Minimising Capital Injections with and without Regime-Switching

Julia Eisenberg

TU Wien

14.10.2011

1. Motivation
   - Examples
   - Markov Switching

2. Dividends with Bounded Dividend Rates
   - The Model
   - HJB Equation
   - Solution for a 2-Regimes Model

3. Capital Injections
   - The Problem without Switching
   - The General Model
   - Special Case $n = 2$

Markov Regime-Switching models are based on the assumption that the considered system has two or more regimes (states).

Markov Regime-Switching models are based on the assumption that the considered system has two or more regimes (states).

Application Areas:

- Financial crisis (e.g. the crisis of 2007)
- Changes in the legislative or political framework
- Business cycles

Markov Regime-Switching models are based on the assumption that the considered system has two or more regimes (states).

Application Areas:

- Financial crisis (e.g. the crisis of 2007)
- Changes in the legislative or political framework
- Business cycles

Essentially one uses a MRSM to describe the deterioration of a set of macroeconomic variables, e.g. continuously increasing public debt.

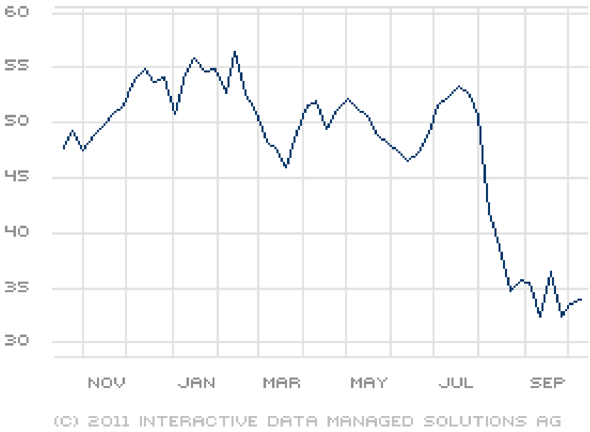
Markov Regime-Switching models are based on the assumption that the considered system has two or more regimes (states).

Application Areas:

- Financial crisis (e.g. the crisis of 2007)
- Changes in the legislative or political framework
- Business cycles

Essentially one uses a MRSM to describe the deterioration of a set of macroeconomic variables, e.g. continuously increasing public debt.

Daimler AG Stock



DAX Performance-Index

- “Für 2011 wird eine Schaden-Kosten-Quote von 107,9 Prozent der Beiträge für die Rückversicherungsbranche, nach 94,7 Prozent im Jahr 2010, erwartet.”

- “Eine positive Entwicklung kann nur mit höheren Preisen erzielt werden. Rückversicherungspreise sind an einem Scheideweg, und eine Steigerung ist der Faktor, der am ehesten die mittelfristigen Gewinnaussichten des Sektors verbessert”

Analyst Chris Watermann to Financial Times Deutschland

- Let $M = (M_t)_{t \geq 0}$ be a jump process on $(\Omega, \mathcal{F}, \mathbb{P})$ with state space $\mathscr{S} = \{1, ..., n\}$. Then M is a Markov chain, if

$$\mathbb{P}[M_t = i | M_s : s \leq r] = \mathbb{P}[M_t = i | M_r]$$

for all $0 \leq r \leq t$ and $i \in \mathscr{S}$.

- Let $M = (M_t)_{t \geq 0}$ be a jump process on $(\Omega, \mathcal{F}, \mathbb{P})$ with state space $\mathscr{S} = \{1, ..., n\}$. Then M is a Markov chain, if

$$\mathbb{P}[M_t = i | M_s : s \leq r] = \mathbb{P}[M_t = i | M_r]$$

  for all $0 \leq r \leq t$ and $i \in \mathscr{S}$.
- For arbitrary $i, j \in \mathscr{S}$ we let

$$\begin{aligned} q_{ij} &= \lim_{h \to 0} \frac{\mathbb{P}[M_{t+h} = j | M_t = i]}{h} \quad \text{for } i \neq j \\ q_i := q_{ii} &= -\sum_{k \neq i}^{n} q_{ik} . \end{aligned}$$

- Let $M = (M_t)_{t\geq 0}$ be a jump process on $(\Omega, \mathcal{F}, \mathbb{P})$ with state space $\mathscr{S} = \{1, ..., n\}$. Then M is a Markov chain, if

$$\mathbb{P}[M_t = i | M_s : s \leq r] = \mathbb{P}[M_t = i | M_r]$$

  for all $0 \leq r \leq t$ and $i \in \mathscr{S}$.
- For arbitrary $i, j \in \mathscr{S}$ we let

$$\begin{aligned} q_{ij} &= \lim_{h\to 0} \frac{\mathbb{P}[M_{t+h} = j | M_t = i]}{h} \quad \text{for } \; i \neq j \\ q_i := q_{ii} &= -\sum_{k\neq i}^{n} q_{ik} \; . \end{aligned}$$

- The matrix $Q = (q_{ij})$ is called generator of M.

A generator Q is said to be strongly irreducible, if the system

$$\begin{aligned} fQ &= 0 \\ \sum_{i=1}^{n} f_i &= 1 \end{aligned}$$

has a unique solution $f = (f_1, ..., f_n)$ with $f_i > 0 \; \forall i \in \mathscr{S}$.

SDE with Markov-Switching

Let W be a standard Brownian Motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, M $\{\mathcal{F}_t\}$ adapted and independent from W. Consider an SDE with Markov-Switching of the form

$$dX_t = f(X_t, M_t, t)\,\mathrm{d}t + g(X_t, M_t, t)\,\mathrm{d}W_t \tag{1}$$

with $X_0 = x$ and $M_0 = i$, $f, g : \mathbb{R} \times \{1, ..., n\} \times \mathbb{R}_+ \to \mathbb{R}$.
An $\mathbb{R}$-valued stochastic process $X = \{X_t\}$ is said to be a solution to (1), if

- X is $\{\mathcal{F}_t\}$-adapted;
- $\{f(X_t, M_t, t)\} \in \mathcal{L}(\mathbb{R}_+, \mathbb{R})$ and $g(X_t, M_t, t) \in \mathcal{L}^2(\mathbb{R}_+, \mathbb{R})$;
- it holds

$$X_t = x + \int f(X_s, M_s, s)\,\mathrm{d}s + \int_0^t g(X_s, M_s, s)\,\mathrm{d}W_s$$

with probability 1.

Theorem:

Assume there exist two positive constants K_1 and K_2 such that for all $x, y \in \mathbb{R}$ and $i \in \mathscr{S}$

Lipschitz condition

$$|f(x,i,t) - f(y,i,t)|^2 \vee |g(x,i,t) - g(y,i,t)|^2 \leq K_1|x-y|^2$$

Linear growth condition

$$|f(x,i,t)|^2 \vee |g(x,i,t)|^2 \leq K_2(1+|x|^2)\ .$$

Then there exists a unique solution X to

$$dX_t = f(X_t, M_t, t)\,\mathrm{d}t + g(X_t, M_t, t)\,\mathrm{d}W_t\ .$$

- We consider an insurance company and model the surplus process as a diffusion.

- We consider an insurance company and model the surplus process as a diffusion.
- The uncertainty is integrated into the model via a standard Brownian motion W and a Markov chain M with a finite state space $\mathscr{S}$.

- We consider an insurance company and model the surplus process as a diffusion.
- The uncertainty is integrated into the model via a standard Brownian motion W and a Markov chain M with a finite state space $\mathscr{S}$.
- The process W describes the uncertainty about future states due to randomly occurring claims.

- We consider an insurance company and model the surplus process as a diffusion.
- The uncertainty is integrated into the model via a standard Brownian motion W and a Markov chain M with a finite state space $\mathscr{S}$.
- The process W describes the uncertainty about future states due to randomly occurring claims.
- M models the long-term macroeconomic changes.

- We consider a filtration $\mathbb{F} = \{\mathcal{F}_t\}$, generated by a standard Brownian Motion W and Markov chain M.

- We consider a filtration $\mathbb{F} = \{\mathcal{F}_t\}$, generated by a standard Brownian Motion W and Markov chain M.
- We assume that the surplus process X fulfils the following SDE

$$\mathrm{d}X_t = \mu_{M(t)}\,\mathrm{d}t + \sigma_{M(t)}\,\mathrm{d}W_t - \,\mathrm{d}Z_t$$

with $X_0 = x$ and $M_0 = i$, where the drift function $\{\mu_i, i \in \mathscr{S}\}$ and the volatility function $\{\sigma_i, i \in \mathscr{S}\}$ are positive constants.

- We consider a filtration $\mathbb{F} = \{\mathcal{F}_t\}$, generated by a standard Brownian Motion W and Markov chain M.
- We assume that the surplus process X fulfils the following SDE

$$dX_t = \mu_{M(t)}\, dt + \sigma_{M(t)}\, dW_t - \, dZ_t$$

with $X_0 = x$ and $M_0 = i$, where the drift function $\{\mu_i, i \in \mathscr{S}\}$ and the volatility function $\{\sigma_i, i \in \mathscr{S}\}$ are positive constants.

- The process $Z = \{Z_t\}$, is caglad and $\, dZ_t = u_t\, dt$, denotes the cumulated dividend payments until t. The non-negative, $\mathbb{F}$ adapted process $u_t \in [0, K]$ denotes the dividend rate; $K \in \mathbb{R}_+$.
A process with the properties mentioned above is called admissible. The set of all admissible strategies we denote by $\mathscr{U}$.

- We consider a filtration $\mathbb{F} = \{\mathcal{F}_t\}$, generated by a standard Brownian Motion W and Markov chain M.
- We assume that the surplus process X fulfils the following SDE

$$dX_t = \mu_{M(t)}\, dt + \sigma_{M(t)}\, dW_t - \, dZ_t$$

  with $X_0 = x$ and $M_0 = i$, where the drift function $\{\mu_i, i \in \mathscr{S}\}$ and the volatility function $\{\sigma_i, i \in \mathscr{S}\}$ are positive constants.
- The process $Z = \{Z_t\}$, is caglad and $\, dZ_t = u_t\, dt$, denotes the cumulated dividend payments until t. The non-negative, $\mathbb{F}$ adapted process $u_t \in [0, K]$ denotes the dividend rate; $K \in \mathbb{R}_+$.
  A process with the properties mentioned above is called admissible.
  The set of all admissible strategies we denote by $\mathscr{U}$.
- The time of ruin will be denoted by $\Theta := \inf\{t \geq 0 : X_t \leq 0\}$.

The surplus process with dividends has the following form

$$X_t^u = x + \int_0^t \mu_{M_s} - u_s \, \mathrm{d}s + \int_0^t \sigma_{M_s} \, \mathrm{d}W_s$$

for $t \in [0, \Theta)$.

The surplus process with dividends has the following form

$$X_t^u = x + \int_0^t \mu_{M_s} - u_s \,\mathrm{d}s + \int_0^t \sigma_{M_s} \,\mathrm{d}W_s$$

for $t \in [0, \Theta)$.

Problem: Find the maximiser u of $J(x,i;u) := \mathbb{E}_{x,i}\Big[\int_0^\Theta e^{-\delta t} u_t \,\mathrm{d}t\Big]$.

The surplus process with dividends has the following form

$$X_t^u = x + \int_0^t \mu_{M_s} - u_s \, \mathrm{d}s + \int_0^t \sigma_{M_s} \, \mathrm{d}W_s$$

for $t \in [0, \Theta)$.

Problem: Find the maximiser u of $J(x, i; u) := \mathbb{E}_{x,i}\left[\int_0^\Theta e^{-\delta t} u_t \, \mathrm{d}t \right]$.

The surplus process with dividends has the following form

$$X_t^u = x + \int_0^t \mu_{M_s} - u_s \,\mathrm{d}s + \int_0^t \sigma_{M_s} \,\mathrm{d}W_s$$

for $t \in [0, \Theta)$.

Problem: Find the maximiser u of $J(x,i;u) := \mathbb{E}_{x,i}\Big[\int_0^\Theta e^{-\delta t} u_t \,\mathrm{d}t\Big]$.

The surplus process with dividends has the following form

$$X_t^u = x + \int_0^t \mu_{M_s} - u_s \, \mathrm{d}s + \int_0^t \sigma_{M_s} \, \mathrm{d}W_s$$

for $t \in [0, \Theta)$.

Problem: Find the maximiser u of $J(x, i; u) := \mathbb{E}_{x,i}\Big[\int_0^\Theta e^{-\delta t} u_t \, \mathrm{d}t\Big]$.

For this purpose we define

$$V(x, i) := \sup_{u \in \mathscr{U}} \mathbb{E}_{x,i}\left[\int_0^\Theta e^{-\delta t} u_t \, \mathrm{d}t\right] .$$

The surplus process with dividends has the following form

$$X_t^u = x + \int_0^t \mu_{M_s} - u_s \,\mathrm{d}s + \int_0^t \sigma_{M_s} \,\mathrm{d}W_s$$

for $t \in [0, \Theta)$.

Problem: Find the maximiser u of $J(x,i;u) := \mathbb{E}_{x,i}\Big[\int_0^\Theta e^{-\delta t} u_t \,\mathrm{d}t \Big]$.

For this purpose we define

$$V(x,i) := \sup_{u \in \mathscr{U}} \mathbb{E}_{x,i}\Big[\int_0^\Theta e^{-\delta t} u_t \,\mathrm{d}t \Big] .$$

Note that $V(0,i) = 0$ and $V(x,i) \le \frac{K}{\delta}$.

Hamilton–Jacobi–Bellman (HJB) Equation

The problem can be solved via the HJB equation:

HJB

$$\sup_{u\in[0,K]} \frac{\sigma_i^2}{2}V''(x,i) + (\mu_i - u)V'(x,i) + u - \delta V(x,i) = q_i V(x,i) - \sum_{j\in\mathscr{S}\setminus i} q_{ij}V(x,j)$$

Hamilton–Jacobi–Bellman (HJB) Equation

The problem can be solved via the HJB equation:

HJB

$$\sup_{u\in[0,K]} \frac{\sigma_i^2}{2} V''(x,i) + (\mu_i - u)V'(x,i) + u - \delta V(x,i) = q_i V(x,i) - \sum_{j\in\mathscr{S}\setminus i} q_{ij} V(x,j)$$

The HJB equation can be transformed as follows

$$\frac{\sigma_i^2}{2} V''(x,i) + \mu_i V'(x,i) - \delta V(x,i) + \sup_{u\in[0,K]} \{u(1 - V'(x,i))\} = q_i V(x,i) - \sum_{j\in\mathscr{S}\setminus i} q_{ij} V(x,j) \, .$$

We assume $b_1 < b_2$.

The value function and the optimal strategy are given by

$$V(x,i) = \begin{cases} \sum_{k=1}^{4} A_{ik} e^{\alpha_k (x-b_1)} & x \in [0, b_1) \\ \sum_{k=1}^{4} \tilde{A}_{ik} e^{\tilde{\alpha}_k (x-b_2)} + F_1 & x \in [b_1, b_2) \\ \sum_{k=1}^{2} \hat{A}_{ik} e^{\gamma_k x} + K/\delta & x \in [b_2, \infty) \end{cases}$$

with uniquely determined A_{ik}, $\tilde{A}_{ik}$, $k \in \{1,2,3,4\}$ and $\hat{A}_{ik}$, $k \in \{1,2\}$.

$$u_t = \begin{cases} 0 & M_t = i \text{ and } X_t \in [0, b_i) \\ K & M_t = i \text{ and } X_t \in [b_i, \infty) \end{cases}.$$

for $t \in [0, \Theta)$; and $u_t = 0$ for $t \in [\Theta, \infty)$.

Diffusion Approximation

The simplest diffusion approximation can be obtained as follows

Diffusion Approximation

The simplest diffusion approximation can be obtained as follows

- Let $Z_i \geq 0$ be iid, $\mu_k = \mathbb{E}[Z_i^k]$ for $k > 1$ and $\mu = \mathbb{E}[Z_i]$;

Diffusion Approximation

The simplest diffusion approximation can be obtained as follows

- Let $Z_i \geq 0$ be iid, $\mu_k = \mathbb{E}[Z_i^k]$ for $k > 1$ and $\mu = \mathbb{E}[Z_i]$;
- Choose $\eta > 0$ and $\lambda > 0$ and let $\{N_t^{(n)}\}$ be Poisson processes with intensity $n\lambda$;

Diffusion Approximation

The simplest diffusion approximation can be obtained as follows

- Let $Z_i \geq 0$ be iid, $\mu_k = \mathbb{E}[Z_i^k]$ for $k > 1$ and $\mu = \mathbb{E}[Z_i]$;
- Choose $\eta > 0$ and $\lambda > 0$ and let $\{N_t^{(n)}\}$ be Poisson processes with intensity $n\lambda$;
- Construct a sequence of classical risk models $X_t^{(n)}$ as follows:

$$X_t^{(n)} = x + \Big(1 + \frac{\eta}{\sqrt{n}}\Big)\lambda\mu\sqrt{n}t - \sum_{i=1}^{N_t^{(n)}} Z_i/\sqrt{n}\,.$$

Diffusion Approximation

The simplest diffusion approximation can be obtained as follows

- Let $Z_i \geq 0$ be iid, $\mu_k = \mathbb{E}[Z_i^k]$ for $k > 1$ and $\mu = \mathbb{E}[Z_i]$;
- Choose $\eta > 0$ and $\lambda > 0$ and let $\{N_t^{(n)}\}$ be Poisson processes with intensity $n\lambda$;
- Construct a sequence of classical risk models $X_t^{(n)}$ as follows:

$$\boldsymbol{X_t^{(n)} = x + (1 + \frac{\eta}{\sqrt{n}})\lambda\mu\sqrt{n}t - \sum_{i=1}^{N_t^{(n)}} Z_i/\sqrt{n}\,.}$$

As a weak limit we obtain

$$X_t = x + \lambda\mu\eta t + \sqrt{\lambda\mu_2}W_t\ ,$$

where W is a standard Brownian motion.

Expected Discounted Capital Injections

- Capital injections prevent the surplus entering $(0, -\infty)$.

Expected Discounted Capital Injections

- Capital injections prevent the surplus entering $(0, -\infty)$.
- The discounting factor $\delta > 0$ describes the investment preferences of the insurer.

Expected Discounted Capital Injections

- Capital injections prevent the surplus entering $(0, -\infty)$.
- The discounting factor $\delta > 0$ describes the investment preferences of the insurer.
- A process $\mathrm{d}Z_t = \mu(Z_t)\,\mathrm{d}t + \rho(Z_t)\,\mathrm{d}W_t$ with capital injection $\{Y\}$ fulfils

$$\mathbf{d}\boldsymbol{Z}_t^{\boldsymbol{Y}} = \mathbf{d}\boldsymbol{Z}_t + \mathbf{d}\boldsymbol{Y}_t\,.$$

Expected Discounted Capital Injections

- Capital injections prevent the surplus entering $(0, -\infty)$.
- The discounting factor $\delta > 0$ describes the investment preferences of the insurer.
- A process $\mathrm{d}Z_t = \mu(Z_t)\,\mathrm{d}t + \rho(Z_t)\,\mathrm{d}W_t$ with capital injection $\{Y\}$ fulfils

$$\mathbf{d}\boldsymbol{Z}_t^{\boldsymbol{Y}} = \mathbf{d}\boldsymbol{Z}_t + \mathbf{d}\boldsymbol{Y}_t\,.$$

- Shreve et al. [3] showed that the function $f(x) = \mathbb{E}_x[\int_0^\infty e^{-\delta t}\,\mathrm{d}Y_t]$, $\delta > 0$ solves

$$\frac{\rho(x)^2}{2} f''(x) + m(x) f'(x) - \delta f(x) = 0 \qquad (2)$$

for $x \geq 0$, and fulfils $f'(0) = -1$, $\lim\limits_{x\to\infty} f(x) = 0$. Every solution $f(x)$ to (2) with $\lim\limits_{x\to\infty} f(x) = 0$ has the form

$$\boldsymbol{f(x) = f'(0)}\mathbb{E}_{\boldsymbol{x}}\Big[\int_{\mathbf{0}}^{\boldsymbol{\infty}} \boldsymbol{e^{-\delta t}}\,\mathbf{d}\boldsymbol{Y}_t\Big]\,.$$

Surplus Process with Reinsurance and Capital Injections

We denote

- the retention level by $b \in [0, \tilde{b}]$. The first insurer can change his retention level continuously in time, i.e. $B = \{b_t\}$ describes the "reinsurance behaviour" of the first insurer in t;
- the self-insurance function by $r(z, b)$. We assume that r is continuous and increasing in both variables;
- the premium rate by $c(b)$.

$$X_t^B = x + \int_0^t c(b_s)\,\mathrm{d}s + \int_0^t \sqrt{\lambda \mathbb{E}[r(Z, b_s)^2]}\,\mathrm{d}W_s$$

where

Surplus Process with Reinsurance and Capital Injections

We denote

- the retention level by $b \in [0, \tilde{b}]$. The first insurer can change his retention level continuously in time, i.e. $B = \{b_t\}$ describes the "reinsurance behaviour" of the first insurer in t;
- the self-insurance function by $r(z, b)$. We assume that r is continuous and increasing in both variables;
- the premium rate by $c(b)$.

$$X_t^B = x + \int_0^t c(b_s)\,\mathrm{d}s + \int_0^t \sqrt{\lambda \mathbb{E}[r(Z, b_s)^2]}\,\mathrm{d}W_s$$

where

- $B = \{b_t\}$, $b_t \in [0, \tilde{b}]$ reinsurance strategy;

Surplus Process with Reinsurance and Capital Injections

We denote

- the retention level by $b \in [0, \tilde{b}]$. The first insurer can change his retention level continuously in time, i.e. $B = \{b_t\}$ describes the "reinsurance behaviour" of the first insurer in t;
- the self-insurance function by $r(z, b)$. We assume that r is continuous and increasing in both variables;
- the premium rate by $c(b)$.

$$X_t^B = x + \int_0^t c(b_s)\,\mathrm{d}s + \int_0^t \sqrt{\lambda \mathbb{E}[r(Z, b_s)^2]}\,\mathrm{d}W_s$$

where

- $B = \{b_t\}$, $b_t \in [0, \tilde{b}]$ reinsurance strategy;
- $\int_0^t c(b_s)\,\mathrm{d}s$ premium until t;

Surplus Process with Reinsurance and Capital Injections

We denote

- the retention level by $b \in [0, \tilde{b}]$. The first insurer can change his retention level continuously in time, i.e. $B = \{b_t\}$ describes the "reinsurance behaviour" of the first insurer in t;
- the self-insurance function by $r(z, b)$. We assume that r is continuous and increasing in both variables;
- the premium rate by $c(b)$.

$$X_t^{B,Y} = x + \int_0^t c(b_s)\,\mathrm{d}s + \int_0^t \sqrt{\lambda \mathbb{E}[r(Z, b_s)^2]}\,\mathrm{d}W_s + Y_t^B$$

where

- $B = \{b_t\}$, $b_t \in [0, \tilde{b}]$ reinsurance strategy;
- $\int_0^t c(b_s)\,\mathrm{d}s$ premium until t;

Model Assumptions

To simplify the presentation we consider just the **proportional reinsurance** and the **expected value principle** for the premium calculation, i.e.

$$c(b) = \lambda\mu(1+\theta)b - \lambda\mu(\theta-\eta)$$
$$\mathbb{E}[r(Z,b)^2] = \mathbb{E}[Z^2]b^2 \ ,$$

where θ and η are the safety loadings of the first insurer and reinsurer respectively!

The Value Function

Assumptions:

- The filtration $\{\mathcal{F}_t\}$ is generated by W;
- A strategy B is said to be admissible, if B is cadlag and $\{\mathcal{F}_t\}$ adapted.

The Value Function

Assumptions:

- The filtration $\{\mathcal{F}_t\}$ is generated by W;
- A strategy B is said to be admissible, if B is cadlag and $\{\mathcal{F}_t\}$ adapted.

As a risk measure connected to some admissible reinsurance strategy B we choose the value of expected discounted capital injections with some discounting factor $\delta \geq 0$.

$$\underbrace{V(x)}_{\text{value function}} = \inf_B \underbrace{V^B(x)}_{\text{return function}} = \inf_B \mathbb{E}_x\left[\int_0^\infty e^{-\delta t}\,\mathrm{d}Y_t^B\right].$$

Properties of the Value Function

- V is decreasing with $\lim_{x \to \infty} V(x) = 0$;
- $V'(0) = -1$;
- V is convex.

Die HJB equation has the form

HJB

$$\inf_{b\in[0,1]} \frac{\lambda\mu_2 b^2}{2} V''(x) + \lambda\mu\{b\theta - \theta + \eta\} V'(x) - \delta V(x) = 0 \; .$$

The unique solution to the problem is given by the differential equation

$$-\frac{\lambda\mu^2\theta^2}{2\mu_2}\frac{V'(x)^2}{V''(x)} - \lambda\mu(\theta - \eta)V'(x) - \delta V(x) = 0$$

Die HJB equation has the form

HJB

$$\inf_{b\in[0,1]} \frac{\lambda\mu_2 b^2}{2} V''(x) + \lambda\mu\{b\theta - \theta + \eta\}V'(x) - \delta V(x) = 0 \,.$$

The unique solution to the problem is given by the differential equation

$$-\frac{\lambda\mu^2\theta^2}{2\mu_2}\frac{V'(x)^2}{V''(x)} - \lambda\mu(\theta-\eta)V'(x) - \delta V(x) = 0$$

It holds $V(x) = \frac{1}{\beta}e^{-\beta x}$ with $\beta \in \mathbb{R}_+$ and the optimal strategy is constant!

Consider the process

$$dX_t^B = \left(\theta_{M_t} b_t + \lambda_{M_t}\mu_{M_t}(\theta_{M_t} - \eta_{M_t})\right) dt + b_t\sqrt{\lambda_{M_t}\mu_{M_t,2}}\, dW_t$$

with Filtration $\{\mathcal{F}_t\}$ generated by W and M.

HJB Equation

$$\inf_{b\in[0,1]} \frac{\lambda_i \mu_{i,2} b^2}{2} V''(x,i) + \lambda_i \mu_i \{b\theta_i - \theta_i + \eta_i\} V'(x,i) - (\delta - q_i) V(x,i) = -\sum_{j\neq i} q_{ij} V(x,j)\ .$$

The optimal strategy for all $n \in \mathbb{N}$ is given by the relation

$$b^*(x,i) = -\frac{V'(x,i)\mu_i\theta_i}{V''(x,i)\mu_{i,2}} \wedge 1\ .$$

HJB Equation

$$\inf_{b\in[0,1]} \frac{\lambda_i\mu_{i,2}b^2}{2}V''(x,i) + \lambda_i\mu_i\{b\theta_i - \theta_i + \eta_i\}V'(x,i) - (\delta - q_i)V(x,i) = -\sum_{j\neq i} q_{ij}V(x,j) \; .$$

The optimal strategy for all $n \in \mathbb{N}$ is given by the relation

$$b^*(x,i) = -\frac{V'(x,i)\mu_i\theta_i}{V''(x,i)\mu_{i,2}} \wedge 1 \; .$$

Assume $b^*(x,i) < 1$. Inserting the optimal strategy yields

$$-\frac{\lambda_i\mu_i^2\theta_i^2}{2\mu_{i,2}}\frac{V'(x,i)^2}{V''(x,i)} - \lambda_i\mu_i(\theta_i - \eta_i)V'(x,i) - (\delta - q_i)V(x,i) = -\sum_{j\neq i} q_{ij}V(x,j) \; .$$

Constant Strategies

Consider the case $n = 2$ and the strategy $B \equiv 1$. We have to solve the following system of differential equations

$$\frac{\lambda_i \mu_{i,2}}{2} V_1''(x,i) + \lambda_i \mu_i \eta_i V_1'(x,i) - (\delta - q_i) V_1(x,i) = q_i V_1(x,j) \; .$$

Constant Strategies

Consider the case $n = 2$ and the strategy $B \equiv 1$. We have to solve the following system of differential equations

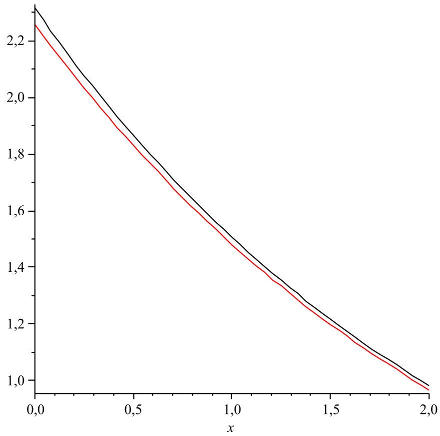
$$\frac{\lambda_i \mu_{i,2}}{2} V_1''(x,i) + \lambda_i \mu_i \eta_i V_1'(x,i) - (\delta - q_i) V_1(x,i) = q_i V_1(x,j) \; .$$

As a solution we obtain

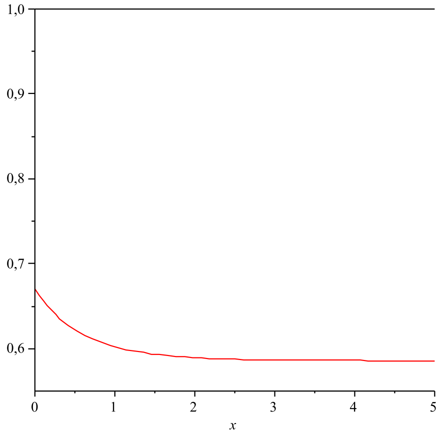
$$\begin{aligned} V_1(x,1) &= C_1 e^{\kappa_1 x} + C_2 e^{\kappa_2 x} \\ V_1(x,2) &= \frac{\lambda_1 \mu_{1,2}}{2q_2} \Big(C_1 \kappa_1^2 e^{\kappa_1 x} + C_2 \kappa_2^2 e^{\kappa_2 x} \Big) + \frac{q_1 - \delta}{q_2} \Big(C_1 e^{\kappa_1 x} + C_2 e^{\kappa_2 x} \Big) \\ &\quad + \frac{-\lambda_1 \mu_1 (\theta_1 - \eta_1)}{q_2} \Big(C_1 \kappa_1 e^{\kappa_1 x} + C_2 \kappa_2 e^{\kappa_2 x} \Big) \; , \end{aligned}$$

with unique $\kappa_1, \kappa_2 < 0$.

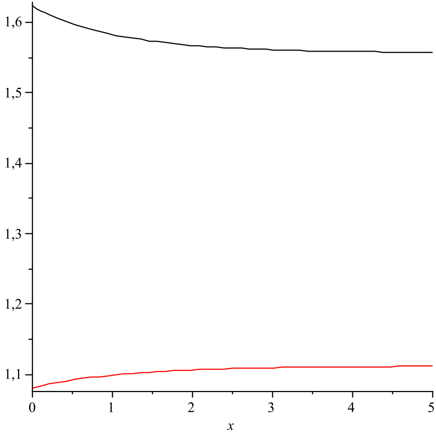
$\lambda = \mu = 1$, $\mu_2 = 2$, $\theta_1 = 0.5$, $\theta_2 = 1.4$, $\eta_1 = 0.3$, $\eta_2 = 0.4$ and $\delta = 0.04$.



$\lambda = \mu = 1$, $\mu_2 = 2$, $\theta_1 = 0.5$, $\theta_2 = 1.4$, $\eta_1 = 0.3$, $\eta_2 = 0.4$ and $\delta = 0.04$. Functions $-\frac{V'(x,i)\mu_i\theta_i}{V''(x,i)\mu_{i,2}}$.



$\lambda = \mu = 1$, $\mu_2 = 2$, $\theta_1 = \mathbf{1.3}$, $\theta_2 = 1.4$, $\eta_1 = 0.3$, $\eta_2 = 0.4$ and $\delta = 0.04$. Functions $-\frac{V'(x,i)\mu_i\theta_i}{V''(x,i)\mu_{i,2}}$.



In contrast to the model with dividends: there is no closed expression for a solution!

In contrast to the model with dividends: there is no closed expression for a solution!

But it is possible to transform the system of the second order differential equations into a first order differential equation. We just divide the HJB equation by the first derivative and obtain

$$\frac{\lambda_i \mu_i^2 \theta_i^2}{2\mu_{i,2}} \Big(\underbrace{-\frac{V'(x,i)}{V''(x,i)}}_{=: f_i(x)} \Big) - \lambda_i \mu_i (\theta_i - \eta_i) = (\delta - q_i) \frac{V(x,i)}{V'(x,i)} - \sum_{j \neq i} q_{ij} \frac{V(x,j)}{V'(x,i)} \,.$$

Derivation with respect to x yields

$$\frac{\lambda_i \mu_i^2 \theta_i^2}{2\mu_{i,2}} f_i'(x) + \frac{\lambda_i \mu_i (\theta_i - \eta_i)}{f_i(x)} - \frac{\lambda_i \mu_i^2 \theta_i^2}{2\mu_{i,2}} - \delta = -q_i - \underbrace{\sum_{j \neq i} q_{ij} \frac{V'(x,j)}{V'(x,i)}}_{>0} \,.$$

Reinsurance and Surplus Investment

The surplus process has the following form:

Reinsurance and Surplus Investment

The surplus process has the following form:

$$\mathrm{d}X_t^B = \left\{\lambda\theta\mu b_t - \lambda\mu(\theta-\eta) + mX_t^B\right\}\mathrm{d}t + \sqrt{\lambda\mu_2}b_t\,\mathrm{d}W_t + \sigma X_t^B\,\mathrm{d}\tilde{W}_t\ .$$

Reinsurance and Surplus Investment

The surplus process has the following form:

$$\mathrm{d}X_t^B = \left\{\lambda\theta\mu b_t - \lambda\mu(\theta-\eta) + mX_t^B\right\}\,\mathrm{d}t + \sqrt{\lambda\mu_2}b_t\,\mathrm{d}W_t + \sigma X_t^B\,\mathrm{d}\tilde{W}_t\,.$$

The solution is

$$X_t^B = U_t\Big(x + \lambda\int_0^t \{\theta\mu b_s - \mu(\theta-\eta)\}U_s^{-1}\,\mathrm{d}s + \int_0^t \sqrt{\lambda\mu_2}b_s U_s^{-1}\,\mathrm{d}W_s\Big)\,,$$

where $U_t = \exp\{(m - \frac{\sigma^2}{2})t + \sigma\tilde{W}_t\}$.

Hamilton–Jacobi–Bellman Equation

For the HJB equation corresponding to the considered problem we get

$$0 = \inf_{b \in [0,1]} \Big(\frac{\lambda \mu_2 b^2}{2} + \frac{\sigma^2 x^2}{2} \Big) V''(x) + \Big(\lambda \theta \mu b - \lambda \mu (\theta - \eta) \Big) V'(x) + m x V'(x) - \delta V(x) \,.$$

The optimal strategy is the unique solution to the following differential equation:

$$f'(x) - \delta = w f(x) \,,$$
$$f(x) = \frac{\lambda \mu^2 \theta^2}{2 \mu_2} \frac{1}{w(x)} - \frac{\sigma^2 x^2}{2} w(x) + (m x - \lambda \mu (\theta - \eta)) \,.$$

Let $\beta_i(x) := \frac{\lambda_i \mu_i^2 \theta_i^2}{2\mu_{i,2}} f_i'(x) + \frac{\lambda_i \mu_i (\theta_i - \eta_i)}{f_i(x)} - \frac{\lambda_i \mu_i^2 \theta_i^2}{2\mu_{i,2}} - \delta + q_i$

Let $\beta_i(x) := \frac{\lambda_i \mu_i^2 \theta_i^2}{2\mu_{i,2}} f_i'(x) + \frac{\lambda_i \mu_i(\theta_i - \eta_i)}{f_i(x)} - \frac{\lambda_i \mu_i^2 \theta_i^2}{2\mu_{i,2}} - \delta + q_i$

For $n = 1$

$$\beta_1(x) = 0$$

Let $\beta_i(x) := \frac{\lambda_i \mu_i^2 \theta_i^2}{2\mu_{i,2}} f_i'(x) + \frac{\lambda_i \mu_i(\theta_i - \eta_i)}{f_i(x)} - \frac{\lambda_i \mu_i^2 \theta_i^2}{2\mu_{i,2}} - \delta + q_i$

For $n = 1$

$$\beta_1(x) = 0$$

For $n = 2$

$$\beta_1(x)\beta_2(x) = q_1 q_2 \ .$$

Let $\beta_i(x) := \frac{\lambda_i \mu_i^2 \theta_i^2}{2\mu_{i,2}} f_i'(x) + \frac{\lambda_i \mu_i (\theta_i - \eta_i)}{f_i(x)} - \frac{\lambda_i \mu_i^2 \theta_i^2}{2\mu_{i,2}} - \delta + q_i$

For $n = 1$

$$\beta_1(x) = 0$$

For $n = 2$

$$\beta_1(x)\beta_2(x) = q_1 q_2 \ .$$

For $n = 3$

$$\prod_{k=1}^{3} \beta_k(x) = \sum_{k=1}^{3} \beta_k(x) \cdot \prod_{\substack{i,j \neq k \\ i \neq j}}^{3} q_{ij} - q_{12}q_{23}q_{31} - q_{13}q_{21}q_{32} \ .$$

$n = 2$, $i, j \in \{1, 2\}$, $i \neq j$, $b^*(x, i) < 1$.

The value function and the optimal strategy obey the following equations

HJB

$$-\frac{\lambda_i \mu_i^2 \theta_i^2}{2\mu_{i,2}} \frac{V'(x,i)^2}{V''(x,i)} - \lambda_i \mu_i (\theta_i - \eta_i) V'(x,i) - (\delta - q_i) V(x,i) = q_i V(x,j)$$

$$\Downarrow$$

Optimal strategy $b^*(x,i) = \frac{\mu\theta_i}{\mu_2} f_i(x)$

$$f_i'(x) + 2\frac{1 - \frac{\eta_i}{\theta_i}}{b^*(x,i)} - 1 - \frac{2\mu_{i,2}\delta}{\lambda_i \mu_i^2 \theta_i^2} = \left(1 - \frac{V'(x,j)}{V'(x,i)}\right) \frac{-2\mu_{i,2} q_i}{\lambda_i \mu_i^2 \theta_i^2}\ .$$

Deriving the right hand side of the equation for the optimal strategy with respect to x gives the relation

$$\frac{-2\mu_{i,2}q_i}{\mu_i^2\theta_i^2}\frac{V'(x,j)}{V'(x,i)}\Big(\frac{1}{f_j(x)} - \frac{1}{f_i(x)}\Big) .$$

Thus, we obtain an instrument to get information about the optimal strategy.

Deriving the right hand side of the equation for the optimal strategy with respect to x gives the relation

$$\frac{-2\mu_{i,2}q_i}{\mu_i^2\theta_i^2}\frac{V'(x,j)}{V'(x,i)}\Big(\frac{1}{f_j(x)} - \frac{1}{f_i(x)}\Big)\,.$$

Thus, we obtain an instrument to get information about the optimal strategy.

The strategies for i and j have an opposite behaviour.

$n = 2,\ i, j \in \{1, 2\},\ i \neq j,\ b^*(x, i) = 1.$

Repeating all the calculations for $B \equiv 1$ and letting $g(x) = -\frac{V_1''(x,i)}{V_1'(x,i)}$ yields

$$-\frac{\lambda_i \mu_{i,2}}{2} g'(x) + \frac{\lambda_i \mu_{i,2}}{2} g(x)^2 - \lambda_i \mu_i \eta_i g(x) - \delta = -q_i + q_i \frac{V'(x,j)}{V_1'(x,i)}\ .$$

$n = 2,\ i, j \in \{1, 2\},\ i \neq j,\ b^*(x, i) = 1.$

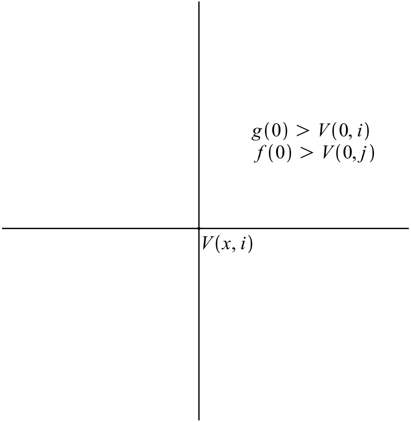
Repeating all the calculations for $B \equiv 1$ and letting $g(x) = -\frac{V_1''(x,i)}{V_1'(x,i)}$ yields

$$-\frac{\lambda_i \mu_{i,2}}{2} g'(x) + \frac{\lambda_i \mu_{i,2}}{2} g(x)^2 - \lambda_i \mu_i \eta_i g(x) - \delta = -q_i + q_i \frac{V'(x, j)}{V_1'(x, i)} .$$

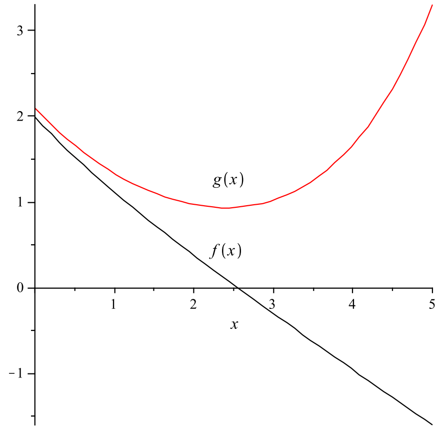
For given parameters it is possible to see whether the strategy $B \equiv 1$ is optimal or not.

Numerical Calculation of the Value Function

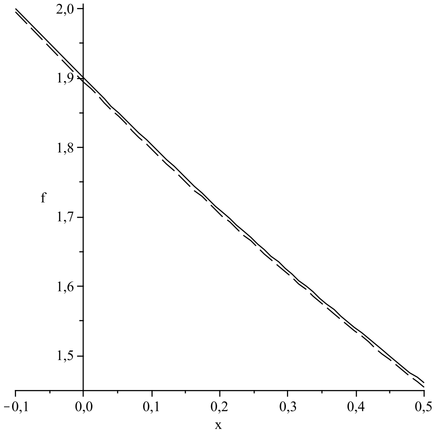
Numerical Calculation of the Value Function



"False" Initial Values



$\lambda = \mu = 1$, $\mu_2 = 2$, $\theta_1 = \mathbf{0.5}$, $\theta_2 = 1.4$, $\eta_1 = 0.3$, $\eta_2 = 0.4$ and $\delta = 0.04$.



References

- Mao, X. and Yuan, C. (2006). Stochastic Differential Equations with Markovian Switching.Imperial College Press, London.
- Sotomayor, L.R. and Cadenillas, A. (2011). Classical and singular stochastic control for the optimal dividend policy when there is regime switching. *Insurance: Mathematics and Economics* **48**, 344–354.
- Shreve, S.E., Lehoczky, J.P. and Gaver, D.P. (1984). Optimal consumption for general diffusions with absorbing and reflecting barriers. *SIAM J. Control and Optimization* **22**, 55–75.

Thank You for Your Attention