Brownian Moving Averages and Applications Towards Interst Rate Modelling

F. Hubalek, T. Blümmel

October 14, 2011

Table of contents

1 Data and Observations

2 Brownian Moving Averages

3 BMA-driven Vasicek-Model

4 Literature

The Data

For different time lags $h \in \mathbb{N}$, we are interested in the (overlapping) increments of the interest rates

$$IR(t+h) - IR(t), \qquad t \in \mathbb{N}.$$

Interest Rate	Day 1	Day 2	Day 3	Day 4	Day 5	...
EURIBOR 01M	4.97	4.95	4.96	4.98	5.00	...
EURIBOR 03M	4.92	4.88	4.90	4.89	4.91	...
EURIBOR 06M	4.85	4.81	4.83	4.82	4.84	...
GBP LIBOR 01M	5.86	5.86	5.87	5.90	5.90	...
GBP LIBOR 03M	5.90	5.90	5.90	5.89	5.89	...
GBP LIBOR 06M	5.92	5.93	5.93	5.93	5.94	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

non-overlapping increments $[IR(t*h+1) - IR((t-1)*h+1)]$

Graphic I

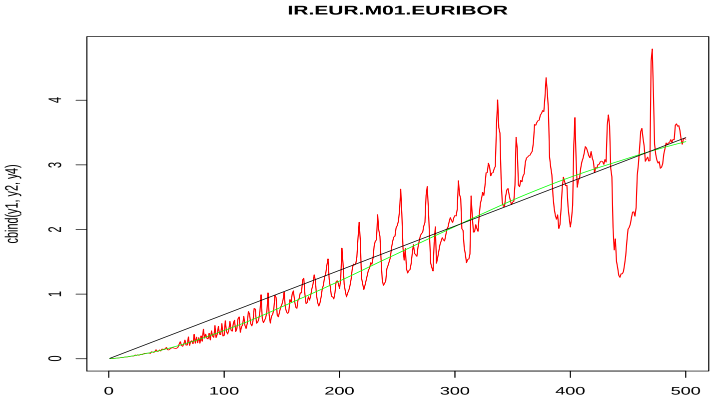


Figure: overlapping-increments, non-ol-increments, straight line

Graphic II

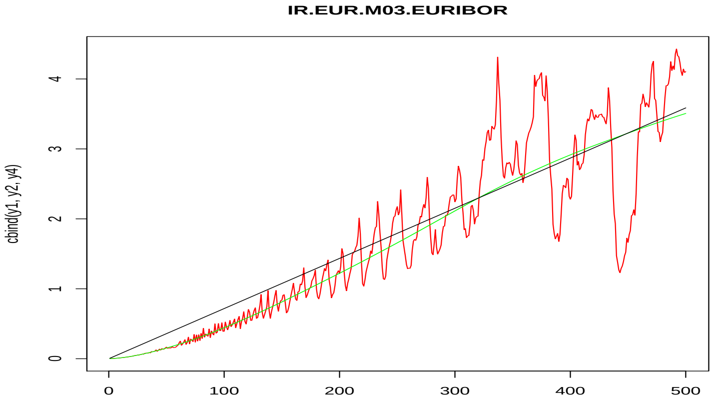


Figure: overlapping-increments, non-ol-increments, straight line

Graphic III

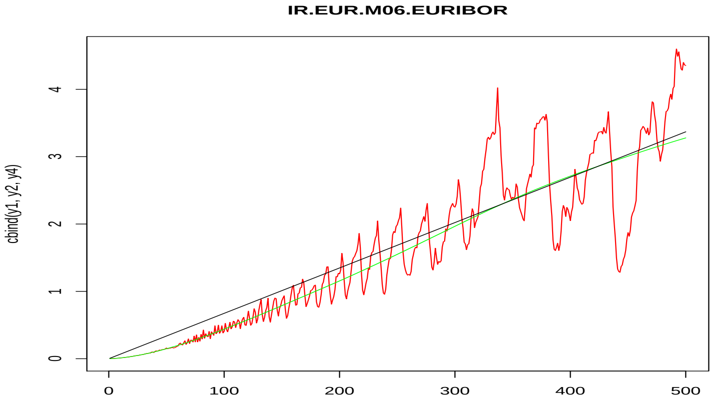


Figure: overlapping-increments, non-ol-increments, straight line

Definition and Properties

Definition: Brownian Moving Average

Let $(B_u)_{u \in \mathbb{R}}$ be a two-sided Brownian motion and φ a Borel-measurable function, which is zero on $(0, \infty)$ and $\varphi(. - t) - \varphi(.) \in L^2(\mathbb{R})$ for all $t \geq 0$. The Brownian moving average (BMA) concerning φ is defined as

$$X_t^{\varphi} := \int_{\mathbb{R}} \left(\varphi\left(u - t\right) - \varphi\left(u\right)\right) dB_u.$$

Properties:

- Its variance is given by

$$Var\left(X_t^{\varphi}\right) = \int_{\mathbb{R}} \left(\varphi\left(u - t\right) - \varphi\left(u\right)\right)^2 du, \qquad t \geq 0.$$

- X^{φ} is a centered Gaussian process with stationary increments.

Examples

Brownian Moving Average

$$X_t^{\varphi} := \int_{\mathbb{R}} (\varphi(u-t) - \varphi(u))\, dB_u \quad t \geq 0$$

- Brownian Motion (BM): $\varphi(u) = \mathbb{1}_{\{u \leq 0\}}$.
- Fractional BM (FBM): $\varphi(u) = c_H(-u)^{H-\frac{1}{2}} \mathbb{1}_{\{u \leq 0\}}$ for $H \in (0,1)$.

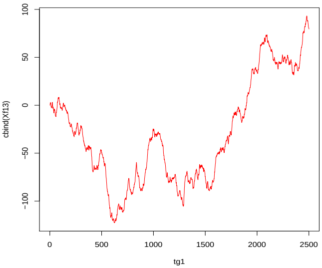


Figure: path of FBM ($H = 0.8$)

BMA-Semimartingales

Theorem [Cherny]

1. X^φ is a $\left(\mathcal{F}_t^B\right)$-semimartingale if and only if there exist $\alpha \in \mathbb{R}$ and $\psi \in L^2\left(\mathbb{R}\right)$ such that

$$\varphi\left(u\right) = \alpha + \int_u^0 \psi\left(v\right) dv, \quad u \leq 0.$$

2. If X^φ is a $\left(\mathcal{F}_t^B\right)$-semimartingale it is continuous, and its canonical decomposition is given by

$$X_t^\varphi = \int_{\mathbb{R}} \left(\chi\left(u-t\right) - \chi\left(u\right)\right) dB_u + \alpha B_t,$$

where $\chi\left(u\right) = \varphi\left(u\right) - \alpha \mathbb{1}_{\{u \leq 0\}}$.

Application

Fractional Brownian Motion

$$\varphi(u) = c_H(-u)^{H-\frac{1}{2}}\mathbb{1}_{\{u\leq 0\}}$$

↓ Modification

Regularized FBM (Rogers)

$$\varphi(u) = c_H(\beta - u)^{H-\frac{1}{2}}\mathbb{1}_{\{u\leq 0\}}$$

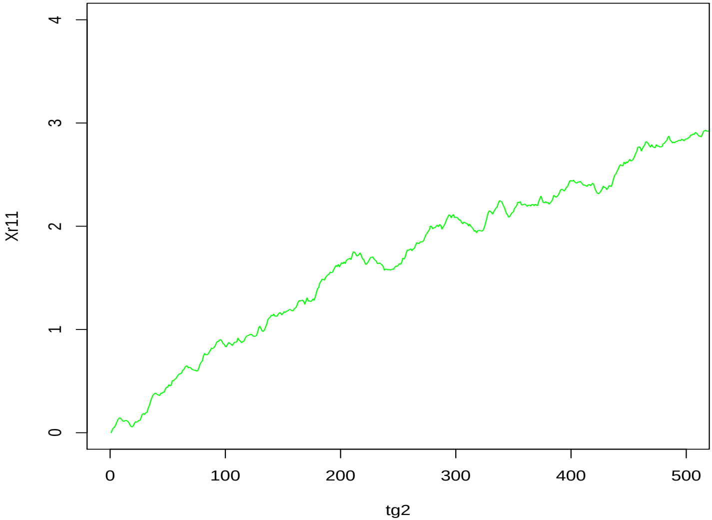
Brownian Motion

$$\varphi(u) = \mathbb{1}_{\{u\leq 0\}}$$

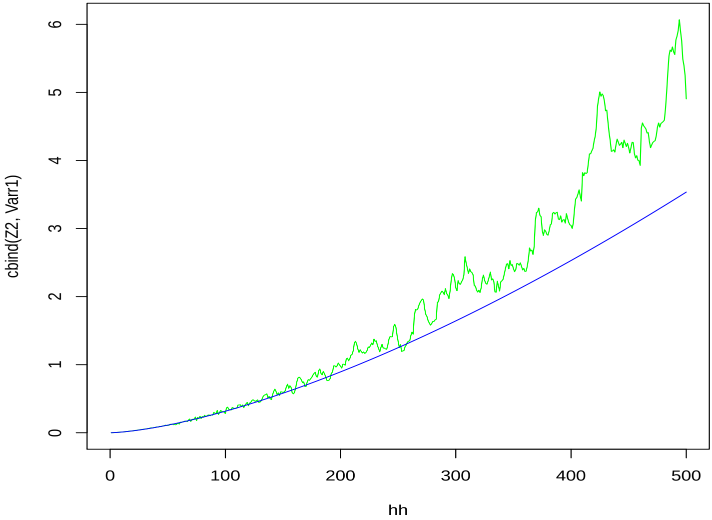
Brownian Motion + Regularized FBM (Rogers) →

$$\varphi(u) = c_H\left(\frac{\beta-u}{1-cu}\right)^{H-\frac{1}{2}}\mathbb{1}_{\{u\leq 0\}}$$

Path of BMA



Variance of BMA



The Dynamics

Dynamics of the BMA-driven Vasicek-model

$$dr = (b - ar)\, dt + \sigma dX^{\varphi}$$

Remarks:

- a, b and σ are positive constants.
- For $\varphi(u) = \mathbb{1}_{\{u \leq 0\}}$ this is the classical Vasicek-model.

Zero coupon bond prices

Due to Gaussianity we have

$$B(t,T) = E\left[e^{-\int_t^T r(s)ds} | \mathcal{F}_t\right]$$
$$= \exp\left[E\left[\int_t^T r(s)ds | \mathcal{F}_t\right] - \frac{1}{2} Var\left[\int_t^T r(s)ds | \mathcal{F}_t\right]\right]$$

Representation of $\int_t^T r(s)ds$

$$\int_t^T r(s)ds = \frac{1}{a}\left[b\left(T-t\right) + \left(1 - e^{-a(T-t)}\right)\left(r\left(t\right) - \frac{b}{a}\right)\right] +$$
$$+ \frac{\sigma}{a}\int_t^T \left(1 - e^{-a(T-u)}\right) dX_u^{\varphi}$$

$dr = (b - ar)\,dt + \sigma dX^{\varphi}$

Conditional Expectation of X^{φ}

$$E\left[X_T^{\varphi}|\mathcal{F}_t\right] = X_t^{\varphi} + \int_{\mathbb{R}} (\varphi(u-T) - \varphi(u-t))\,\mathbb{1}_{\{u \leq t\}} dB_u =: Y_t^{T,\varphi}$$

Conditional Variance of X^{φ}

$$Var\left[X_T^{\varphi}|\mathcal{F}_t\right] = Var\,X_T^{\varphi} - Var\left[E\left[X_T^{\varphi}|\mathcal{F}_t\right]\right]$$
$$= Var\,X_T^{\varphi} - Var\,Y_t^{T,\varphi}$$

$dr = (b - ar)\, dt + \sigma dX^{\varphi}$

Conditional Expectation of $\int_t^T \left(1 - e^{-a(T-u)}\right) dX_u^{\varphi}$

$$E\left[\int_t^T \left(1 - e^{-a(T-u)}\right) dX_u^{\varphi} | \mathcal{F}_t\right] = \int_t^T \left(1 - e^{-a(T-u)}\right) dY_u^{T,\varphi}$$

Conditional Variance of $\int_t^T \left(1 - e^{-a(T-u)}\right) dX_u^{\varphi}$

$$Var\left[\int_t^T \left(1 - e^{-a(T-u)}\right) dX_u^{\varphi} | \mathcal{F}_t\right] =$$

$$= Var \int_t^T \left(1 - e^{-a(T-u)}\right) dX_u^{\varphi} - Var \int_t^T \left(1 - e^{-a(T-u)}\right) dY_u^{T,\varphi}$$

Thank you for your attention!

Literature

- *Cherny "When is a moving average a semimartingale?"*
- *Rogers "Arbitrage with fractional Brownian motion"*
- *Klüppelberg, et al. "Conditional characteristic functions of processes related to fractional Brownian motion"*
- *Cheridito "Regularizing fractional Brownian motion with a view towards stock price modelling"*
- *Basse "Gaussian moving averages and semimartingales"*