# Design of Optimal Cost-Efficient Payoffs

## and Corresponding Investment Strategies

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PRisMa Day 2011

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# Outline

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# Assumptions

- We consider an investor with fixed investment period $I = [0, T]$ or $I = \{0, 1, \ldots, T\}$ and there is no intermediate consumption.
- The investor is just interested in the (probability) distribution function of terminal wealth (law-invariant preferences).
- We consider a perfectly liquid, frictionless and arbitrage-free market with $d$ assets $S^1, S^2, \ldots, S^d$ and a numéraire $B$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in I}, \mathbb{P})$.
- We assume that there exists a state-price process.

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- We assume that there exists a state-price process.

# State-price process

**Definition: State-price process**

A $\mathbb{P}$-a.s. non-negative adapted stochastic (càdlàg in the continuous-time case) process $\{\xi_t\}_{t\in I}$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in I}, \mathbb{P})$ is called a state-price process if $\{\xi_t S_t\}_{t\in I}$ is a $\mathbb{P}$-martingale.

Under the existence of an equivalent martingale $\mathbb{Q}$ measure a version of the state-price process is given by (a càdlàg version in the continuous-time case) of $\frac{1}{B_t}\mathbb{E}_{\mathbb{P}}[d\mathbb{Q}/d\mathbb{P}|\mathcal{F}_t]$.

# What is cost-efficiency?

**Definition: The initial cost**

The initial cost of a terminal (at time $T$) payoff $h \in L^0(\Omega, \mathcal{F}_T, \mathbb{P})$ with $\mathbb{E}_\mathbb{P}[(\xi_T h)^-] < \infty$ is given by

$$c(h) := \mathbb{E}_\mathbb{P}[\xi_T h]\Big( = \mathbb{E}_\mathbb{Q}\Big[\frac{1}{B_T} h\Big]\Big).$$

**Definition: Cost-efficiency**

A terminal payoff $h \in L^0(\Omega, \mathcal{F}_T, \mathbb{P})$ with initial cost $c(h)$ is cost-efficient if every other payoff $\tilde{h}$, with $\mathbb{E}_\mathbb{P}[(\xi_T \tilde{h})^-] < \infty$, which has the same distribution as $h$ at time $T$ does not have a lower initial cost, i.e.

$$c(h) \leq c(\tilde{h})$$

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# History

- Dybvig (1988) uses a preference-free framework to compare two terminal payoffs by analysing their cost. He gives a characterization of the optimal payoff in complete one-dimensional markets.
- Vanduffel (2009) shows similar results for incomplete one-dimensional Lévy markets using the Esscher transform.
- Bernard, Boyle and Vanduffel (2011) give an explicit representation of cost-efficient payoffs (mostly under the assumption that the state-price density $\xi_T$ has a continuous distribution function). Moreover they extend the theory by adding state-dependent constraints.

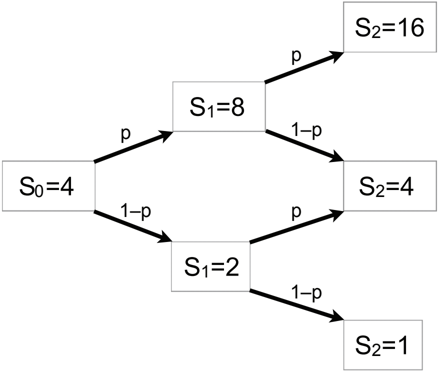
# Introductory example

Let us consider a simple example, with the following properties:

- A market with a bond $B$ and a stock $S$.
- The bond has an effective deterministic interest rate of $0\%$.
- The dynamics of the stock are given by a two-period binomial model.

# Introductory example

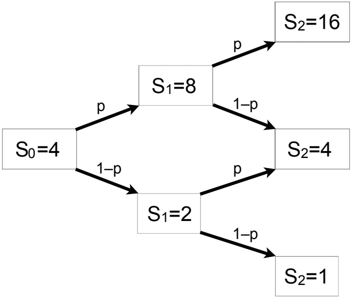
Hence the up and down movements of the underlying stock are independent with a given physical probability of $p = 1/2$.



# Introductory example

Thus we have $\mathbb{P}(S_2 = 16) = 1/4, \mathbb{P}(S_2 = 4) = 1/2$ and $\mathbb{P}(S_2 = 1) = 1/4$. The risk-neutral probability for the stock to double is given by

$$q = \frac{1 - \frac{1}{2}}{2 - \frac{1}{2}} = \frac{1}{3}.$$



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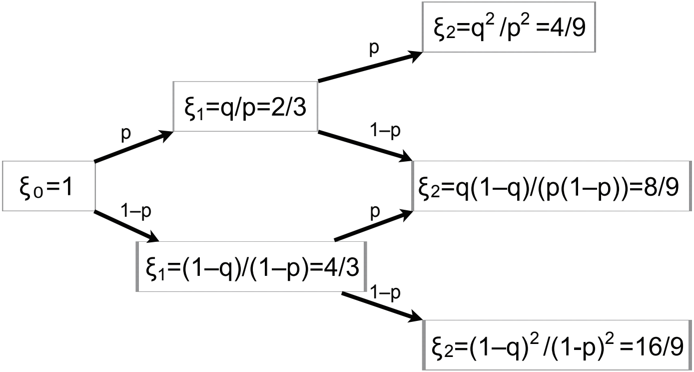
Then the state-price process can be calculated as follows.



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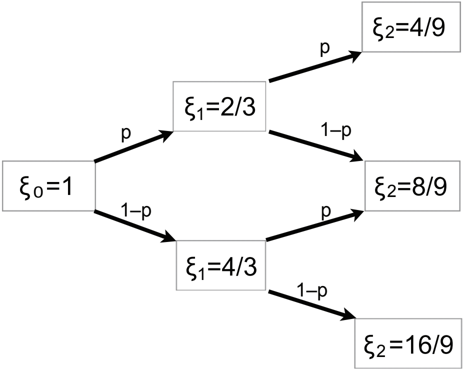


Consider two payoffs $h$ and $\tilde{h}$.

$$h := \begin{cases} 1 & \text{for } S_2 = 16, \\ 2 & \text{for } S_2 = 4, \\ 3 & \text{for } S_2 = 1, \end{cases}$$

$$\tilde{h} := \begin{cases} 3 & \text{for } S_2 = 16, \\ 2 & \text{for } S_2 = 4, \\ 1 & \text{for } S_2 = 1. \end{cases}$$

It is immediate that $h$ and $\tilde{h}$ have the same distribution under $\mathbb{P}$.

# Introductory example

Hence for every Borel-measurable function $u : \{1, 2, 3\} \to \mathbb{R}$, in particular for every utility function, we have

$$\mathbb{E}_{\mathbb{P}}[u(h)] = \mathbb{E}_{\mathbb{P}}[u(\tilde{h})].$$

BUT, for the initial costs of the two payoffs we get

$$c(h) = \mathbb{E}_{\mathbb{P}}[\xi_2 h] = \frac{q^2}{p^2} p^2 1 + \frac{q(1-q)}{p(1-p)} 2p(1-p)2 + \frac{(1-q)^2}{(1-p)^2}(1-p)^2 3$$
$$= \frac{1}{9} + \frac{8}{9} + \frac{12}{9} \approx 2.33,$$

and

$$c(\tilde{h}) = \frac{3}{9} + \frac{8}{9} + \frac{4}{9} \approx 1.67.$$

# Outline

# An important definition

## Definition: The Set M

For a real-valued random variable $\xi$ on $(\Omega, \mathcal{F}, \mathbb{P})$ and a distribution function $F$ we define

$$M(F, \xi) := \{Y \in L^0(\Omega, \mathcal{F}, \mathbb{P}) \,|\, \mathbb{E}_{\mathbb{P}}[(\xi Y)^-] < \infty, \mathbb{P}Y^{-1} = F\}.$$

Let $X \in L^0(\Omega, \mathcal{F}, \mathbb{P})$ be a given random variable. Then for notational purposes we write $M(X, \xi)$ instead of $M(\mathbb{P}X^{-1}, \xi)$.

# The goal

As before, let $\xi \in L^0(\Omega, \mathcal{F}, \mathbb{P})$ be a real-valued random variable and let $F$ be a distribution function. Then the goal is to find an explicit representation of random variables $Y^*, Z^* \in M(F, \xi)$ for which

1. $$\mathbb{E}[\xi Z^*] = \sup_{Z \in M(F,\xi)} \mathbb{E}[\xi Z],$$

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# Co- and countermonotonicity

### Definition: For subsets

For a subset $A \subseteq \mathbb{R}^2$ we define the following:

1. $A$ is called comonotonic if $(x_1 - x_2)(y_1 - y_2) \geq 0$ for all $(x_i, y_i) \in A$ with $i \in \{1, 2\}$.
2. $A$ is called countermonotonic if $(x_1 - x_2)(y_1 - y_2) \leq 0$ for all $(x_i, y_i) \in A$ with $i \in \{1, 2\}$.

### Definition: For random pairs

A random pair $(\xi, X) \in L^0(\Omega, \mathcal{F}, \mathbb{P}) \times L^0(\Omega, \mathcal{F}, \mathbb{P})$ is called comonotonic (countermonotonic) if there exists a comontonic (countermontonic) Borel-measurable subset $A \subseteq \mathbb{R}^2$ with $\mathbb{P}((\xi, X) \in A) = 1$.

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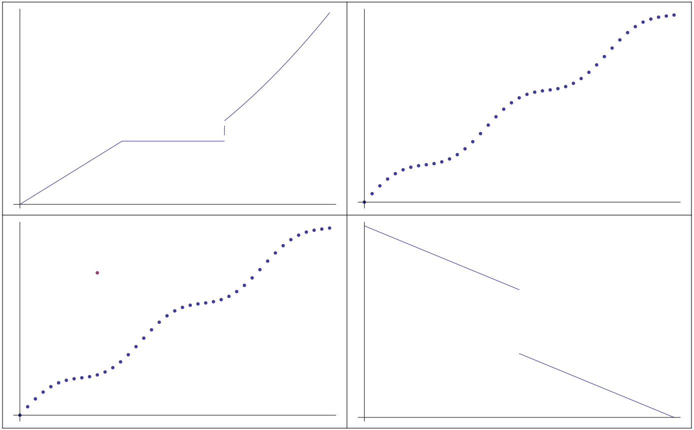
# Illustration of co- and countermonotonicity

Comonotonic | Comonotonic



Neither-nor | Countermonotonic

# A generalized concept

Consider an increasing function $g : \mathbb{R} \to \mathbb{R}$ and a real-valued random variable $\xi \in L^0(\Omega, \mathcal{F}, \mathbb{P})$.

If $\xi(\omega_1) < \xi(\omega_2)$ for $\omega_1, \omega_2 \in \Omega$, then it immediately follows that, by the monotonicity of $g$, $g(\xi(\omega_1)) \leq g(\xi(\omega_2))$.

$\Rightarrow (\xi, g(\xi))$ is a comonotonic pair.

Correspondingly: $g$ decreasing $\Rightarrow (\xi, g(\xi))$ is a countermonotonic pair.

# The central lemma

**Lemma: Optimal bounds for payoffs**

Consider two real-valued random variables $\xi, X \in L^0(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}[\xi^-] < \infty$ and $\mathbb{E}[(\xi X)^-] < \infty$.

1. Then, if the pair $(\xi, X)$ is comonotonic, then
$$\mathbb{E}[\xi X] \geq \mathbb{E}[\xi Y], \quad Y \in M(X, \xi).$$
2. Then, if the pair $(\xi, X)$ is countermonotonic, then
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3. If in addition $\mathbb{E}[|\xi X|] < \infty$, then the equalities above hold if and only if $(\xi, X)$ is comonotonic or countermonotonic, respectively.

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# The lower quantile function

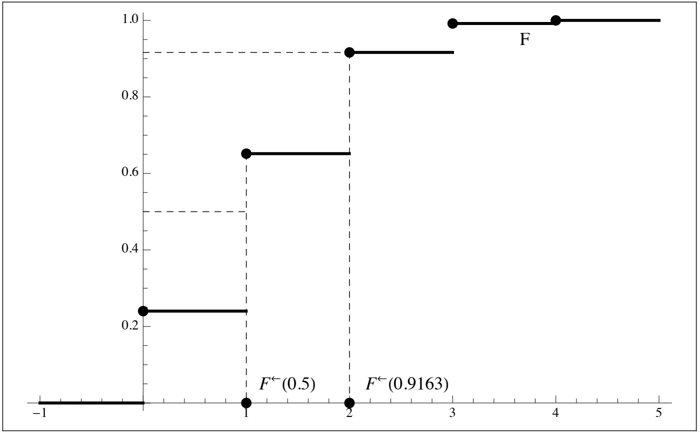
**Definition: The lower quantile function**

Let $F$ be a distribution function. Then the lower quantile function $F^{\leftarrow} : [0,1] \to \overline{\mathbb{R}}$ of $F$ is defined as

$$F^{\leftarrow}(y) := \inf\{x \in \overline{\mathbb{R}} | F(x) \geq y\}.$$

# The lower quantile function

Example: Two values of the lower quantile function of a binomial distribution $F$ with parameters $n = 4$ and $p = 0.4$.



# Theorem 1

## Theorem: General representation of optimal payoffs

Consider a real-valued random variable $\xi \in L^0(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}[\xi^-] < \infty$ and a given distribution function $F$. Then, if there exists a real-valued random variable $\tilde{\xi} \in L^0(\Omega, \mathcal{F}, \mathbb{P})$ (let $\tilde{G}$ denote its distribution function) such that the pair $(\xi, \tilde{\xi})$ is comonotonic and such that

1. $\mathrm{im}(F) \subseteq \overline{\mathrm{im}(\tilde{G})}$ and $\mathbb{E}[(\xi Z^\star)^-] < \infty$ for $Z^\star := F^{\leftarrow}(\tilde{G}(\tilde{\xi}))$, then $Z^\star \in M(F, \xi)$ and

$$\mathbb{E}[\xi Z^\star] \geq \mathbb{E}[\xi Z], \quad Z \in M(F, \xi).$$

2. $\mathrm{im}(1 - F) \subseteq \overline{\mathrm{im}(\tilde{G})}$ and $\mathbb{E}[(\xi Y^\star)^-] < \infty$ for $Y^\star := F^{\leftarrow}(1 - \tilde{G}(\tilde{\xi}-))$, then $Y^\star \in M(F, \xi)$ and

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## Corollary: Explicit representation of optimal payoffs

Consider a real-valued random variable $\xi \in L^0(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}[\xi^-] < \infty$ and let $G$ denote its distribution function. Consider a given distribution function $F$.

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2. Then, if $\operatorname{im}(1-F) \subseteq \overline{\operatorname{im}(G)}$ and if $\mathbb{E}[(\xi Y^\star)^-] < \infty$ for $Y^\star := F^\leftarrow(1 - G(\xi-))$, then $Y^\star \in M(F,\xi)$ and

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3. If in addition $\mathbb{E}[|\xi Y^\star|] < \infty$ or $\mathbb{E}[|\xi Z^\star|] < \infty$, then $Y^*$ or $Z^*$, respectively, is the a.s. unique optimal payoff in $M(F,\xi)$.

# Remarks to Corollary 1

- If $G$ is continuous, then $\overline{\mathrm{im}(G)} = [0,1]$. Thus, for every distribution function $F$,

$$\mathrm{im}(F) \subseteq \overline{\mathrm{im}(G)}$$

  and

$$\mathrm{im}(1-F) \subseteq \overline{\mathrm{im}(G)}.$$

- Suppose that $\xi = 1$ a.s. Then $\overline{\mathrm{im}(G)} = \{0,1\}$. Thus, in general, for distribution function $F$ we have

$$\mathrm{im}(F) \not\subseteq \overline{\mathrm{im}(G)}$$

  and

$$\mathrm{im}(1-F) \not\subseteq \overline{\mathrm{im}(G)}.$$

# Preparation for Corollary 2

Consider a random variable $\xi \in L^0(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}[\xi^-] < \infty$, where $G$ denotes its distribution function. Let $F$ be a given distribution function such that $\mathrm{im}(F) \nsubseteq \overline{\mathrm{im}(G)}$ or $\mathrm{im}(1-F) \nsubseteq \overline{\mathrm{im}(G)}$. Then let

- $\{d_j\}_{j\in J}$ ... different points of discontiuity of $G$ for which $(G(d_j-), G(d_j)) \cap \mathrm{im}(F) \neq \varnothing$ or $(G(d_j-), G(d_j)) \cap \mathrm{im}(1-F) \neq \varnothing$, respectively.
- $\{p_j\}_{j\in J}$ ... the corresponding magnitudes.

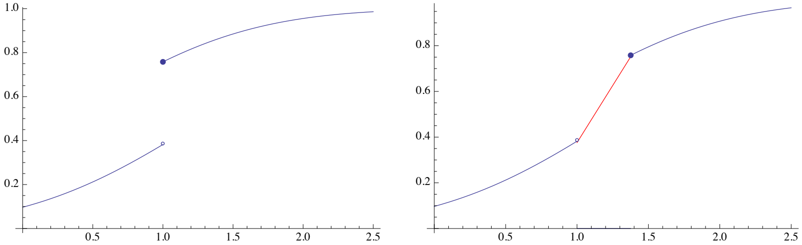
Then the idea is to 'expand' $\xi$ to $\tilde{\xi}$ (where $\tilde{G}$ denotes its distribution function) such that $\mathrm{im}(F) \subseteq \overline{\mathrm{im}(\tilde{G})}$ or $\mathrm{im}(1-F) \subseteq \overline{\mathrm{im}(\tilde{G})}$, respectively. For example, if possible, define

$$\tilde{\xi} := \xi + \sum_{j\in J} p_j 1_{(d_j,\infty)}(\xi) + \sum_{j\in J} p_j E_j 1_{\{d_j\}}(\xi)$$

where $\{E_j\}_{j\in J}$ is a set of random variables with values in $[0,1]$ such that $x \mapsto \mathbb{P}(E_j \leq x \,|\, \xi = d_j)$ is continuous.

# Illustration for Corollary 2

| One discontinuity in the distribution | Removal of it via expansion |
| --- | --- |



# Corollary 2: Expansion

## Corollary: Removal of unwanted atoms

Consider the ‘expanded’ real-valued random variable $\tilde{\xi} \in L^0(\Omega, \mathcal{F}, \mathbb{P})$ and let $\tilde{G}$ denote its distribution function. Moreover, consider a distribution function $F$. Then,

1. if $\mathbb{E}[(\xi Z^\star)^-] < \infty$ for $Z^\star := F^\leftarrow(\tilde{G}(\tilde{\xi}))$, then $Z^\star \in M(F, \xi)$ and
$$\mathbb{E}[\xi Z^\star] \geq \mathbb{E}[\xi Z], \quad Z \in M(F, \xi).$$
2. if $\mathbb{E}[(\xi Z^\star)^-] < \infty$ for $Y^\star := F^\leftarrow(1 - \tilde{G}(\tilde{\xi}-))$, then $Y^\star \in M(F, \xi)$ and
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# Outline

# Increasing von Neumann–Morgenstern preferences

Assume an objective function of the investor, $V$. Then $V$ is said to satisfy increasing von Neumann–Morgenstern preferences if:

- The investor prefers 'more to less', that is $V$ preserves first-order stochastic dominance order. I.e., if for two terminal payoffs $h, \tilde{h} \in L^0(\Omega, \mathcal{F}_T, \mathbb{P})$ we have $F_{\tilde{h}}(x) \geq F_h(x)$ for all $x \in \mathbb{R}$, then $V(\tilde{h}) \leq V(h)$.
- The investor has 'law-invariant preferences', that is if $\mathbb{P}h^{-1} = \mathbb{P}\tilde{h}^{-1}$ for two payoffs $h, \tilde{h} \in L^0(\Omega, \mathcal{F}_T, \mathbb{P})$ at time $T$, then $V(h) = V(\tilde{h})$.

Under these fairly general assumptions, together with a deterministic numéraire, such an investor will prefer a cost-efficient payoff $Y^*$ to any other payoff $h$ with the same terminal payoff distribution. (In particular for expected utility maximizers with increasing utility functions).

# A time-dependent Black–Scholes market

Let $I := [0,T]$ and consider a Brownian motion $\{W_t\}_{t\in I}$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in I}, \mathbb{P})$. Let the constant $S_0 > 0$ denote the initial stock price. Then the underlying stock price process $\{S_t\}_{t\in I}$ is given by the SDE

$$dS_t = \sigma(t)S_t\,dW_t + \mu(t,\cdot)S_t\,dt, \quad t \in I,$$

where $\sigma$ and $\mu$ are assumed to satisfy usual measurability and integrability conditions. Moreover, we define the numéraire $B$ by

$$B_t = \exp\biggl(\int_0^t r(u)\,du\biggr), \quad t \in I,$$

where $r : I \to \mathbb{R}$ is assumed to be sufficiently regular such that there exists a sufficiently regular (Novikov condition) $\theta : I \times \Omega \to \mathbb{R}$ with

$$\sigma(u)\theta(u,\omega) = \mu(u,\omega) - r(u), \quad \text{for almost all } (u,\omega) \in I \times \Omega.$$

# A time-dependent Black–Scholes market

## Lemma: Dynamics of the market

Consider the previous assumptions. Then the up to indistinguishability unique, strong and pathwise continuous solution of the SDE is given by the process

$$S_t = S_0 \exp\Big( \int_0^t \sigma(u)\,dW_u + \int_0^t \Big(\mu(u,\cdot) - \frac{1}{2}\sigma^2(u)\Big)\,du\Big), \quad t \in I,$$

which is strictly positive.

# A deterministic market

Now assume deterministic time-dependent coefficients in our model and define $\Sigma_{s,t} = \sqrt{\int_s^t \sigma^2(u)\,du}$ and $\Theta_{s,t} = \sqrt{\int_s^t \theta^2(u)\,du}$ for all $s,t \in I$ with $s \leq t$.

Then the state-price process $\{\xi_t\}_{t\in I}$ is given by

$$\xi_t = \begin{cases} \frac{1}{B_t}\exp\left(-\int_0^t \theta(u)\,dW_u - \frac{1}{2}\Theta_{0,t}^2\right) & \text{for } t \in I \text{ with } \Theta_{0,t} > 0, \\ \frac{1}{B_t} & \text{for } t \in I \text{ with } \Theta_{0,t} = 0. \end{cases}$$

# A call option

Consider a European call option with terminal payoff $h \in L^1_+(\Omega, \mathcal{F}_T, \mathbb{P})$ given by

$$h := (S_T - K)^+, \quad \text{with } K > 0.$$

Then the continuous version of the cost process (arbitrage-free price) of it is given by

$$c_t(h) = S_t \Phi(d_1(S_t, t)) - K \frac{B_t}{B_T} \Phi(d_2(S_t, t)), \quad t \in I,$$

where $d_{1,2}(x,t) : (0, \infty) \times I \to \mathbb{R}$ is defined by

$$d_{1,2}(x,t) := \frac{\log\left(\frac{x}{K}\right) + \int_t^T r(u)\,du \pm \frac{1}{2}\Sigma^2_{t,T}}{\Sigma_{t,T}}.$$

# A call option

If $\Theta_{0,T} > 0$, then the a.s. unique cost-efficient payoff $Y^*$ of the call option $h$ is given by

$$\begin{aligned} Y^* &= \left( S_T \exp\left( \frac{\Sigma_{0,T}}{\Theta_{0,T}} \int_0^T \theta(u)\, dW_u - \int_0^t \sigma(u)\, dW_u \right) - K \right)^+ \\ &= \left( S_0 \exp\left( \frac{\Sigma_{0,T}}{\Theta_{0,T}} \int_0^T \theta(u)\, dW_u + \int_0^T \mu(u)\, du - \frac{1}{2}\Sigma_{0,T}^2 \right) - K \right)^+ . \end{aligned}$$

# A call option

Moreover define

$$\hat{S}_t := S_0 \exp\left(\frac{\Sigma_{0,T}}{\Theta_{0,T}} \int_0^t \theta(u)\, dW_u + \int_0^t \mu(u)\, du - \frac{1}{2}\Sigma_{0,t}^2\right), \quad t \in I.$$

If $\sigma(u) > 0$ for all $u \in I$, then a version of the cost process of the cost-efficient payoff $Y^*$ is given by

$$c_t(Y^*) = \begin{cases} \frac{B_t}{B_T} Y^* & \text{for } t \geq T_0, \\ \exp\left(\frac{1}{2}\delta_t + \varepsilon_t\right) \hat{S}_t\, \Phi(\hat{d}_1(\hat{S}_t, t)) - K\, \frac{B_t}{B_T} \Phi(\hat{d}_2(\hat{S}_t, t)) & \text{for } t < T_0, \end{cases}$$

where $T_0 := \inf\{t \in I \mid \Theta_{t,T} = 0\}$, $\varepsilon_t := \int_t^T (\mu(u) - r(u))\, du - \frac{\Sigma_{0,T}}{\Theta_{0,T}} \Theta_{t,T}^2$ and $\delta_t := \frac{\Sigma_{0,T}^2}{\Theta_{0,T}^2} \Theta_{t,T}^2 - \Sigma_{t,T}^2$ for all $t \in I$, as well as for all $t \in [0, \infty)$

$$\hat{d}_i(\hat{S}_t, t) := \begin{cases} \frac{\Theta_{0,T}\Sigma_{t,T}}{\Theta_{t,T}\Sigma_{0,T}} \left(d_1(\hat{S}_t, t) + \frac{\delta_t + \varepsilon_t}{\Sigma_{t,T}}\right) & \text{for } i = 1, \\ \frac{\Theta_{0,T}\Sigma_{t,T}}{\Theta_{t,T}\Sigma_{0,T}} \left(d_2(\hat{S}_t, t) + \frac{\varepsilon_t}{\Sigma_{t,T}}\right) & \text{for } i = 2. \end{cases}$$

# A geometric Asian option

Consider a continuously monitored geometric Asian option with terminal payoff $g \in L^1_+(\Omega, \mathcal{F}_T, \mathbb{P})$ given by

$$g := \Big( \exp\Big( \frac{1}{T} \int_0^T \log(S_t)\, dt \Big) - K \Big)^+, \quad \text{with } K > 0.$$

This payoff is dominated by the usual arithmetic Asian option and thus provides a lower bound.

Then the initital cost (arbitrage-free price) of it is given by

$$c(g) = \frac{1}{B_T} \Big( S_0 \exp\Big( \mu_{g,r} + \frac{1}{2}\sigma_g^2 \Big) \Phi(d + \sigma_g^2) - K\,\Phi(d) \Big)$$

where $d := \frac{\log(\frac{S_0}{K}) + \mu_{g,r}}{\sigma_g}$, $\sigma_g = \frac{2}{T^2} \int_0^T \int_0^t \int_0^u \sigma^2(x)\, dx\, du\, dt$ and $\mu_{g,r} = \frac{1}{T} \int_0^T \int_0^t \big( r(u) - \frac{1}{2}\sigma^2(u) \big)\, du\, dt$.

# A geometric Asian option

If $\Theta_{0,T} > 0$, then the a.s. unique cost-efficient payoff $Y^*$ of the geometric Asian option $g$ is given by

$$Y^* = \left( S_0 \exp\left( \frac{\sigma_g}{\Theta_{0,T}} \int_0^T \theta(u)\, dW_u + \mu_g \right) - K \right)^+$$

where $\mu_g = \frac{1}{T} \int_0^T \int_0^t \left( \mu(u) - \frac{1}{2}\sigma^2(u) \right) du\, dt$. The initial cost of it is given by

$$c(Y^*) = \frac{1}{B_T} \Big( S_0 \exp\left( \mu_g + \frac{1}{2}\sigma_g^2 - \sigma_g \Theta_{0,T} \right) \Phi(\hat{d} + \sigma_g^2) - K\, \Phi(\hat{d}) \Big)$$

where $\hat{d} := \frac{\log(\frac{S_0}{K}) + \mu_g - \sigma_g \Theta_{0,T}}{\sigma_g}$.

# A jump in the state-price density

Now assume the following stochastic drift: Let $t_0 \in I$, let $n \in \mathbb{N}$ and let $\theta_i : [t_0, T] \to \mathbb{R}$ be Borel-measurable functions for all $i \in \{1, 2, \dots n\}$ such that $\int_{t_0}^{T} \theta_i^2(u)\, du < \infty$ and such that $\int_{t_0}^{T} |\sigma(u)\theta_i(u)|\, du < \infty$ for all $i \in \{1, 2, \dots, n\}$. Then define

$$\mu(u, \omega) := r(u) + \sum_{i=1}^{n} 1_{[t_0,T]\times A_i}(u, \omega)\mu_i(u), \quad (u, \omega) \in I \times \Omega,$$

where

$$\mu_i(u) := \sigma(u)\theta_i(u), \quad (i, u) \in \{1, 2, \dots, n\} \times [t_0, T],$$

where the $A_i$ are $\mathcal{F}_{t_0}$-measurable and mutually disjoint.

# A jump in the state-price density

Then the state-price density at time $T$ is given by

$$\xi_T = \frac{1}{B_T} \exp\left( -\sum_{i=1}^{n} 1_{A_i} \left( \int_{t_0}^{T} \theta_i(u)\, dW_u + \frac{1}{2}\hat{\Theta}_i^2 \right)\right),$$

where $\hat{\Theta}_i := \sqrt{\int_{t_0}^{T} \theta_i^2(u)\, du}$ for all $i \in \{1, 2, \ldots, n\}$. Moreover, there exists a subset $J \subseteq \{1, 2, \ldots, n\}$ such that $\hat{\Theta}_i > 0$ for all $i \in J$ and such that $\hat{\Theta}_i = 0$ for all $i \in \{1, 2, \ldots, n\} \setminus J$. Then the distribution function $G$ of $\xi_T$ is given by

$$G(x) = \begin{cases} 1 - \sum_{i\in J} \mathbb{P}(A_i) + \sum_{i \in J} \mathbb{P}(A_i) G_i(x) & \text{for } x \geq \frac{1}{B_T}, \\ \sum_{i\in J} \mathbb{P}(A_i) G_i(x) & \text{for } 0 < x < \frac{1}{B_T}, \\ 0 & \text{for } x \leq 0, \end{cases}$$

where $G_i(x) := \Phi((\log(x) + \int_0^T r(u)\, du + \frac{1}{2}\hat{\Theta}_i^2)/\hat{\Theta}_i)$ for all $(x, i) \in (0, \infty) \times J$. Thus $G$ has a point of discontinuity in $1/B_T$.

# A jump in the state-price density

Thus, in general, for a given payoff $h$ with distribution function $F$, we have $\mathrm{im}(1-F) \nsubseteq \overline{\mathrm{im}(G)}$. One possible way is to expand $\xi_T$ as in Corollary 2:

$$\tilde{\xi}_T := \xi_T + p\,1_{(\frac{1}{B_T},\infty)}(\xi_T) + p\,E1_{\{\frac{1}{B_T}\}}(\xi_T)$$

where we define

$$E := \frac{1}{1+\exp(W_T - W_{t_0})}.$$

Now if $\tilde{G}$ denotes the distribution function of $\tilde{\xi}$, then the payoff given by

$$Y^* := F^{\leftarrow}(1-\tilde{G}(\tilde{\xi}-))$$

is cost-efficient.

# Thank you!