

Transaction Costs, Trading Volume, and the Liquidity Premium

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Introduction

Optimal portfolios in frictionless markets

Merton (1971, 1973): Consider a safe and a risky asset, following

$$dS_t^0/S_t^0 = rdt, \quad dS_t/S_t = (\mu + r)dt + \sigma dW_t, \quad \mu, \sigma > 0$$

For an investor with constant relative risk aversion γ and arbitrary time horizon:

- ▶ Optimal policy: constant risky fraction $\pi_* = \mu/\gamma\sigma^2$, only depends on (r, μ, σ) through **mean-variance ratio** μ/σ^2
- ▶ Utility grows at certainty equivalent rate $\beta = r + \mu^2/2\gamma\sigma^2$
- ▶ Hence, assets are ranked by **Sharpe ratio** μ/σ
- ▶ Same optimal strategy for utility from consumption
- ▶ Explicit solution in terms of model parameters and preferences
- ▶ Results shape much of our intuition, but are not robust w.r.t. transaction costs since the implied trading volume is infinite

Introduction

Optimal portfolios with transaction costs

- ▶ Black-Scholes model for S^0 , S as before
- ▶ But now, buy at **ask price** S , sell at **bid price** $(1 - \epsilon)S$, where $\epsilon \in (0, 1)$ is the width of the bid-ask spread.

Simplest possible benchmark optimization problem?

- ▶ Investment depends on time horizon \Rightarrow infinite horizon
- ▶ Utility from consumption (Magill and Constantinidis (1976), Davis and Norman (1990), Shreve and Soner (1994)): Keep risky fraction in **no-trade region** $[\pi_-, \pi_+]$ around π_*
- ▶ π_-, π_+ characterized by a free boundary problem
- ▶ Janeček and Shreve (2004): First-order asymptotics via viscosity approach
- ▶ Further simplification: Drop consumption!

Introduction

Long-run optimal portfolios with transaction costs

Maximize the **certainty equivalent rate**

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \frac{1}{1 - \gamma} \log E \left[\left(\varphi_T^0 + \varphi_T^+ (1 - \epsilon) S_T - \varphi_T^- S_T \right)^{1 - \gamma} \right] \rightarrow \max!$$

With transaction costs:

- ▶ Simplest case: log-utility ($\gamma = 1$). Solution in Taksar et al. (1989), asymptotics in SG, Muhle-Karbe & Schachermayer (2011)
- ▶ No trade region $[\pi_-, \pi_+]$, optimal growth rate known via the solution of a one-dimensional equation
- ▶ $\gamma \neq 1$: Similar heuristic results of Dumas and Luciano (1991)
- ▶ Starting point of our analysis

Main results and Implications

Definition of the key extra parameter, the **gap**

Lemma

For small $\epsilon > 0$, there is a unique $\lambda > 0$ for which the solution to

$$w'(x) + (1 - \gamma)w(x)^2 + \left(\frac{2\mu}{\sigma^2} - 1\right) w(x) - \gamma \left(\frac{\mu - \lambda}{\gamma\sigma^2}\right) \left(\frac{\mu + \lambda}{\gamma\sigma^2}\right) = 0$$

$$w(0) = \frac{\mu - \lambda}{\gamma\sigma^2},$$

also satisfies the following terminal condition:

$$w(\log(u_\lambda/l_\lambda)) = \frac{\mu + \lambda}{\gamma\sigma^2} \quad \text{where} \quad u_\lambda/l_\lambda = \frac{1}{(1-\epsilon)} \frac{(\mu + \lambda)(\mu - \lambda - \gamma\sigma^2)}{(\mu - \lambda)(\mu + \lambda - \gamma\sigma^2)}$$

- ▶ Scalar equation for λ , implicit function theorem yields:

$$\lambda = \gamma\sigma^2 \left(\frac{3}{4\gamma} \pi_*^2 (1 - \pi_*)^2 \right)^{1/3} \epsilon^{1/3} + O(\epsilon).$$

- ▶ λ/σ^2 only depends on μ, σ through μ/σ^2

Main results and Implications

Optimal policy

Theorem

It is a long-run optimal policy to keep the risky weight within the buy and sell bounds

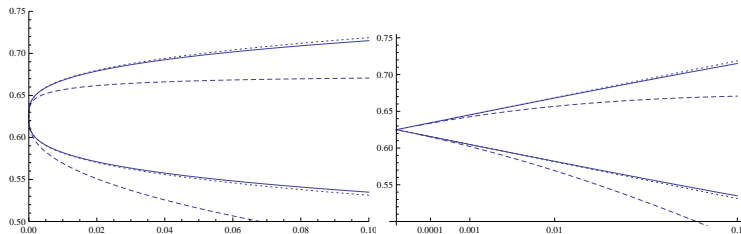
$$\pi_{\pm} = \frac{\mu \pm \lambda}{\gamma \sigma^2} = \pi_* \pm \left(\frac{3}{4\gamma} \pi_*^2 (1 - \pi_*)^2 \right)^{1/3} \epsilon^{1/3} + O(\epsilon).$$

- ▶ By evaluating π_- , π_+ in terms of the trading prices, the no-trade region becomes symmetric around $\pi_* = \mu/\gamma\sigma^2$
- ▶ Boundaries known explicitly in terms of μ, σ, γ and the gap λ
- ▶ At the first order, same result as in Janeček and Shreve (2004), i.e., investment and consumption separate
- ▶ As without transaction costs, policy only depends on investment opportunities μ, σ via mean-variance ratio μ/σ^2

Main results and Implications

Optimal policy

Buy and sell boundaries as functions of the spread ϵ :



- ▶ $\mu = 8\%$, $\sigma = 16\%$, $\gamma = 5$, zero discount rate for consumption
- ▶ First term of the expansion almost perfectly matches exact optimal weights
- ▶ Boundaries of Davis and Norman (1990) with consumption also close for $\epsilon < 1\%$, but start diverging for larger values

Main results

Welfare

Theorem

The certainty equivalent rate is given by $\beta = r + (\mu^2 - \lambda^2)/2\gamma\sigma^2$

- ▶ Long-run power investor is indifferent between
 - ▶ trading the risky asset with transaction costs
 - ▶ trading a frictionless asset with the same volatility σ but with lower expected excess return $\sqrt{\mu^2 - \lambda^2}$
- ▶ Hence, $\mu - \sqrt{\mu^2 - \lambda^2}$ is the **liquidity premium** introduced by Constantinidis (1986)
- ▶ Alternative interpretation: Need additional uncorrelated frictionless asset with volatility σ and excess return λ to offset transaction costs
- ▶ Certainty equivalent rate no longer solely depends on the Sharpe ratio μ/σ , due to different trading costs

Main results and Implications

Trading volume

Theorem

Relative turnover, i.e., the number $d\|\varphi\|_t$ of shares traded as a fraction of the number $|\varphi_t|$ of shares held, has long-term average

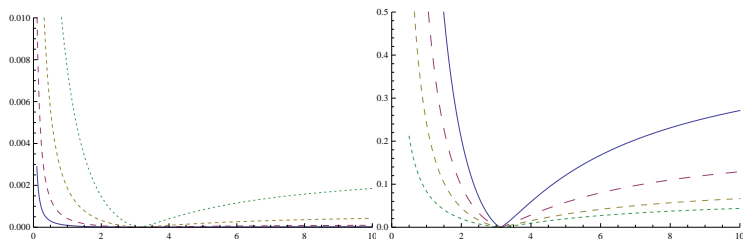
$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{d\|\varphi\|_t}{|\varphi_t|} = \left(1 - \frac{\mu - \lambda}{\gamma\sigma^2}\right) \frac{\sigma^2}{2} \left(\frac{\frac{2\mu}{\sigma^2} - 1}{(u_\lambda/l_\lambda)^{\frac{2\mu}{\sigma^2} - 1} - 1} \right) + \left(1 - \frac{\mu + \lambda}{\gamma\sigma^2}\right) \frac{\sigma^2}{2} \left(\frac{1 - \frac{2\mu}{\sigma^2}}{(u_\lambda/l_\lambda)^{1 - \frac{2\mu}{\sigma^2}} - 1} \right)$$

- ▶ Proposed in the empirical literature to measure trading activity
- ▶ Also determined completely in terms of μ, σ, γ and the gap λ , again is a function of mean-variance ratio in business time
- ▶ In our model, real-world trading volume is a puzzle for the opposite reason as the equity premium: The implied risk aversion is too low

Main results and Implications

Trading volume

Liquidity premium and share turnover as functions of risk aversion:



- ▶ $\mu = 8\%$, $\sigma = 16\%$, and $\epsilon = 0.01\%$, 0.1% , 1% , 10%
- ▶ Small liquidity premium (as in Constantinides (1986)) and trading volume without leverage ($\gamma > \mu/\sigma^2$)
- ▶ No trading and hence no liquidity premium for full investment
- ▶ Much larger liquidity premia and trading volume with leverage
- ▶ Similar behaviour of both quantities. Connection?

Main results and Implications

Trading volume

- ▶ Asymptotic expansion of relative share turnover:

$$\text{ShT} = \frac{\sigma^2}{2}(1 - \pi_*)^2 \pi_* \left(\frac{3}{4\gamma} \pi_*^2 (1 - \pi_*)^2 \right)^{-1/3} \epsilon^{-1/3} + O(\epsilon^{1/3})$$

- ▶ Asymptotic expansion of the liquidity premium:

$$\text{LiP} = \frac{\mu}{2\pi_*^2} \left(\frac{3}{4\gamma} \pi_*^2 (1 - \pi_*)^2 \right)^{2/3} \epsilon^{2/3} + O(\epsilon^{4/3})$$

- ▶ Universal relation, independent of model parameters and preferences:

$$\text{LiP} = \frac{3}{4} \epsilon \text{ShT} + O(\epsilon^{4/3}),$$

- ▶ As $\gamma \uparrow \infty$, relative turnover decreases due to lower risky weight, but increases due to smaller no-trade region
- ▶ Effects offset, giving a finite limit. Interpretation?

Heuristics and Proofs

Shadow Price

- ▶ Determine a “shadow price” \tilde{S} lying in the bid-ask spread, whose long-run optimal portfolio only buys stocks when $\tilde{S} = S$ resp. sells when $\tilde{S} = (1 - \epsilon)S$
- ▶ Optimizer for \tilde{S} is then also optimal for $(\underline{S}, \overline{S})$
- ▶ Previously used for log-utility in Kallsen and M-K (2010), SG, Muhle-Karbe, and Schachermayer (2011)

Heuristics and Proofs

Shadow Price

Theorem

*The same growth rate **and** portfolio are long-run optimal in the BS model with transaction costs and in a frictionless market with **shadow price***

$$d\tilde{S}_t/\tilde{S}_t = (\tilde{\mu}(Y_t) + r)dt + \tilde{\sigma}(Y_t)dW_t$$

- ▶ $\tilde{\mu}(\cdot)$ and $\tilde{\sigma}(\cdot)$ are functions known explicitly in terms of λ
- ▶ Y is reflected BM with drift on an interval, corresponds to log-stock-cash ratio
- ▶ Tracks position of the portfolio in the no-trade region
- ▶ Constant coefficient market with transaction costs corresponds to frictionless market with stochastic coefficients
- ▶ Key tool for verification as in Kallsen & Muhle-Karbe (2010), SG, Muhle-Karbe & Schachermayer (2011) for log-utility

Heuristics and Proofs

Heuristics

Standard control arguments:

- ▶ Long-run ansatz $V(t, x, y) = x^{1-\gamma} v(y/x) e^{-(1-\gamma)(r+\beta)t}$ formally yields linear reduced form HJB equation:

$$\frac{\sigma^2}{2} z^2 v''(z) + \mu z v'(z) - (1-\gamma)\beta v(z) = 0 \quad \text{if } l < z < u,$$

$$((1+\epsilon) + l)v'(l) - (1-\gamma)v(l) = 0$$

$$(1/(1+\epsilon) + u)v'(u) - (1-\gamma)v(u) = 0$$

- ▶ Growth rate β , free boundaries l, u unknown, therefore add smooth-pasting conditions:

$$((1+\epsilon) + l)v''(l) + \gamma v'(l) = 0$$

$$(1/(1+\epsilon) + u)v''(u) + \gamma v'(u) = 0$$

Heuristics and Proofs

Heuristics

As in Dumas and Luciano (1991), plug the boundary conditions into the ODE and simplify. But reparametrize using...

- ▶ the risky fraction $\pi_- = \frac{l}{1+l}$ of wealth in terms of the ask price S when it is time to buy
- ▶ the risky fraction $\pi_+ = \frac{(1-\epsilon)u}{1+(1-\epsilon)u}$ of wealth in terms of the bid price $(1-\epsilon)S$ when it is time to sell

Then:

$$\pi_{\pm} = \frac{\mu}{\gamma\sigma^2} \pm \frac{\sqrt{\mu^2 - 2\beta\gamma\sigma^2}}{\gamma\sigma^2},$$

and

$$\beta =: \frac{\mu^2 - \lambda^2}{2\gamma\sigma^2} \Rightarrow \pi_{\pm} = \frac{\mu \pm \lambda}{\gamma\sigma^2}$$

- ▶ Remains to determine λ

Heuristics and Proofs

Heuristics

- ▶ Plug u, l in terms of λ into the boundary conditions and do some substitutions for the HJB equation
- ▶ Find that λ has to be chosen so that the solution to the Riccati initial value problem

$$w'(x) + (1 - \gamma)w(x)^2 + \left(\frac{2\mu}{\sigma^2} - 1\right)w(x) - \gamma\left(\frac{\mu - \lambda}{\gamma\sigma^2}\right)\left(\frac{\mu + \lambda}{\gamma\sigma^2}\right) = 0$$
$$w(0) = \frac{\mu - \lambda}{\gamma\sigma^2},$$

also satisfies the second boundary condition

$$w(\log(u_\lambda/l_\lambda)) = \frac{\mu + \lambda}{\gamma\sigma^2} \quad \text{where} \quad u_\lambda/l_\lambda = \frac{1}{(1-\epsilon)} \frac{(\mu + \lambda)(\mu - \lambda - \gamma\sigma^2)}{(\mu - \lambda)(\mu + \lambda - \gamma\sigma^2)}$$

- ▶ This is the characterization for λ from above, asymptotics follow from implicit function theorem

Heuristics and Proofs

Proofs

- ▶ Reduced value function with transaction costs is a function of the stock-cash ratio, $e^Y = \varphi S / \varphi^0$, normalized to lie in $[1, u/l]$
- ▶ Assume the same holds for the ratio \tilde{S}/S
- ▶ Since S is a multiple of e^Y in the no-trade region, this is most conveniently written as $\tilde{S} = S e^{-Y} g(e^Y)$, as in SG, Muhle-Karbe, and Schachermayer (2011)
- ▶ Y follows Brownian motion in the no-trade region
- ▶ To keep it in $[0, \log(u/l)]$, complement this by reflection at the boundaries:

$$dY_t = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dW_t + dL_t - dU_t,$$

where L, U are the local times at 0 and $\log(u/l)$

Heuristics and Proofs

Proofs

So how to determine g in the ansatz $\tilde{S} = Se^{-Y}g(e^Y)$?

- ▶ To ensure $\tilde{S} = \bar{S}$ resp. $\tilde{S} = \underline{S}$ at the boundaries and make \tilde{S} an Itô process, impose:

$$g(1) = 1, \quad g'(1) = 1, \quad g(u/l) = (1 - \epsilon)u/l, \quad g'(u/l) = 1 - \epsilon$$

- ▶ Using long-run ansatz for frictionless markets, derive reduced HJB equation for generic g
- ▶ Then, to find the shadow price, match the corresponding value function to its counterpart in the transaction cost problem
- ▶ This gives a (complicated) ODE for g
- ▶ But we are lucky...

Heuristics and Proofs

Proofs

- ▶ There is an explicit solution for g in terms of λ , namely

$$g(e^y) = \begin{cases} \frac{1}{1-\epsilon} \left(1 + \frac{\gamma\sigma^2}{\mu-\lambda} \left(\left(1 + \frac{\mu-\lambda}{\gamma\sigma^2} \left(\frac{b}{b^2-a^2} - \frac{a}{b^2-a^2} \tanh \left[\tanh^{-1} \left(\frac{b}{a} \right) - ay \right] \right) \right)^{-1} - 1 \right) \right) \\ \frac{1}{1-\epsilon} \left(1 + \frac{\gamma\sigma^2}{\mu-\lambda} \left(\left(1 + \frac{\mu-\lambda}{\gamma\sigma^2} \left(\frac{b}{b^2+a^2} - \frac{a}{a^2+b^2} \tan \left[\tan^{-1} \left(\frac{b}{a} \right) + ay \right] \right) \right)^{-1} - 1 \right) \right) \\ \frac{1}{1-\epsilon} \left(1 + \frac{\gamma\sigma^2}{\mu-\lambda} \left(\left(1 + \frac{\mu-\lambda}{\gamma\sigma^2} \left(\frac{b}{b^2-a^2} - \frac{a}{b^2-a^2} \coth \left[\coth^{-1} \left(\frac{b}{a} \right) - ay \right] \right) \right)^{-1} - 1 \right) \right) \end{cases}$$

where the different cases depend on μ, σ, γ , and

$$a = \frac{1}{2} \sqrt{\left| 4(\gamma-1) \frac{\mu^2 - \lambda^2}{\gamma\sigma^4} - \left(1 - \frac{2\mu}{\sigma^2} \right)^2 \right|}, \quad b = \frac{1}{2} - \frac{\mu}{\sigma^2} + (\gamma-1) \frac{\mu-\lambda}{\gamma\sigma^2}$$

- ▶ This function satisfies the smooth pasting conditions and the ODE derived above

Heuristics and Proofs

Proofs

- ▶ \tilde{S} is an Itô process, its drift and diffusion coefficients are known functions of reflected BM with Drift Y
- ▶ Solution to the ergodic HJB equation for \tilde{S} known explicitly by construction, yields optimal growth rate β and portfolio for \tilde{S}
- ▶ Straightforward calculations: Corresponding number φ of stocks is

$$\frac{d\varphi_t}{\varphi_t} = \left(1 - \frac{\mu - \lambda}{\gamma\sigma^2}\right) L_t - \left(1 - \frac{\mu + \lambda}{\gamma\sigma^2}\right) U_t$$

- ▶ Local times L and U only increase when $Y = 0$ resp. when $Y = \log(u/l)$, i.e., when $\tilde{S} = S$ resp. $\tilde{S} = (1 - \epsilon)S$, hence \tilde{S} is a shadow price
- ▶ Same portfolio and growth rate are also long-run optimal with transaction costs

Summary

Results

Portfolio choice problem of a CRRA investor with long horizon:

- ▶ Black-Scholes model or, more generally, models with constant mean-variance ratio
- ▶ No-trade region, certainty equivalent rate, liquidity premium, and trading volume determined in terms of model parameters and the gap λ , which is the solution to a scalar equation
- ▶ For small transaction costs $\epsilon \downarrow 0$: Asymptotic expansions of arbitrary order for all quantities
- ▶ Rigorous verification by determining a shadow price

Summary

Implications

- ▶ The no-trade region is symmetric around the frictionless solution, when evaluated using trading prices
- ▶ Trading boundaries only depend on the model parameters through the mean-variance ratio
- ▶ At the first order, trading volume, liquidity premium, and spread are linked independent of preference and model parameters, which leads to testable implications
- ▶ With leverage: Much bigger impact of transaction costs
- ▶ In our model, real-world trading volume is a puzzle for the opposite reason as the equity premium: The implied risk aversion is too low