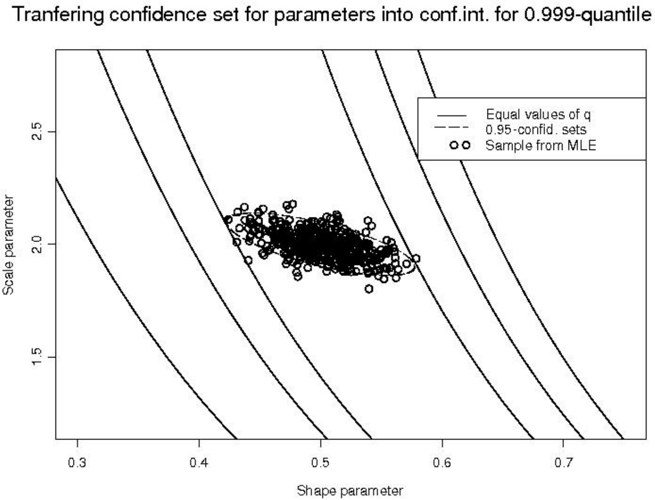
$$T_Q(y) = \mathbf{P}[Q_{\boldsymbol{\theta}}(\alpha, \sigma) \leq y] = \int \ldots \int_{\boldsymbol{\theta}:\, q(\boldsymbol{\theta}) \leq y} T_{\boldsymbol{\theta}}(\alpha,\, \sigma) d\alpha d\sigma$$

$$\text{On the other hand, } 0.95 = \int \ldots \int_{\boldsymbol{\theta} \in \Omega_{0.95}} T_{\boldsymbol{\theta}}(\alpha,\, \sigma) d\alpha d\sigma\,.$$

$$\int \ldots \int_{\boldsymbol{\theta} \in \Omega_{0.95}} T_{\boldsymbol{\theta}}(\alpha,\, \sigma) d\alpha d\sigma \;>\; \int \ldots \int_{\boldsymbol{\theta}:\, q_1 \leq q(\boldsymbol{\theta}) \leq q_2} T_{\boldsymbol{\theta}}(\alpha,\, \sigma) d\alpha d\sigma\,,$$

for such $q_1 = \inf\{q \in \mathbb{R} : \; Q(\boldsymbol{\theta}) = q\,,\, \boldsymbol{\theta} \in \Theta \; (\Theta \subset \Omega)\}$
and $q_2 = \sup\{q \in \mathbb{R} : \; Q(\boldsymbol{\theta}) = q\,,\, \boldsymbol{\theta} \in \Theta \; (\Theta \subset \Omega)\}$.

Quantile as a function of the model parameters



Confidence intervals for the quantile

Line N	**Parameters / Error bounds**	**VaR** (FFT)	lower bound	upper bound
Line 1	$\xi = 1.12\ (0.95\,,\ 1.29)$ $\beta = 7460\ (6326\,,\ 8594)$	656.12	115	3738
Line 4	$\xi = 0.52\ (0.58\,,\ 0.46)$ $\beta = 1.38 \cdot 10^6\ (1.21\,,\ 1.55) \cdot 10^6$	27.3	18	44
Line 7	$\xi = 1.2\ (1.1\,,\ 1.3)$ $\beta = 15600\ (14352\,,\ 16848)$	209.47	94	468

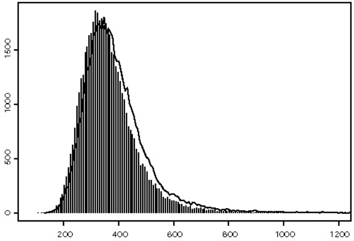
Table: VaR bounds from confidence intervals

Bayesian inference and and MCMC for modelling VaR (accounting for uncertainty)

$$\pi_{\theta\,|\,X}(\theta\,|\,\mathbf{x}) = \frac{f_{X\,|\,\theta}(\mathbf{x}\,|\,\theta)\pi(\theta)}{\int f_{X\,|\,\theta}(\mathbf{x}\,|\,\theta)\pi(\theta)d\theta}, \tag{6}$$

$\pi_{\theta\,|\,X}(\theta\,|\,\mathbf{x})$ — posterior, $\pi(\theta)$ — prior

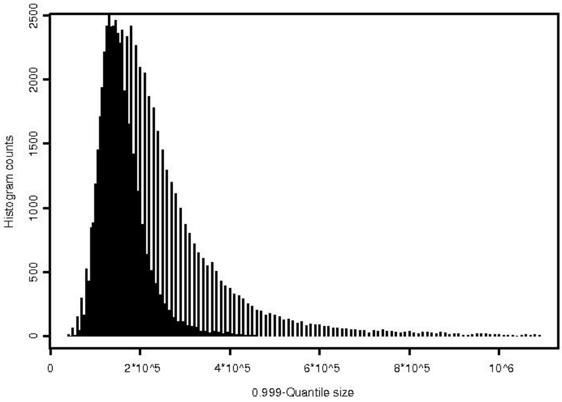
Even in the case of **Pareto** $(F(x) = 1 - (1 + \frac{x}{\beta})^{-1/\xi})$ and **Poisson** joint model, $\pi_{\theta\,|\,X}(\theta\,|\,\mathbf{x}) = \text{explicit}(\pi(\theta))$ is not always possible



Histogram of the full predictive distribution (10^4 points) using whole and censored data

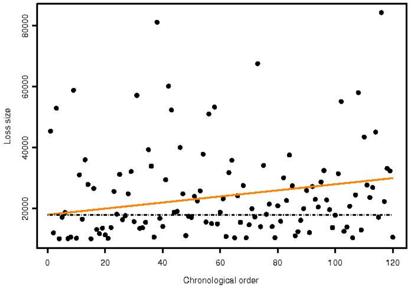
Quantiles of the full predictive distribution

A sample from the predictive distribution is considered is simulated by MCMC. As the size **N** of the observed sample $\mathbf{X} = \{X_i\}_{i=1,\ldots,N}$ increases, asymptotically, $h(z \mid \mathbf{X}) \xrightarrow[N\to\infty]{} g(z \mid \overline{\theta})$



Fig

Varying threshold



$$\pi_{\Theta\,|\,\mathbf{X}}(\theta\,|\,\mathbf{x}) \propto f_{\mathbf{X}\,|\,\Theta}(\mathbf{x}\,|\,\theta)\pi(\theta). \tag{7}$$

$$f_{\mathbf{X}\,|\,\Theta}(\mathbf{x}\,|\,\theta) = \prod_{i=1}^{N} f^{(T)}(X_i\,|\,L_{t_i},\alpha)g_{\lambda_i}(\tau_i\,|\,\sigma),\ f^{(T)}(\cdot) = \frac{f(X_i\,|\,\alpha)}{1-F(L_i\,|\,\alpha)}\,.$$

Influence of the inflation

Investigations of mis-specified models

Peter Grandits & GT, 2010

$$\widetilde{G}_{\alpha,\sigma}(y) = 1 - \left(1 + \frac{y}{\sigma}\right)^{-\alpha}, \quad l_{\alpha,\sigma}(\mathbf{Y}) = \frac{\alpha^n}{\sigma^n} \prod_{i=1}^{n} \left(1 + \frac{Y_i}{\sigma}\right)^{-\alpha-1}$$

The system of ML equations

$$\begin{cases} \widehat{\alpha}_n - \dfrac{n}{\sum\limits_{i=1}^{n} \ln\left(1+\frac{\widetilde{Y}_i}{\widehat{\sigma}_n}\right)} = 0, \\ \dfrac{1}{\sum\limits_{i=1}^{n} \ln\left(1+\frac{\widetilde{Y}_i}{\widehat{\sigma}_n}\right)} \sum\limits_{i=1}^{n} \dfrac{\widetilde{Y}_i}{\widehat{\sigma}_n^2+\widehat{\sigma}_n \widetilde{Y}_i} + \dfrac{1}{n} \sum\limits_{i=1}^{n} \dfrac{\widetilde{Y}_i}{\widehat{\sigma}_n^2+\widehat{\sigma}_n \widetilde{Y}_i} - \dfrac{1}{\widehat{\sigma}_n} = 0. \end{cases} \tag{8}$$

Inflation incoming $Y_i \to \widetilde{Y}_i \equiv Y_i q^i = Y_i e^{\frac{rTi}{n}}$

r **yearly inflation rate** (if you do not take into account inflation at all)
or an error in the estimation of inflation

Results: Influence of inflation and trends

Inflation

$Y_i = X_i e^{\frac{rTi}{n}}$

$$\begin{cases} \Delta\alpha_r &= 0, \\ \Delta\sigma_r &= \frac{r\sigma^* T}{2}. \end{cases}$$

P. Grandits, GT, 2010

$Y_i = X_i q(r, T_i) + d(r, T_i)$

(trend in scaling and location parameter)

A special case: $Y_i = X_i e^{rT_i} + ArT_i$

$$\begin{aligned} \Delta\alpha_r &= \frac{(\alpha^* + 1)(\alpha^*)^2}{\sigma^*} A \frac{rT_{\max}}{2} \\ \Delta\sigma_r &= \sigma^* \frac{rT_{\max}}{2} + \left(\alpha^* + (1 + \alpha^*)^2\right) A \frac{rT_{\max}}{2}. \end{aligned}$$

P. Grandits, R. Kainhofer & GT, 2010

Influence of inflation : case with positive threshold

$$\begin{aligned}
\overline{\alpha}_r(0) &= \frac{(\alpha^*)^2 \cdot T}{2} \cdot \frac{A_2 C_1 - A_1 C_2}{(\alpha^*)^2 A_1^2 - A_2} \\
\overline{\sigma}_r(0) &= \frac{\sigma^* \cdot T}{2} \cdot \frac{\alpha^* A_1 C_1 - C_2}{(\alpha^*)^2 A_1^2 - A_2}
\end{aligned}$$

$$\begin{aligned}
A_1 &:= \left(\frac{\sigma^*}{\sigma^* + L}\right) \cdot \frac{1}{\alpha^* + 1} \\
A_2 &:= \left(\frac{\sigma^*}{\sigma^* + L}\right)^2 \cdot \frac{\alpha^*}{\alpha^* + 2} \\
C_1 &:= A_1 + \frac{L}{L + \sigma^*} \\
C_2 &:= A_2 + \frac{\alpha^* L}{\sigma^*} \left(\frac{\sigma^*}{L + \sigma^*}\right)^2
\end{aligned}$$

Inflation and threshold - Illustration of the impact

Infl. rate 0.03, period 5 years, true param. $(1, 3)$
True quantile 2999 , True **aggregate** quantile 11374 $(\lambda = 7)$

Misconsideration			Resulting effects			
Trunc. before scaling	Loss scaling	Threshold scaling	α	σ	Quantile	Aggregate quantile
X	X	X	0.94	4.16	6075.4	23242
X		X	0.94	3.78	5520	19730
X	X		1	3.75	3748	15200
	X	X	1	3.4	3399	13720

Actuarial and Financial data : difference and similarities

Financial data :

- Evidence of stability ("scale-invariance", "self-similarity")
- Relevance of the whole distribution

Actuarial data :

- "Heavy-tailedness", relevance of the right tail

Similarities :

- Aggregation of Actuarial data and Increments of Financial data

Extensions of Stable distribution

Regular stability

$$X_1+\cdots+X_n = S_n \stackrel{d}{=} b_n X + a_n \; ; \quad \textbf{Characteristic function} \quad e^{\lambda(-it)^{\gamma}} =: f_{St}(t)$$

Stability under random summation

$$X \stackrel{d}{=} X_1^{(n)} + \cdots + X_{\nu_n}^{(n)} \; ; \quad \textbf{Characteristic function} \quad \mathcal{L}\gamma(-\ln f_{St}(t))$$

Discrete stability

"Binomial operation" $\quad \alpha \circ X = \sum_{j=1}^{X} B_j$, where $B_j \sim$ Bernoulli(α)

and X is some *discrete* r.v. $\quad (X \in \mathbf{Z}_+)$

Then the r.v. X is discrete stable if

$$X \stackrel{d}{=} \frac{1}{N^{1/\gamma}} \circ (X_1 + \cdots + X_N) \; ; \quad \textbf{Ch. f.} \quad e^{\lambda(e^{it}-1)^{\gamma}}$$

Discrete models for financial data (random summation revisited)

Recall: characteristic function of a Poisson process

$$g(t) = \exp\{\lambda(e^{it} - 1)\}, \ .$$

"Discrete Brownian Motion"

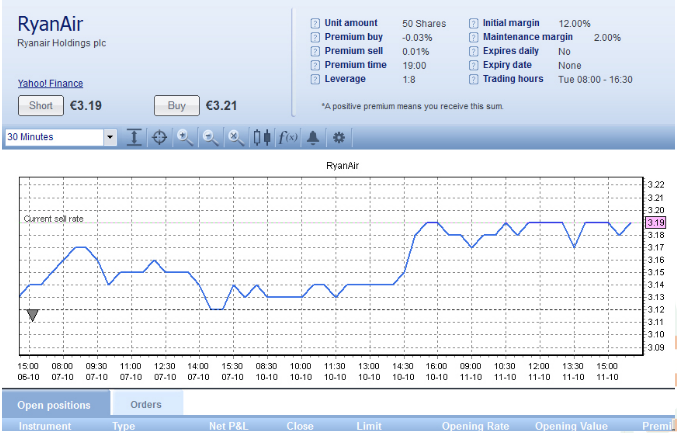
Its characteristic function is

$$g(t) = \exp\{\lambda_1(e^{ia_1 t} - 1) + \lambda_2(e^{ia_2 t} - 1)\} \ .$$

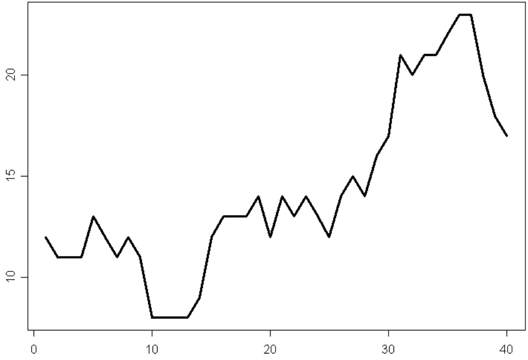
Advantages :

- Explicit link both with random summation and with regular Brownian Motion
- Simple analytic results for first hitting time etc.

Real data example



Simulated "Discrete Br. Motion"



Thanks

Thank you very much for your attention!