On the Lower Arbitrage Bound of American Contingent Claims

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Market model

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\dots,T}, \mathbb{P})$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = \mathcal{F}$, $T \in \mathbb{N}$
- $(S_t^i)_{t=0,\dots,T}, i = 1, \dots, d$, non-negative, adapted process: (discounted) price evolution of d risky assets
- $\xi = (\xi_t^1, \dots, \xi_t^d)_{t=1,\dots,T}$ is a self-financing trading strategy if it is predictable and

$$\sum_{i=1}^{d} \xi_t^i S_t^i = \sum_{i=1}^{d} \xi_{t+1}^i S_t^i \quad \text{for } t = 1, \dots T-1.$$

The (discounted) value process corresponding to ξ is $(V_t^\xi)_{t=0,\dots,T}$:

$$V_0^\xi = \sum_{i=1}^{d} \xi_1^i S_0^i \quad \text{and} \quad V_t^\xi = \sum_{i=1}^{d} \xi_t^i S_t^i,\ t = 1, \dots, T.$$

$\mathcal{M}$: set of equivalent martingale measures (pricing measures)

Assumption: $\mathcal{M} \neq \emptyset$

European contingent claims

(Discounted) European contingent claim: Y on $(\Omega, \mathcal{F}, \mathbb{P})$, $Y \geq 0$

- Pricing in a **COMPLETE** market: $\mathcal{M} = \{\mathbb{Q}\}$.
  Unique no-arbitrage price: $E^{\mathbb{Q}}[Y]$
- Pricing in an **INCOMPLETE** market.
  Set of no-arbitrage prices:

$$\Pi(Y) = \{E^{\mathbb{Q}}[Y] \mid \mathbb{Q} \in \mathcal{M} \text{ and } E^{\mathbb{Q}}[Y] < \infty\}$$

It is well-known that

$$\boxed{\Pi(Y) = \{p(Y)\} \quad \text{or} \quad \Pi(Y) =]\underline{p}(Y), \overline{p}(Y)[}$$

American contingent claims in a complete market

(Discounted) American contingent claim: $(H_t)_{t=0,\dots,T}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\dots,T}, \mathbb{P})$, $H_t \geq 0 \; \forall t$ such that

$$H_t \in L^1(\Omega, \mathcal{F}, \mathbb{Q}) \quad \text{for all } t, \text{ for all } \mathbb{Q} \in \mathcal{M}$$

- Pricing in a **COMPLETE** market: $\mathcal{M} = \{\mathbb{Q}\}$

Optimal stopping problem: find an optimal exercise time τ s.t.

$$E^{\mathbb{Q}}[H_\tau] = \sup\{E^{\mathbb{Q}}[H_\sigma] \mid \sigma \text{ exercise time}\}$$

$\Rightarrow$ unique **arbitrage-free price**: $\pi = E^{\mathbb{Q}}[H_\tau]$.

It is also the unique **fair price** in the sense that:

- it is not too expensive for the buyer (she may exercise it optimally at τ)
- it is not too cheap for the seller (there is no exercise strategy σ such that the value of the exercised claim H_σ exceeds π)

American contingent claims in an incomplete market

- Pricing in an **INCOMPLETE** market ?

▷ ▷ expect the set of arbitrage-free prices to correspond to the set of prices obtained from the solutions to the **optimal stopping problem** under **any** equivalent martingale measure

▷ ▷ any arbitrage-free price should also be a **fair price** in the above sense

Arbitrage-free prices and optimal stopping

Definition. τ **optimal stopping time** for H under $\mathbb{Q} \in \mathcal{M}$ if

$$E^{\mathbb{Q}}[H_\tau] = \sup_\sigma E^{\mathbb{Q}}[H_\sigma]$$

Evaluations at optimal times:

$$\mathcal{P}(H) := \{E^{\mathbb{Q}}[H_\tau] \mid \mathbb{Q} \in \mathcal{M} \text{ and } \tau \text{ optimal for } H \text{ under } \mathbb{Q}\}$$

Definition. $\pi \in \mathbb{R}$ is an **arbitrage-free price** of H if:

(1) π **not too high** for the **buyer**: there exists some exercise time τ and some $\mathbb{Q} \in \mathcal{M}$ such that $\mathbb{E}^{\mathbb{Q}}[H_\tau] \geq \pi$;

(2) π **not too low** for the **seller**: for each exercise time τ, there exists some $\mathbb{Q} \in \mathcal{M}$ such that $\mathbb{E}^{\mathbb{Q}}[H_\tau] \leq \pi$.

$$\Pi(H): \text{ set of arbitrage-free prices of } H$$

Arbitrage-free prices and optimal stopping

Theorem (Föllmer and Schied (2004))

$$\underline{\pi}(H) := \inf \Pi(H) = \inf_{\mathbb{Q}\in\mathcal{M}} \sup_{\tau} E^{\mathbb{Q}}[H_\tau] = \inf \mathcal{P}(H)$$

$$\overline{\pi}(H) := \sup \Pi(H) = \sup_{\mathbb{Q}\in\mathcal{M}} \sup_{\tau} E^{\mathbb{Q}}[H_\tau] = \sup \mathcal{P}(H)$$

Moreover, as for European contingent claims,

$$\overline{\pi}(H) \in \Pi(H) \quad \Longleftrightarrow \quad \Pi(H) = \{\overline{\pi}(H)\},$$

which is also equivalent to H being **attainable** in the sense that there exists a self-financing strategy ξ and τ s.t. $V_0^\xi = \overline{\pi}(H)$,

$$V_t^\xi \geq H_t \text{ for all } t, \quad \text{and} \quad V_\tau^\xi = H_\tau$$

Arbitrage-free prices and optimal stopping

$$\boxed{\mathcal{P}(H) \subseteq \Pi(H) \subseteq [\underline{\pi}(H), \overline{\pi}(H)]}$$

- if $\overline{\pi}(H) \in \Pi(H) \Rightarrow \mathcal{P}(H) = \Pi(H) = \{\overline{\pi}(H)\}$
- if $\underline{\pi}(H), \overline{\pi}(H) \notin \Pi(H) \Rightarrow \mathcal{P}(H) = \Pi(H) =]\underline{\pi}(H), \overline{\pi}(H)[$

QUESTIONS:

- What about $\underline{\pi}(H)$?
- Can be $\mathcal{P}(H) \neq \Pi(H)$?

Main results

▶ **Theorem.** The set $\Pi(H)$ of all arbitrage-free prices for H coincides with the set $\mathcal{P}(H)$ of evaluations at optimal times:

$$\boxed{\mathcal{P}(H) = \Pi(H)}$$

▶ In contrast to the pricing of a European contingent claim, in case of a non-attainable American contingent claim **both cases**

$$\boxed{\underline{\pi}(H) \in \Pi(H) \quad \text{and} \quad \underline{\pi}(H) \notin \Pi(H)}$$

can occur! (see examples)

▶ We provide characterizations of the case $\underline{\pi}(H)$ arbitrage-free price for H.

Example (Föllmer and Schied (2004))

Complete market on $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$, $T = 2$. Add the states ω^+, ω^-:

$$\Omega := \Omega_0 \times \{\omega^+, \omega^-\}, \quad \mathbb{P}[\{\omega_0, \omega^\pm\}] := \frac{1}{2}\mathbb{P}_0[\{\omega_0\}], \, \omega_0 \in \Omega_0$$

$\Rightarrow$ the enlarged financial market is incomplete, with set of EMM:

$$\mathcal{M} = \{\mathbb{Q}_p \mid 0 < p < 1\}, \quad \text{where} \quad \mathbb{Q}_p[\Omega_0 \times \{\omega^+\}] = p.$$

1. For the American contingent claim H:

$$H_0 \equiv 0, \quad H_1 \equiv 1, \quad H_2(\omega) := \begin{cases} 2 & \text{if } \omega = (\omega_0, \omega^+) \\ 0 & \text{if } \omega = (\omega_0, \omega^-) \end{cases}$$

$$\tau_1 \equiv 1 \, \text{opt. for } p \leq 1/2, \tau_2 \equiv 2 \, \text{for } p > 1/2 \Rightarrow \Pi(H) = [1, 2)$$

2. For the American contingent claim $\widetilde{H}$:

$$\widetilde{H}_0 = H_0, \quad \widetilde{H}_1 \equiv 0, \quad \widetilde{H}_2 = H_2$$

$$\tau_2 \equiv 2 \, \text{is optimal for all } p \Rightarrow \Pi(\widetilde{H}) = (0, 2)$$

Snell envelopes of American contingent claims

Definition. For $\mathbb{Q} \in \mathcal{M}$, the Snell envelope $U^{\mathbb{Q}} = (U_t^{\mathbb{Q}})_{t=0,\dots,T}$ of H w.r.to $\mathbb{Q}$ is defined by

$$U_t^{\mathbb{Q}} = \operatorname*{ess\,sup}_{\tau \geq t} E^{\mathbb{Q}}[H_\tau \mid \mathcal{F}_t], \quad t = 0, \dots, T.$$

In particular, $U_0^{\mathbb{Q}} = \sup_\tau E^{\mathbb{Q}}[H_\tau]$ value obtained by optimally exercising H under $\mathbb{Q}$.

- $U^{\mathbb{Q}}$ is the smallest $\mathbb{Q}$-supermartingale dominating H.
- τ optimal stopping for H under $\mathbb{Q} \iff H_\tau = U_\tau^{\mathbb{Q}}$ and $(U_{t\wedge\tau}^{\mathbb{Q}})_{t=0,\dots,T}$ is a $\mathbb{Q}$-martingale
- the minimal optimal stopping time for H under $\mathbb{Q}$ is given by

$$\tau^{\mathbb{Q}} := \inf\{t \geq 0 \mid U_t^{\mathbb{Q}} = H_t\}$$

$$\boxed{\hat{\tau} := \operatorname*{ess\,inf}_{\mathbb{Q} \in \mathcal{M}} \tau^{\mathbb{Q}}}$$

The lower Snell envelope of American contingent claims

Definition. The lower Snell envelope $U^{\downarrow} = (U_t^{\downarrow})_{t=0,\dots,T}$ of H is defined by

$$U_t^{\downarrow} = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{M}} U_t^{\mathbb{Q}} = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{M}} \operatorname*{ess\,sup}_{\tau\geq t} E^{\mathbb{Q}}[H_\tau \mid \mathcal{F}_t], \quad t = 0, \dots, T.$$

In particular, $U_0^{\downarrow} = \underline{\pi}(H)$.

▷ $(U_{t\wedge\hat{\tau}}^{\downarrow})_{t=0,\dots,T}$ is a $\mathcal{M}$-submartingale

Proposition. The lower Snell envelope process satisfies:

- $U_{\hat{\tau}}^{\downarrow} = H_{\hat{\tau}}$
- if $H_{\hat{\tau}}$ is replicable at price $\underline{\pi}(H) \Rightarrow (U^{\downarrow})^{\hat{\tau}}$ is a $\mathcal{M}$-martingale

Lemma. Let τ be s.t. H_τ is replicable $\Rightarrow p(H_\tau) \leq \underline{\pi}(H)$.
If τ optimal under some $\mathbb{Q} \Rightarrow p(H_\tau) = \underline{\pi}(H)$.

Main theorem

Theorem. The following conditions are equivalent:

1. $\underline{\pi}(H)$ is a NA-price ($\underline{\pi}(H) \in \Pi(H)$)
2. $\underline{\pi}(H)$ is obtained by opt. stopping ($\exists\, \mathbb{Q} \in \mathcal{M} : U_0^{\mathbb{Q}} = \underline{\pi}(H)$)
3. $H_{\hat{\tau}}$ is replicable at price $\underline{\pi}(H)$
4. There exists τ optimal under some $\mathbb{Q}$ such that H_τ is replicable

Remark.

- $\underline{\pi}(H) \in \Pi(H) \iff \forall\, \tau$, the enlarged market $\{S, (\underline{\pi}(H), H_\tau)\}$

with no short-selling of H_τ, admits an ESM (NA).

- $\underline{\pi}(H) = U_0^Q \iff \{S, \{(\underline{\pi}(H), H_\tau), \tau\}\}$ with no short-selling

of any H_τ, admits an ESM.

On the main theorem

Some intermediate steps:

- $\mathcal{M}$ is stable under pasting
- there is $(\mathbb{Q}_n)_{n\in\mathbb{N}} \subset \mathcal{M}$ such that $U_{\hat{\tau}}^{\mathbb{Q}_k} \searrow U_{\hat{\tau}}^{\downarrow} = H_{\hat{\tau}}$
- $\{\tau^{\mathbb{Q}} \mid \mathbb{Q} \in \mathcal{M}\}$ is downward directed
- there is $(\mathbb{Q}_k)_{k\in\mathbb{N}} \subset \mathcal{M}$ such that $\{\tau^{\mathbb{Q}_k} = \hat{\tau}\} \nearrow \Omega$
- $\hat{\tau} = \sum_{k=1}^{\infty} \tau^{\mathbb{Q}_k} 1_{B_k}$
- $\widetilde{\mathbb{P}}$ obtained as pasting in $\hat{\tau}$ of any fixed $\mathbb{P}^* \in \mathcal{M}$ with $\mathbb{Q}_k$'s on B_k's satisfies: $\widetilde{\mathbb{P}} \in \mathcal{M}$ and $U_{\hat{\tau}}^{\widetilde{\mathbb{P}}} = H_{\hat{\tau}}$
- If $H_{\hat{\tau}}$ replicable at price $\underline{\pi}(H)$, then

$$\{\mathbb{Q} \in \mathcal{M} \mid U_{\hat{\tau}}^{\mathbb{Q}} = H_{\hat{\tau}}\} = \{\mathbb{Q} \in \mathcal{M} \mid U_0^{\mathbb{Q}} = \underline{\pi}(H)\}$$

Some remarks

Remark. Our main theorem extends the case of European claims:
Y discounted European contingent claim
H American contingent claim defined by

$$H_t = 0 \text{ for all } t = 0, \dots, T-1, \text{ and } H_T = Y$$

$\Rightarrow H_{\hat{\tau}} = Y$

$\Rightarrow \underline{\pi}(H) = \inf \Pi(Y)$ is arbitrage-free if and only if Y is replicable.

Remark. Föllmer and Schied (2004) consider the stopping time

$$\tau^{\downarrow} := \inf\{t \geq 0 \mid U_t^{\downarrow} = H_t\} \leq \hat{\tau}$$

Questions:

- Is it always $\tau^{\downarrow} = \hat{\tau}$?
- If not, can we carry out our analysis replacing $\hat{\tau}$ by $\tau^{\downarrow}$?

The answer to **both** questions is **no** (see example)

Illustrating example

$X_1, X_2 \sim \mathcal{N}(0,1)$ on $(\Omega_i, \mathcal{A}_i, \mathbb{P}_i), i = 1, 2$. Product space:

$$\Omega = \Omega_1 \times \Omega_2, \quad \mathcal{F} = \mathcal{A}_1 \otimes \mathcal{A}_2, \quad \mathbb{P} = \mathbb{P}_1 \otimes \mathbb{P}_2$$

$\widetilde{X}_i$ on $(\Omega, \mathcal{F}, \mathbb{P})$: $\widetilde{X}_i(\omega_1, \omega_2) = -1 + \sqrt{2} X_i(\omega_i)$, $i = 1, 2$.

Risky asset: $S_0 = 1, \quad S_1 = e^{\widetilde{X}_1}, \quad S_2 = e^{\widetilde{X}_1 + \widetilde{X}_2}$.

Filtration: $\mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \mathcal{F}_1 = \sigma(\widetilde{X}_1), \quad \mathcal{F}_2 = \sigma(\widetilde{X}_1, \widetilde{X}_2)$.

Discounted American contingent claim:

$$H_0 = 0, \quad H_1 = e^{\widetilde{X}_1}, \quad H_2 = e^{\widetilde{X}_1 + \frac{1}{2}\widetilde{X}_2}.$$

- ▷ $\tau^{\mathbb{Q}} \geq 1, \forall \mathbb{Q} \in \mathcal{M}$
- ▷ $\mathbb{P} \in \mathcal{M}$ with $\tau^{\mathbb{P}} = 1$ $(\forall \tau, \ \mathbb{E}^{\mathbb{P}}[H_\tau] \leq 1 = \mathbb{E}^{\mathbb{P}}[H_1])$

$\Rightarrow$ $\hat{\tau} = 1$, $H_{\hat{\tau}} = S_1$ replicable $\Rightarrow$ $\underline{\pi}(H) \in \Pi(H)$

Illustrating example

Another discounted American contingent claim:

$$H_0 = 0, \quad H_1 = e^{\widetilde{X}_1}, \quad H_2 = e^{\widetilde{X}_1} Z$$

where $\quad Z = e^{\widetilde{X}_2} 1_{\{\widetilde{X}_2 > 1\}} + 1_{\{\widetilde{X}_2 \leq 1\}}$

- ▷ $\forall \tau \;\; H_\tau \leq H_2 \Rightarrow \tau^{\mathbb{Q}} = 2 \;\forall \mathbb{Q} \Rightarrow \hat{\tau} = 2$
- ▷ On the other hand, there is $(\mathbb{Q}_n)_{n \in \mathbb{N}} \subset \mathcal{M}$ s.t. $E^{\mathbb{Q}_n}[Z \mid \mathcal{F}_1] \to 1$.

$\Rightarrow U_1^{\downarrow} = \operatorname*{ess\,inf}_{\mathbb{Q} \in \mathcal{M}} \operatorname*{ess\,sup}_{\tau \geq 1} E^{\mathbb{Q}}[H_\tau \mid \mathcal{F}_1] = H_1 \;\Rightarrow\; \tau^{\downarrow} = 1 < 2 = \hat{\tau}$

Moreover, $H_{\tau^\downarrow} = S_1$ is replicable, but $H_{\hat{\tau}}$ is not

$\Rightarrow \underline{\pi}(H) \notin \Pi(H)$

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