Generalization of the Dybvig–Ingersoll–Ross Theorem and Asymptotic Minimality

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References

Philip H. Dybvig, Jonathan E. Ingersoll, and Stephen A. Ross:

Friedrich Hubalek, Irene Klein, and Josef Teichmann:
Outline

1. Notation
2. Dybvig–Ingersoll–Ross Theorem
3. Examples
4. Asymptotic Minimality
Probabilistic Model and Zero-Coupon Bonds

Notation:

- Filtered probability space: \((\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) for discrete time \(t \in \mathbb{N}_0\) or for continuous time \(t \in [0, \infty)\)
- Maturity: \(T \in \mathbb{N}\) or \(T \in (0, \infty)\)
- Zero-coupon bond price process \(P(t, T)\): strictly positive, \(\mathbb{F}\)-adapted process with normalization \(P(T, T) = 1\).
Definition of Zero-Coupon Rates

**Zero-coupon rates** $R(t, T)$ (investment yields):

- **Continuous-time**: For $T > 0$ and $t \in [0, T)$

  $$R(t, T) = -\frac{\log P(t, T)}{T - t}$$

- **Discrete-time**: For $T \in \mathbb{N}$ and $t \in \{0, \ldots, T - 1\}$

  $$R(t, T) = P(t, T)^{-1/(T - t)} - 1$$
Interpretation of Zero-Coupon Rates

Representation of zero-coupon bond prices:

- Continuous-time: For $T > 0$ and $t \in [0, T)$
  \[ P(t, T) = \exp(-(T - t)R(t, T)) \]

- Discrete-time: For $T \in \mathbb{N}$ and $t \in \{0, \ldots, T - 1\}$
  \[ P(t, T) = \frac{1}{(1 + R(t, T))^{(T-t)}} \]

Interpretation:
If we invest 1 at time $t$ in the $T$-bond, then this will accumulate at an average rate of $R(t, T)$ over the period.
Definition of Arbitrage-Free Forward Rates

Arbitrage-free forward rate $F(s, t, T)$:

- Continuous-time: For $T > 0$ and $s \leq t$ in $[0, T)$

$$F(s, t, T) = \frac{1}{T - t} \log \frac{P(s, t)}{P(s, T)}$$

- Discrete-time: For $T \in \mathbb{N}$ and $s \leq t$ in $\{0, \ldots, T - 1\}$

$$F(s, t, T) = \left( \frac{P(s, t)}{P(s, T)} \right)^{1/(T-t)} - 1$$

Interpretation:
In the forward contract we fix at time $s$ the rate of interest $F(s, t, T)$ for a loan between times $t$ and $T$. 
Outline

1 Notation

2 Dybvig–Ingersoll–Ross Theorem
   • Original Version of the DIR Theorem
   • Dybvig–Ingersoll–Ross for Limit Superior

3 Examples

4 Asymptotic Minimality
Dybvig–Ingersoll–Ross Theorem

Dybvig–Ingersoll–Ross theorem (DIR):
Assume that the zero-coupon bond market is arbitrage-free.

- If for $s < t$ the long-term zero-coupon rates
  \[ l(s) = \lim_{T \to \infty} R(s, T) \quad \text{and} \quad l(t) = \lim_{T \to \infty} R(t, T) \]
  exist almost surely, then $l(s) \leq l(t)$ almost surely.

- If for $s \leq t$ the long-term forward rate
  \[ l_F(s, t) = \lim_{T \to \infty} F(s, t, T) \]
  exist a.s., then $l_F(s, t) = l(s)$ a.s. and it holds $l_F(s, s') \leq l_F(t, t')$ for all $s' \geq s$ and $t' \geq t$. 
Why Should the Theorem Be True?

**Economic interpretation of DIR:**

- From time $s$ to a later time $t$ the information increases from $\mathcal{F}_s$ to $\mathcal{F}_t$.
- More informed decision concerning the best zero-coupon bonds for long-term investment can be made at time $t$.
- The earnings during $[s, t]$ are negligible in the limit for $T \to \infty$.

$\Rightarrow$ The long-term zero-coupon rates should never fall!
Disadvantage of Dybvig–Ingersoll–Ross theorem:

- Existence of the limit of the long-term zero-coupon and forward rates has to be shown in advance.
- There exist models, where these limits do not exist!

Our generalization:

Replace the limit by the limit superior of the long-term rates!
Definition of the Long-Term Rates

**Long-term zero-coupon rate** for $t \geq 0$:

$$l(t) := \limsup_{T \to \infty} R(t, T) = \lim_{n \to \infty} \text{ess sup}_{T > n \vee t} R(t, T)$$

**Long-term forward rate** for $0 \leq s \leq t$:

$$l_F(s, t) := \limsup_{T \to \infty} F(s, t, T) = \lim_{n \to \infty} \text{ess sup}_{T > n \vee t} F(s, t, T).$$

**Definition of the essential supremum:**

- $\mathcal{X}$: (uncountable) family of random variables
- $\text{ess sup} \mathcal{X}$: smallest random variable, dominating each random variable in $\mathcal{X}$
Why do we use the limit superior?

- Investor prefers long-term investments with high return.
- Prefers the limit superior of the zero-coupon rates
- Approximation of the limit superior is possible by choosing an appropriate bond maturity

Lemma (G. & Schmock)

Given $t \geq 0$, there exists a sequence of $\mathcal{F}_t$-measurable random maturities $T_n : \Omega \rightarrow (n \lor t, \infty)$, each one taking only a finite number of values, such that

$$l(t) = \lim_{n \to \infty} R(t, T_n).$$
Generalization of the DIR Theorem (Version 1)

Theorem (G. & Schmock)

If there exists a probability measure $\mathbb{Q}_{s,t}$ for $0 \leq s < t$ on $(\Omega, \mathcal{F}_t)$, equivalent to $\mathbb{P}|\mathcal{F}_t$, such that for large $T > t$

$$P(s, T) \geq P(s, t) \mathbb{E}_{\mathbb{Q}_{s,t}}[P(t, T)|\mathcal{F}_s] \quad \text{a.s.}$$

then

- $l(s) \leq l(t)$ a.s. and
- $l_F(s, s') \leq l_F(t, t')$ a.s. for all $s' \geq s$ and $t' \geq t$.

Remarks:

- If equality holds, then $\mathbb{Q}_{s,t}$ is called the **forward (time s)** risk neutral probability measure for maturity $t$.
- Hubalek et al. assume the existence of such a risk neutral measure. Their proof can be adapted for the limit superior.
No Arbitrage in the Limit

Assume for $0 \leq s < t$ and $n \in \mathbb{N}$:

- Maturities $T_n$: $\mathcal{F}_s$-measurable, finite number of values
- Portfolio compositions $(\phi_n, \psi_n)$: $\mathcal{F}_s$-measurable
- Portfolio value $u \in [s, t]$: $V_n(u) = \phi_n P(u, T_n) + \psi_n P(u, t)$

Arbitrage in the limit:

1. $V_n(s) = 0$, a.s. for all $n \in \mathbb{N}$
2. $\mathbb{P}(\lim \inf_{n \to \infty} V_n(t) > 0) > 0$
3. $\lim \inf_{n \to \infty} V_n(t) \geq 0$ a.s.
No Arbitrage in the Limit with Vanishing Risk

Assume for $0 \leq s < t$ and $n \in \mathbb{N}$:

- Maturities $T_n$: $\mathcal{F}_s$-measurable, finite number of values
- Portfolio compositions $(\phi_n, \psi_n)$: $\mathcal{F}_s$-measurable
- Portfolio value $u \in [s, t]$: $V_n(u) = \phi_n P(u, T_n) + \psi_n P(u, t)$

Arbitrage in the limit with vanishing risk:

1. $V_n(s) = 0$, a.s. for all $n \in \mathbb{N}$
2. $P(\liminf_{n \to \infty} V_n(t) > 0) > 0$
3. For each $\varepsilon > 0$ there ex. $n_\varepsilon \in \mathbb{N}$ s.t. $V_n(t) \geq -\varepsilon$ a.s. for all $n \geq n_\varepsilon$
Relation Between Notions of No Arbitrage

No arbitrage in the limit

in general \(\Downarrow\) \(\Uparrow\) \(\mathcal{F}_t\) finite

No arbitrage in the limit with vanishing risk

\(\uparrow\)

Forward (time s) risk neutral measure
Generalization of the DIR Theorem (Version 2)

**Theorem (G. & Schmock)**

*If there is no arbitrage in the limit with vanishing risk for* \(0 \leq s < t\), *then*

- \(l(s) \leq l(t)\) a.s.
- \(l_F(s, s') \leq l_F(t, t')\) a.s. for all \(s' \geq s\) and \(t' \geq t\)

**Remarks:**

- This DIR-version implies version 1 if the existence of a forward risk neutral measure is assumed.
- Dybvig et al. use the notion of no arbitrage in the limit with vanishing risk. Their proof can be adapted.
Outline

1. Notation
2. Dybvig–Ingersoll–Ross Theorem
3. Examples
   - Deterministic Model
   - Vasiček Model
4. Asymptotic Minimality
Model Class with Forward Risk Neutral Measures

Construction of forward risk neutral measure:

- Bank account: \((B_t)_{t \geq 0}\) strictly positive, \(\mathbb{F}\)-adapted, \(B_0 = 1\)
- Assume \(1/B_T\) is \(\mathbb{Q}\)-integrable for every \(T > 0\).
- Zero-coupon bond price

\[
P(t, T) = \mathbb{E}_\mathbb{Q}\left[ \frac{B_t}{B_T} \mid \mathcal{F}_t \right], \quad t \in [0, T]
\]

- Density of forward risk neutral probability measure

\[
\frac{d\mathbb{Q}_{s,t}}{d\mathbb{Q}} = \frac{B_s}{P(s, t)B_t}, \quad s \in [0, t)
\]

\Rightarrow \quad \mathbb{E}_{\mathbb{Q}_{s,t}}[P(t, T)\mid \mathcal{F}_s] = \mathbb{E}_\mathbb{Q}\left[ \frac{B_s}{P(s, t)B_t} \mathbb{E}_\mathbb{Q}\left[ \frac{B_t}{B_T} \mid \mathcal{F}_t \right] \mid \mathcal{F}_s \right]

\[= \frac{P(s, T)}{P(s, t)}
\]
Short-Rate Models

Construction of short-rate models:

- Interest rate intensity: $\{r_t\}_{t \geq 0}$ is $\mathbb{F}$-progressive process with locally integrable paths
- Bank account:

  \[ B_t = \exp\left(\int_0^t r_u \, du\right), \quad t \in [0, \infty). \]

If $1/B_T$ is $\mathbb{Q}$-integrable, then for all $0 \leq t \leq T$:

\[ P(t, T) = \mathbb{E}_\mathbb{Q}\left[\exp\left(-\int_t^T r_u \, du\right) \mid \mathcal{F}_t\right]. \]

\[ R(t, T) = -\frac{1}{T - t} \log \mathbb{E}_\mathbb{Q}\left[\exp\left(-\int_t^T r_u \, du\right) \mid \mathcal{F}_t\right] \]
A Deterministic Short-Rate Model

Define càdlàg interest rate intensity process

\[ r_t = a + b \mathbb{1}_A(t), \quad a, b \in \mathbb{R} \text{ and } t \geq 0, \]

with the set

\[ A = \left[ \frac{1}{3}, 1 \right) \cup \bigcup_{k=0}^{\infty} \left[ 2^{2k+1}, 2^{2k+2} \right). \]

Visualization of \( A \):

![Visualization of A](image)
Deterministic Short-Rate Model \( r_t = a + b 1_A(t) \)

**Zero-coupon rate** for \( 0 \leq t < T \):

\[
R(t, T) = -\frac{1}{T-t} \log \mathbb{E}_Q \left[ \exp \left( -\int_t^T r_u du \right) \big| \mathcal{F}_t \right]
\]

\[
= a + \frac{b}{T-t} \int_t^T 1_A(u) du = a + b \frac{\lambda(A \cap [t, T])}{T-t}.
\]

**Limit of zero-coupon rate does not exist:**

- \( R(0, 2^{2n+1}) = a + \frac{b}{3} \) and \( R(0, 2^{2n+2}) = a + \frac{2b}{3} \) for \( n \in \mathbb{N} \).
- Every point in the interval \( [a + \frac{b}{3}, a + \frac{2b}{3}] \) is an accumulation point of \( \{R(t, T)\}_{T>t} \) as \( T \to \infty \).
Zero-Coupon Rates for Model with $r_t = a + b 1_{A(t)}$

**Figure:** Zero-coupon rate $R(0, T)$ of the deterministic short rate model, with coefficients $a = 1$ and $b = 2$. 
Generalized Vasiček Model

**Short rate process:** Solution of SDE

\[ dr_t = \alpha(\mu_t - r_t)dt + \sigma_t \, dW_t, \quad t \geq 0, \]

where \( \alpha > 0, \mu, \sigma : [0, \infty) \to \mathbb{R} \) bounded and deterministic and \((W_t)_{t \geq 0}\) Brownian motion.

**Zero-coupon rate** for \( 0 \leq t < T \):

\[
R(t, T) = r_t \frac{1 - e^{-\alpha(T-t)}}{\alpha(T-t)} + \frac{1}{T-t} \int_t^T (1 - e^{-\alpha(T-s)}) \mu_s \, ds
\]

\[- \frac{1}{2\alpha^2(T-t)} \int_t^T (1 - e^{-\alpha(T-s)})^2 \sigma_s^2 \, ds \]
Long-Term Rates in the Vasiček Model

Zero-coupon rate for $T \to \infty$:

$$R(t, T) = \frac{1}{T - t} \int_t^T \mu_s \, ds - \frac{1}{2\alpha^2(T - t)} \int_t^T \sigma_s^2 \, ds + O\left(\frac{1}{T}\right)$$

Limit of $R(t, T)$ does not exist:

- Choose $\mu_s = a + b 1_A(s)$ for $s \geq 0$, with the set $A$ from deterministic example, and $\sigma$ constant.
- Set $\mu$ const. For $a, b, c > 0$ with $a \geq b\sqrt{1 + c^2}$ and $s \geq 0$

$$\sigma_s^2 = a + b \sin(c \log(s + 1)) + bc \cos(c \log(s + 1))$$

$$\Rightarrow \quad l(t) = \mu - \frac{a - b}{2\alpha^2}, \quad \text{but } \liminf_{T \to \infty} R(t, T) = \mu - \frac{a + b}{2\alpha^2}$$
Zero-Coupon Rates for Vasiček Model

Set $\mu$ const., $\sigma_s^2 = a + b \sin(c \log(s + 1)) + bc \cos(c \log(s + 1))$

Figure: Zero-coupon rate $R(0, T)$ with $r_0 = 0.5$, $\mu = 2.3$, $\alpha = 3$, $a = b\sqrt{1 + c^2}$, $b = 1/(10\alpha^2)$, and $c = 5.5$. 
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Definition of Lower Envelope

**Definition of lower $\mathcal{G}$-measurable envelope:**

- $(\Omega, \mathcal{F}, \mathbb{P})$ probability space
- $\mathcal{G} \subset \mathcal{F}$ sub-$\sigma$-algebra
- $X$ is $\mathbb{R}$-valued random variable

Define the **lower $\mathcal{G}$-measurable envelope:**

$$X_\mathcal{G} = \text{ess sup}\{Z : \Omega \to \mathbb{R} : \mathcal{G}\text{-measurable r.v., } Z \leq X\}$$
Definition of Asymptotic Minimality

Dybvig–Ingersoll–Ross theorem for $0 \leq s < t$:
No arbitrage in the limit with vanishing risk implies

$$l(s) \leq l(t) \text{ a.s.} \implies l(s) \leq l(t)\mathcal{F}_s \text{ a.s.}$$

Under which conditions holds asymptotic minimality,

$$l(s) = l(t)\mathcal{F}_s, \text{ a.s.}$$
**Theorem (G. & Schmock)**

Assume there is no arbitrage in the limit for $0 \leq s < t$ and $T_n : \Omega \rightarrow (n \wedge t, \infty)$ is $\mathcal{F}_s$-measurable with finitely many values for each $n \in \mathbb{N}$. Then

$$\left( \liminf_{n \to \infty} R(t, T_n) \right)_{\mathcal{F}_s} \leq l(s) \ a.s.$$  

**Corollary**

Assume there is no arbitrage in the limit. Then

$$\left( \liminf_{T \to \infty} R(t, T) \right)_{\mathcal{F}_s} \leq l(s) \leq \left( \limsup_{T \to \infty} R(t, T) \right)_{\mathcal{F}_s} \ a.s.$$  

If the limits exist a.s., then $l(s) = l(t)_{\mathcal{F}_s} \ a.s.$
Asymptotic Minimality for Limit Superior

**Theorem (G. & Schmock)**

Assume there is no arbitrage in the limit for $0 \leq s < t$ and $T_n : \Omega \rightarrow (n \land t, \infty)$ is $\mathcal{F}_s$-measurable with finitely many values for each $n \in \mathbb{N}$. Then

$$\left( \liminf_{n \to \infty} R(t, T_n) \right)_{\mathcal{F}_s} \leq l(s) \text{ a. s.}$$

**Corollary**

Assume there is no arbitrage in the limit. Then

$$\left( \liminf_{T \to \infty} R(t, T) \right)_{\mathcal{F}_s} \leq l(s) \leq \left( \limsup_{T \to \infty} R(t, T) \right)_{\mathcal{F}_s} \text{ a. s.}$$

If the limits exist a. s., then $l(s) = l(t)_{\mathcal{F}_s} \text{ a. s.}$
Results for Asymptotic Minimality

Relation between no arbitrage and asymptotic minimality:

- Limits exist and no arbitrage in the limit  \( \Rightarrow \) Asymptotic minimality holds
- Limits exist and no arbitrage with vanishing risk  \( \Rightarrow \) not sufficient for asymptotic minimality
- Limits and forward risk neutral measure exist  \( \Rightarrow \) not sufficient
- No arbitrage in the limit  \( \Rightarrow \) not sufficient
Reference

V. Goldammer and U. Schmock:
*Generalization of the Dybvig–Ingersoll–Ross Theorem and Asymptotic Minimality.*

Preprint available at:
www.fam.tuwien.ac.at/~schmock/Dybvig-Ingersoll-Ross.html

Thank you for your attention!