Order book resilience, price manipulations, and the positive portfolio problem

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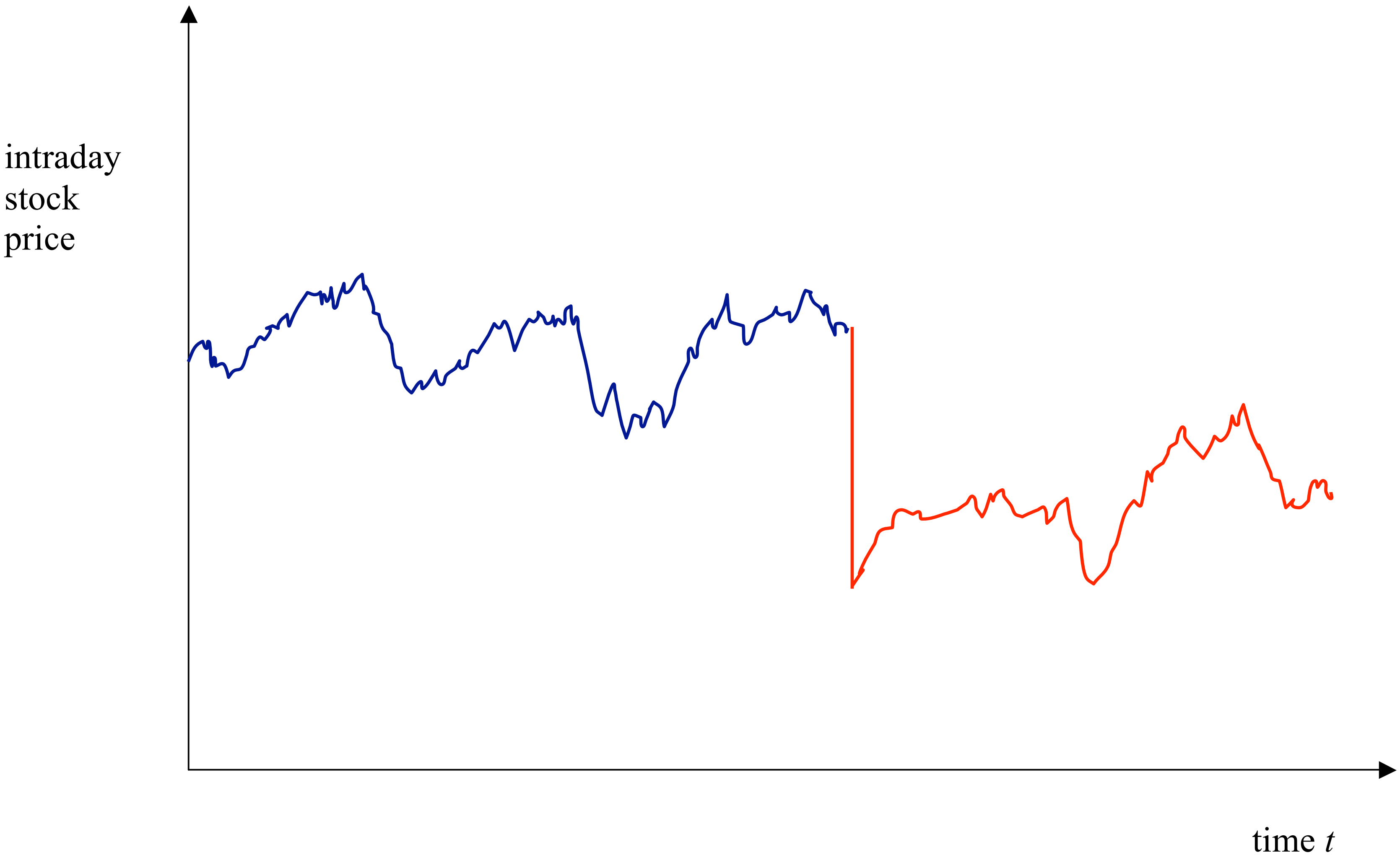
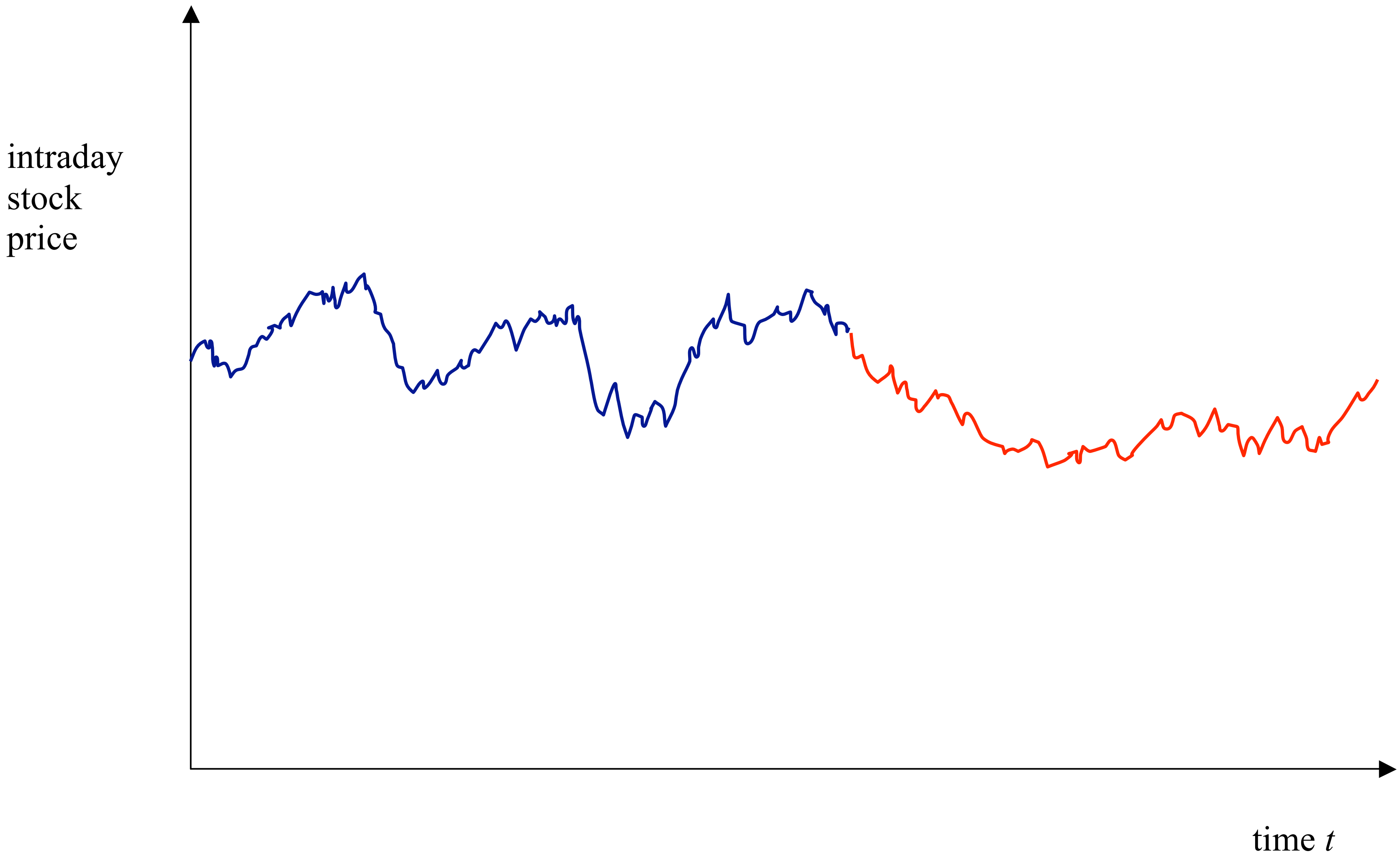
Joint work with Aurélien Alfonsi and Alla Slynko

Large trades can significantly impact prices



Spreading the order can reduce the overall price impact

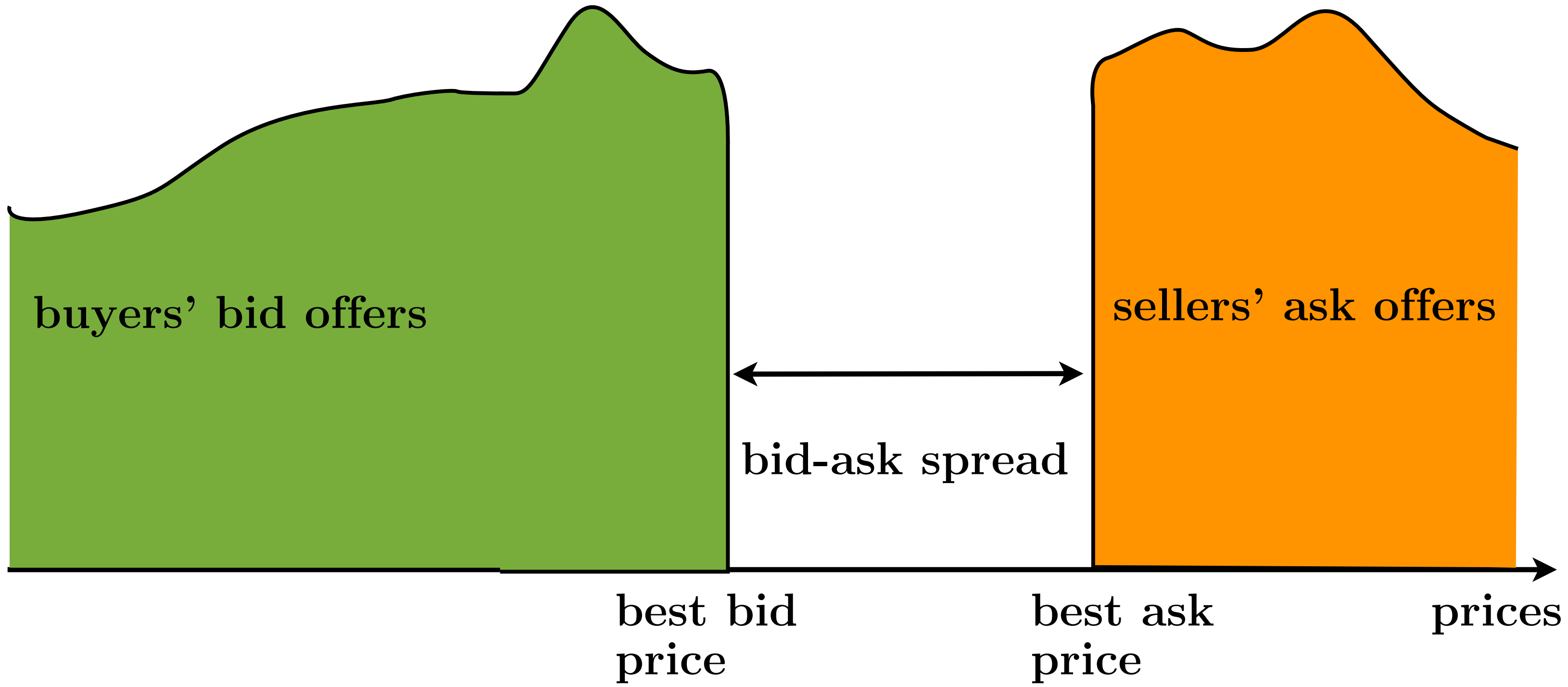


How to execute a single trade of selling X_0 shares?

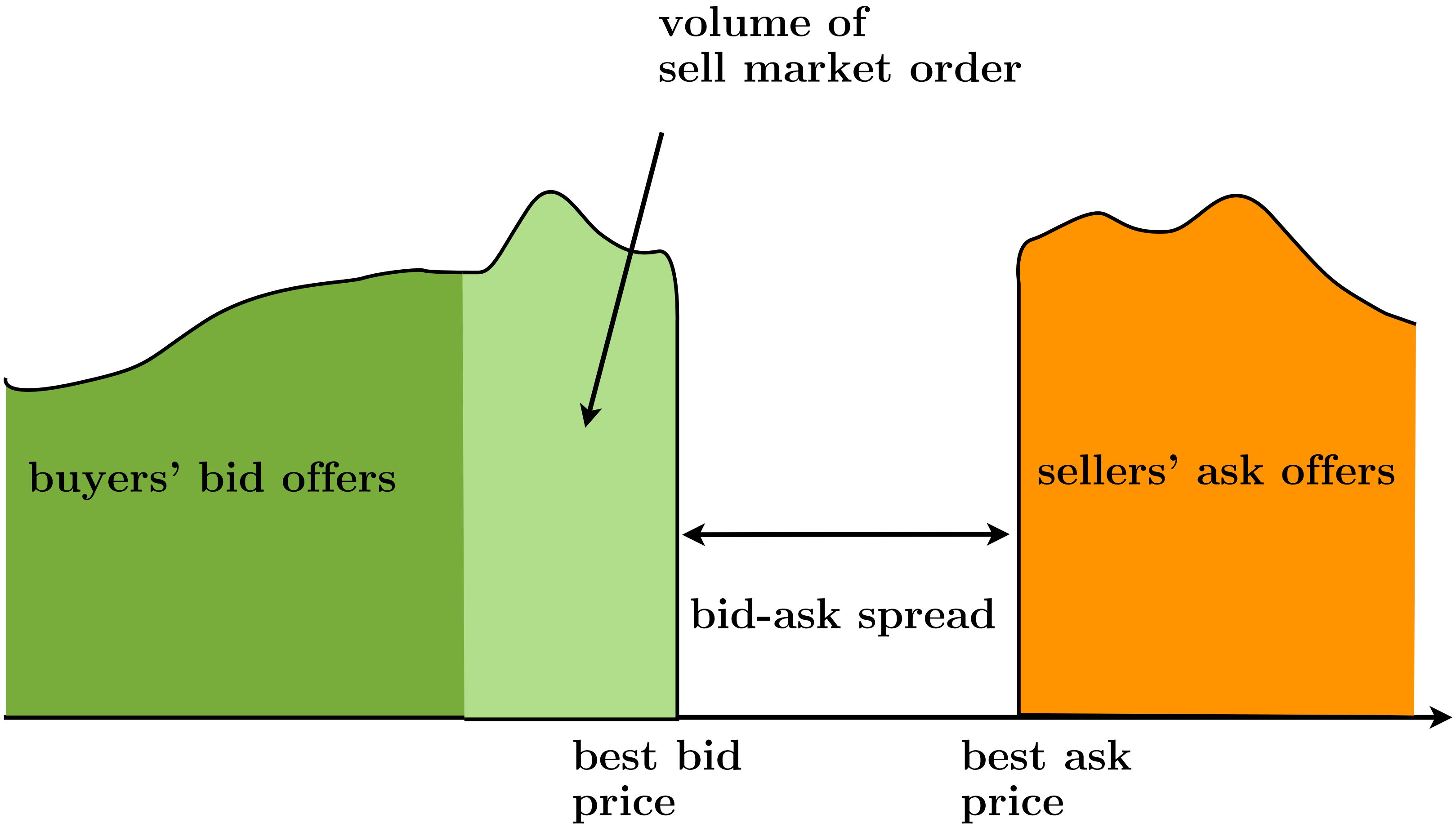
Interesting because:

- **Liquidity/market impact risk in its purest form**
  - development of realistic market impact models
  - checking viability of these models
  - building block for more complex problems
- **Relevant in applications**
  - real-world tests of new models
- **Interesting mathematics**

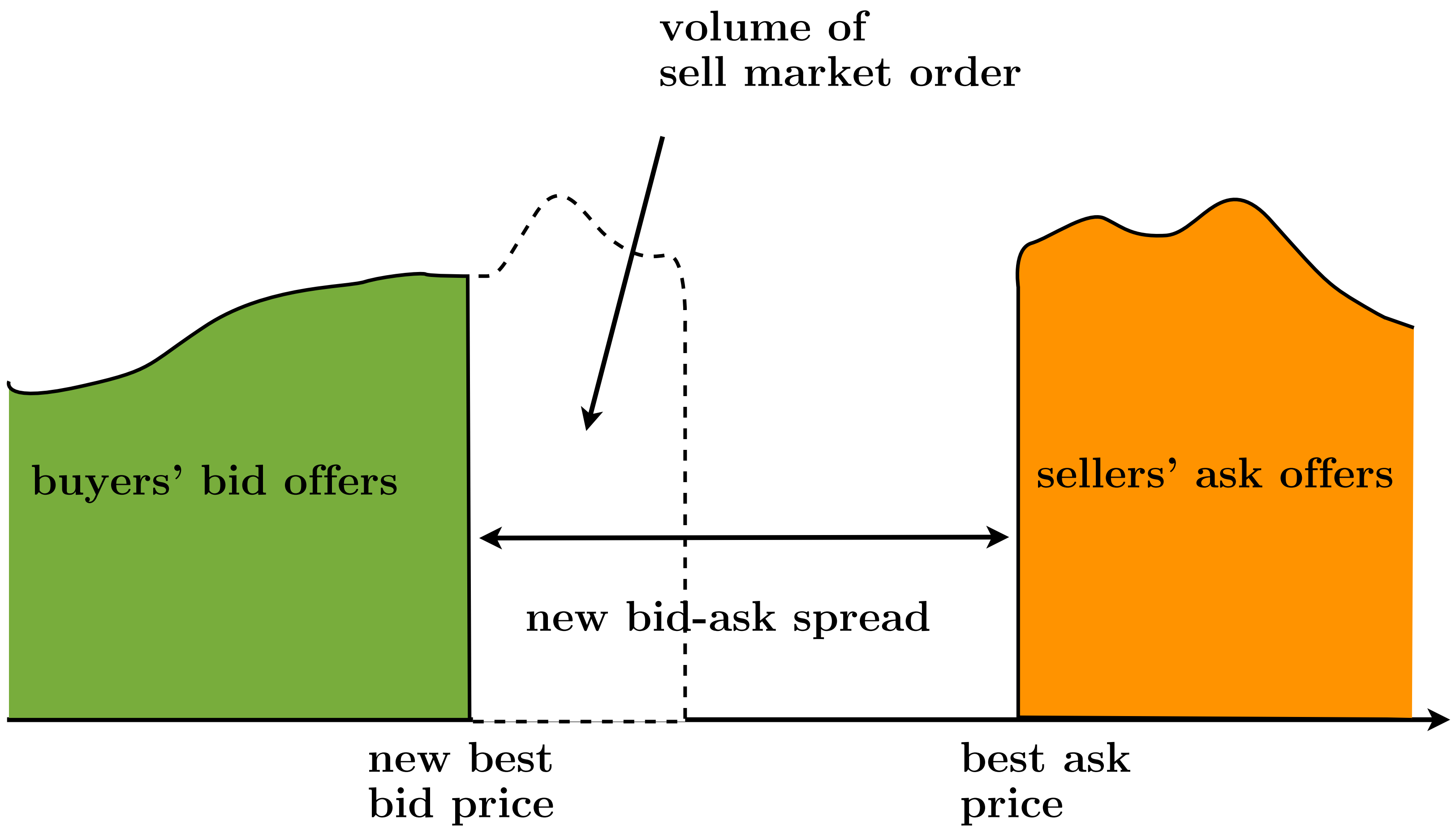
Limit order book before market order



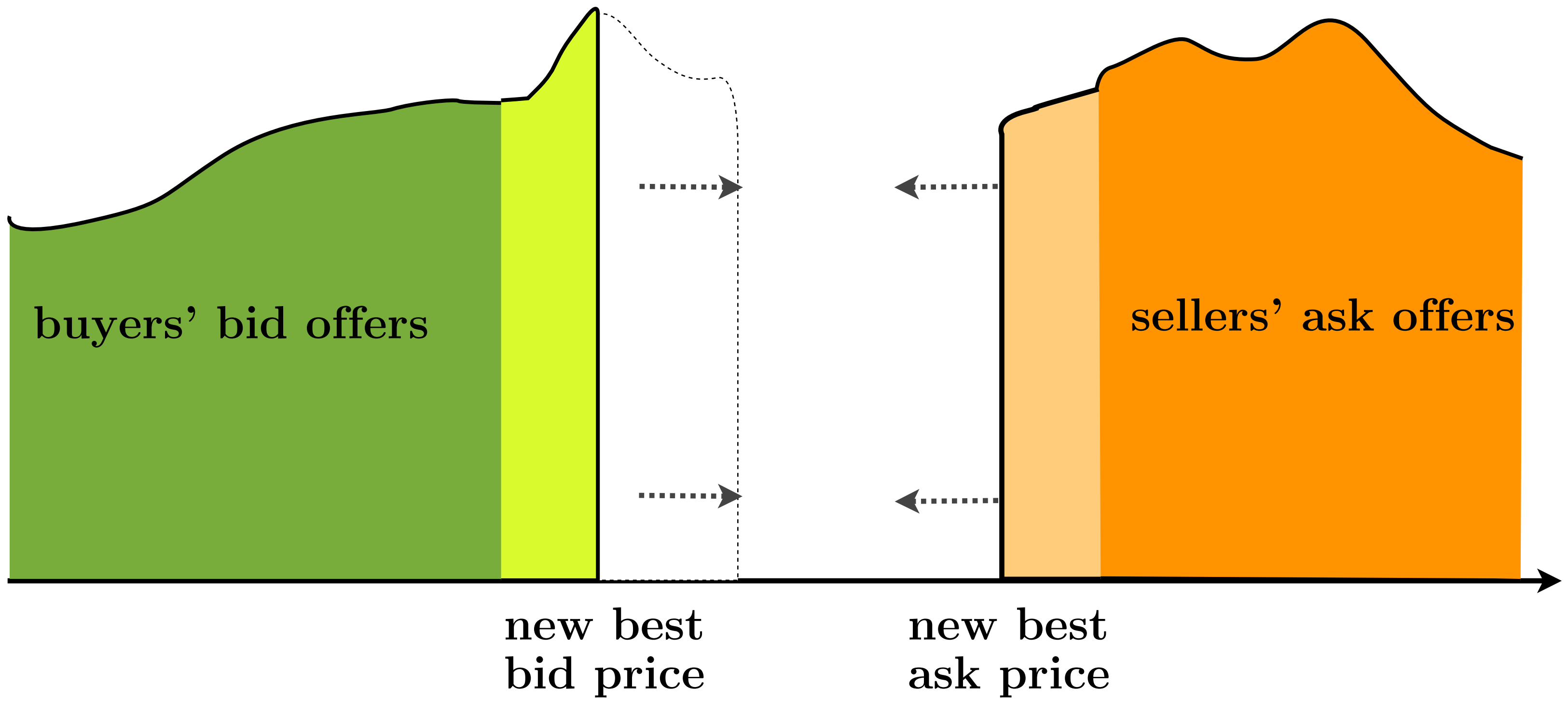
Limit order book before market order



Limit order book after market order



Resilience of the limit order book after market order



Overview

1. Linear impact, general resilience

2. Nonlinear impact,
   exponential resilience

3. Relations with Gatheral's model

References

A. Alfonsi, A. Fruth, and A.S.: *Optimal execution strategies in limit order books with general shape functions.* To appear in Quantitative Finance

A. Alfonsi, A. Fruth, and A.S.: *Constrained portfolio liquidation in a limit order book model.* Banach Center Publ. 83, 9-25 (2008).

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A. Alfonsi, A.S., and A. Slynko: *Order book resilience, price manipulations, and the positive portfolio problem.* Preprint 2009.

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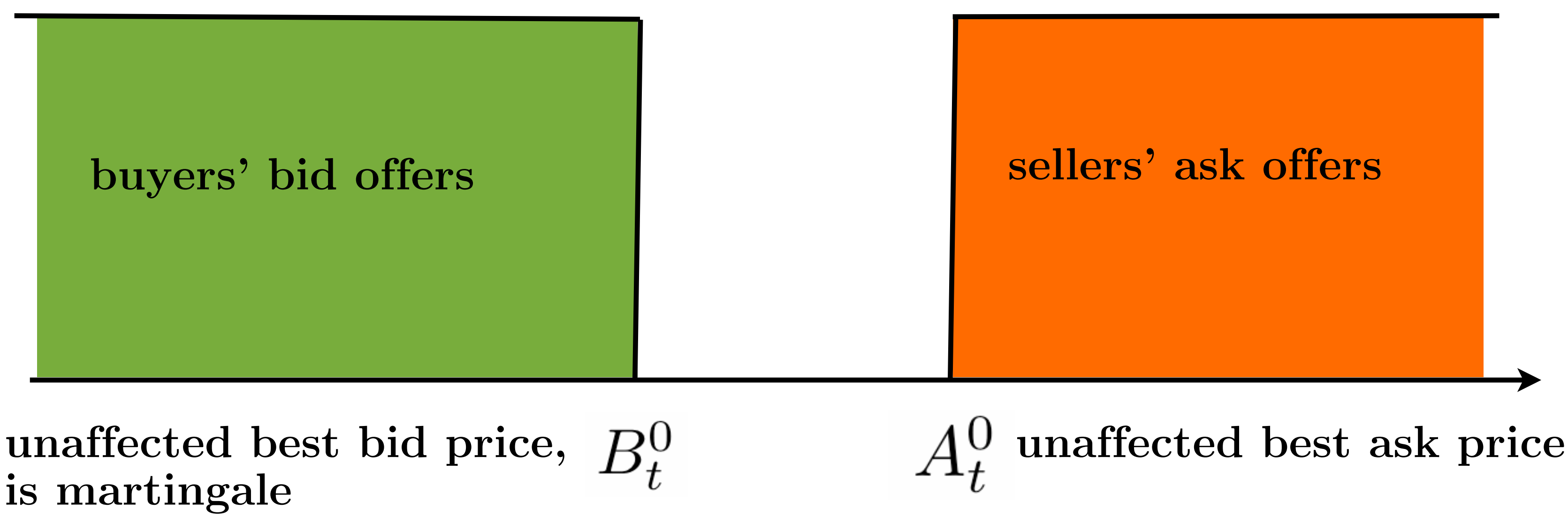
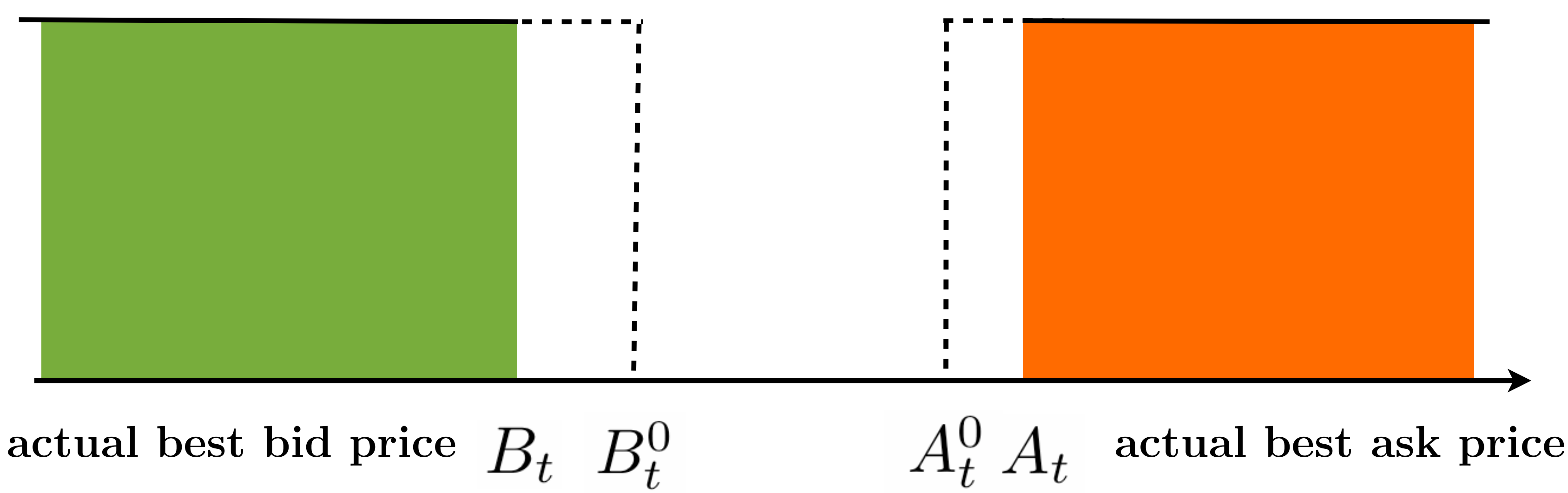
Limit order book model without large trader

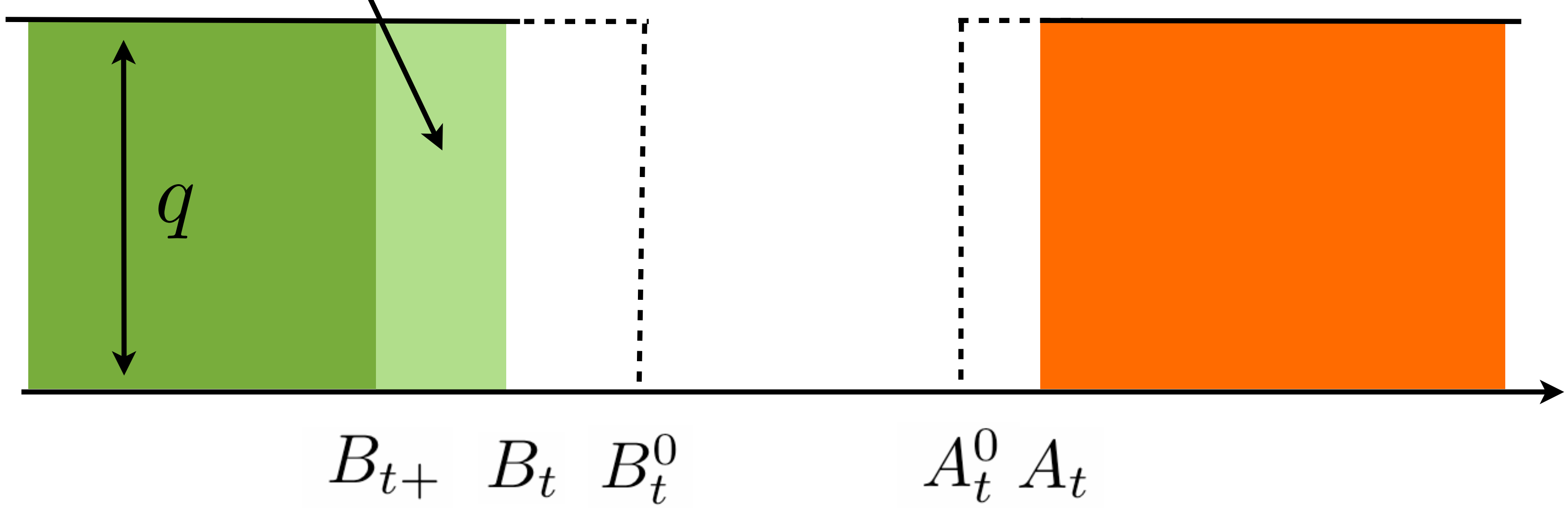


Limit order book model after large trades



Limit order book model at large trade

$$\xi_t = q(B_{t+} - B_t)$$

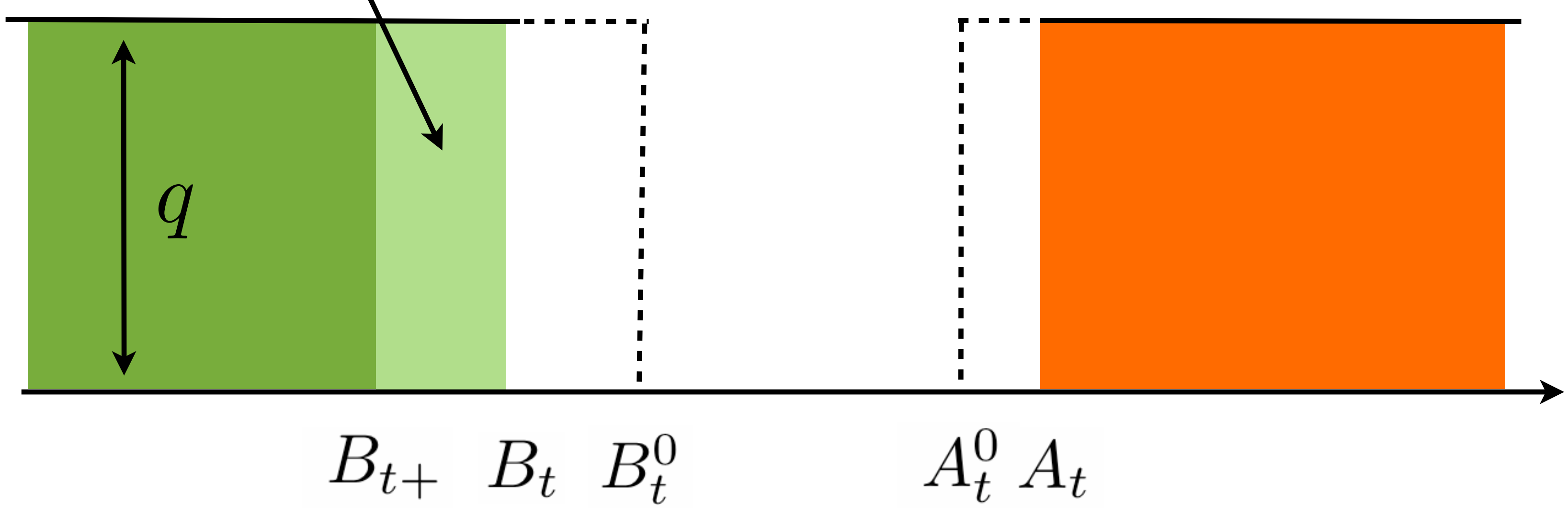


Limit order book model at large trade

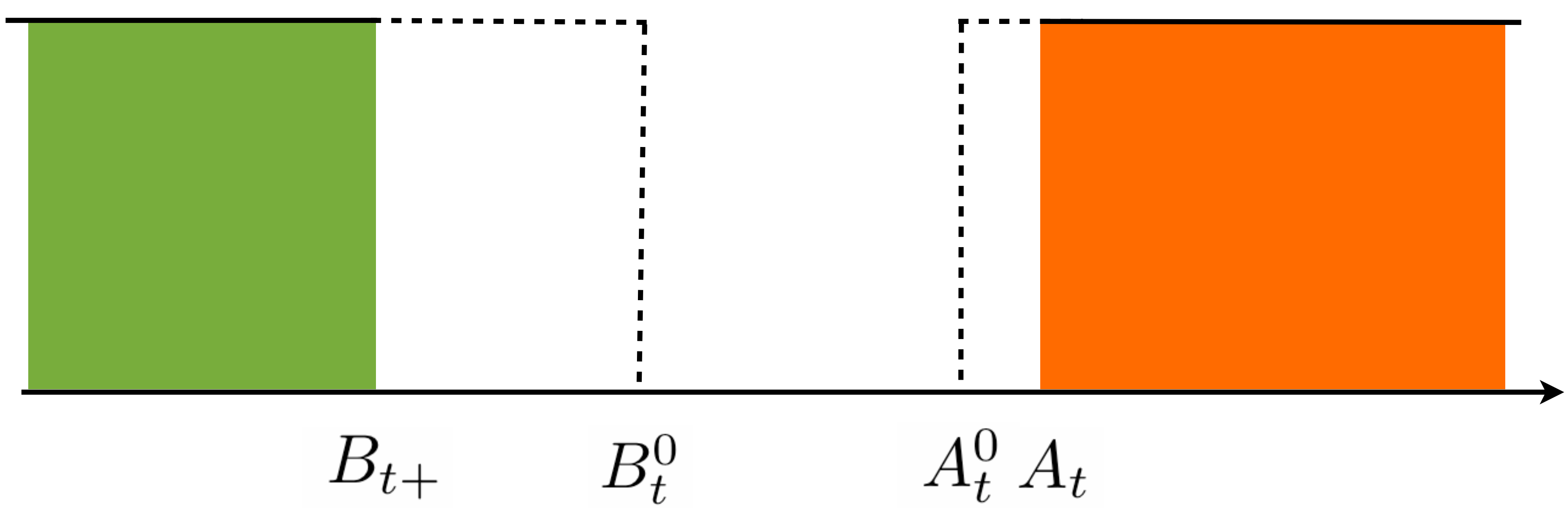
$$\xi_t = q(B_{t+} - B_t)$$

sell order executed at average price $\int_{B_{t+}}^{B_t} xq\,dx$

similarly for buy orders



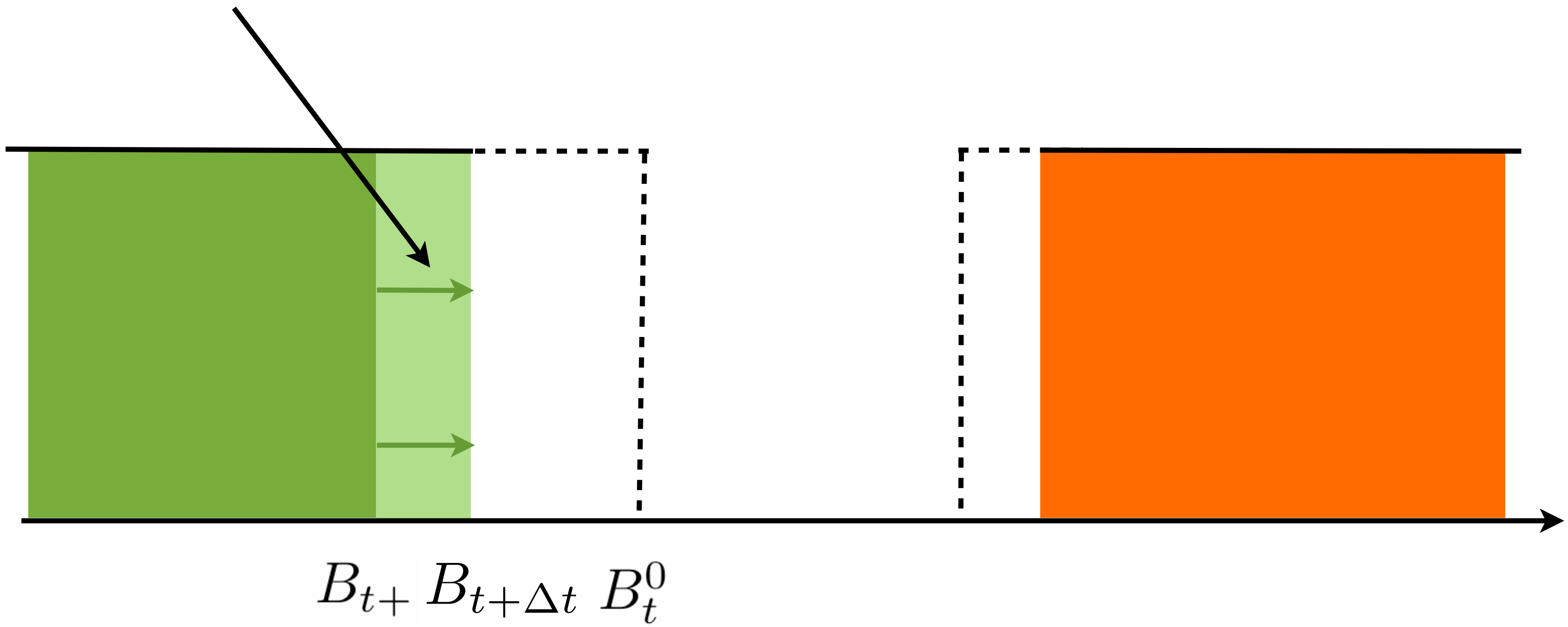
Limit order book model immediately after large trade



Resilience of the limit order book

$\psi : [0, \infty[\to [0, 1]$, $\psi(0) = 1$, decreasing

$\frac{\xi_t}{q} \cdot \psi(\Delta t) +$ decay of previous trades



1. Linear impact, general resilience

Strategy:

$N+1$ market orders: ξ_n shares placed at time t_n s.th.

a) $0 = t_0 \leq t_1 \leq \cdots \leq t_N = T$
(can also be stopping times)

b) ξ_n is $\mathcal{F}_{t_n}$-measurable and bounded from below,

c) we have $\displaystyle\sum_{n=0}^{N} \xi_n = X_0$

Sell order: $\xi_n < 0$

Buy order: $\xi_n > 0$

Actual best bid and ask prices

$$B_t = B_t^0 + \frac{1}{q} \sum_{\substack{t_n < t \\ \xi_n < 0}} \psi(t - t_n)\xi_n$$

$$A_t = A_t^0 + \frac{1}{q} \sum_{\substack{t_n < t \\ \xi_n > 0}} \psi(t - t_n)\xi_n$$

Cost per trade

$$c_n(\boldsymbol{\xi}) = \begin{cases} \displaystyle\int_{A_{t_n}}^{A_{t_n+}} yq\,dy = \frac{q}{2}(A_{t_n+}^2 - A_{t_n}^2) & \text{for buy order } \xi_n > 0 \\[2ex] \displaystyle\int_{B_{t_n}}^{B_{t_n+}} yq\,dy = \frac{q}{2}(B_{t_n+}^2 - B_{t_n}^2) & \text{for sell order } \xi_n < 0 \end{cases}$$

(positive for buy orders, negative for sell orders)

Expected execution costs

$$\mathcal{C}(\boldsymbol{\xi}) = E\Big[\sum_{n=0}^{N} c_n(\boldsymbol{\xi})\Big]$$

A simplified model

No bid-ask spread

S_t^0 = unaffected price, is (continuous) martingale.

$$S_t = S_t^0 + \frac{1}{q}\sum_{t_n<t} \xi_n \psi(t-t_n).$$

Trade ξ_n moves price from S_{t_n} to

$$S_{t_n+} = S_{t_n} + \frac{1}{q}\xi_n.$$

Resulting cost:

$$\overline{c}_n(\boldsymbol{\xi}) := \int_{S_{t_n}}^{S_{t_n+}} yq\,dy = \frac{q}{2}\left[S_{t_n+}^2 - S_{t_n}^2\right] = \frac{1}{2q}\xi_n^2 + \xi_n S_{t_n}$$

(typically positive for buy orders, negative for sell orders)

Lemma 1. *Suppose that $S^0 = A^0$. Then, for any strategy $\boldsymbol{\xi}$,*

$$\overline{c}_n(\boldsymbol{\xi}) \leq c_n(\boldsymbol{\xi}) \qquad \textit{with equality if } \xi_k \geq 0 \textit{ for all } k.$$

Thus: Enough to study the simplified model (as long as all trades ξ_n are positive)

Lemma 1. *Suppose that* $S^0 = A^0$. *Then, for any strategy* $\boldsymbol{\xi}$,

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Thus: Enough to study the simplified model (as long as all trades ξ_n are positive)

Lemma 2. *In the simplified model, the expected execution costs of a strategy* $\boldsymbol{\xi}$ *are*

$$\overline{\mathcal{C}}(\boldsymbol{\xi}) = E\Big[\sum_{n=0}^{N} \overline{c}_n(\boldsymbol{\xi})\Big] = \frac{1}{2q} E\big[\, C_{\boldsymbol{t}}^{\psi}(\boldsymbol{\xi})\,\big] + X_0 S_0^0,$$

where $C_{\boldsymbol{t}}^{\psi}$ *is the quadratic form*

$$C_{\boldsymbol{t}}^{\psi}(\boldsymbol{x}) = \sum_{m,n=0}^{N} x_n x_m \psi(|t_n - t_m|), \qquad \boldsymbol{x} \in \mathbb{R}^{N+1}, \boldsymbol{t} = (t_0, \dots, t_N).$$

First Question:
What are the conditions on ψ under which the (simplified) model is viable?

Requiring the absence of arbitrage opportunities in the usual sense is not strong enough, as examples will show.

Second Question:
Which strategies minimize the expected cost for given X_0?

This is the optimal execution problem. It is very closely related to the question of model viability.

The usual concept of viability from Hubermann & Stanzl (2004):

Definition

A round trip is a strategy $\boldsymbol{\xi}$ with

$$\sum_{n=0}^{N} \xi_n = X_0 = 0.$$

A market impact model admits

price manipulation strategies

if there is a round trip with negative expected execution costs.

In the simplified model, the expected costs of a strategy $\boldsymbol{\xi}$ are

$$\overline{\mathcal{C}}(\boldsymbol{\xi}) = \frac{1}{2q} E\big[\, C_{\boldsymbol{t}}^{\psi}(\boldsymbol{\xi})\,\big] + X_0 S_0^0,$$

where

$$C_{\boldsymbol{t}}^{\psi}(\boldsymbol{x}) = \sum_{m,n=0}^{N} x_n x_m \psi(|t_n - t_m|), \qquad \boldsymbol{x} \in \mathbb{R}^{N+1},\ \boldsymbol{t} = (t_0, \dots, t_N).$$

- There are no price manipulation strategies when $C_{\boldsymbol{t}}^{\psi}$ is nonnegative definite for all $\boldsymbol{t} = (t_0, \dots, t_N)$;
- when the minimizer $\boldsymbol{x}^*$ of $C_{\boldsymbol{t}}^{\psi}(\boldsymbol{x})$ with $\sum_i x_i = X_0$ exists, it yields the optimal strategy in the simplified model; in particular, the optimal strategy is then deterministic;
- when the minimizer $\boldsymbol{x}^*$ has only nonnegative components, it yields the optimal strategy in the order book model.

Bochner's theorem (1932):
$C^{\psi}_{\boldsymbol{t}}$ is always nonnegative definite (ψ is "positive definite") if and only if $\psi(|\cdot|)$ is the Fourier transform of a positive Borel measure μ on $\mathbb{R}$.

$C^{\psi}_{\boldsymbol{t}}$ is even strictly positive definite (ψ is "strictly positive definite") when the support of μ is not discrete.

Bochner's theorem (1932):
$C_{\boldsymbol{t}}^{\psi}$ *is always nonnegative definite (ψ is "positive definite") if and only if $\psi(|\cdot|)$ is the Fourier transform of a positive Borel measure μ on $\mathbb{R}$.*

$C_{\boldsymbol{t}}^{\psi}$ *is even strictly positive definite (ψ is "strictly positive definite") when the support of μ is not discrete.*

- Seems to completely settle the question of model viability;
- for strictly positive definite ψ, the optimal strategy is

$$\boldsymbol{\xi}^* = \boldsymbol{x}^* = \frac{X_0}{\mathbf{1}^\top M^{-1}\mathbf{1}} M^{-1}\mathbf{1} \qquad \text{for } M_{ij} = \psi(|t_i - t_j|).$$

Examples

Example 1: Exponential resilience
[Obizhaeva & Wang (2005), Alfonsi, Fruth, S. (2008)]

For the exponential resilience function

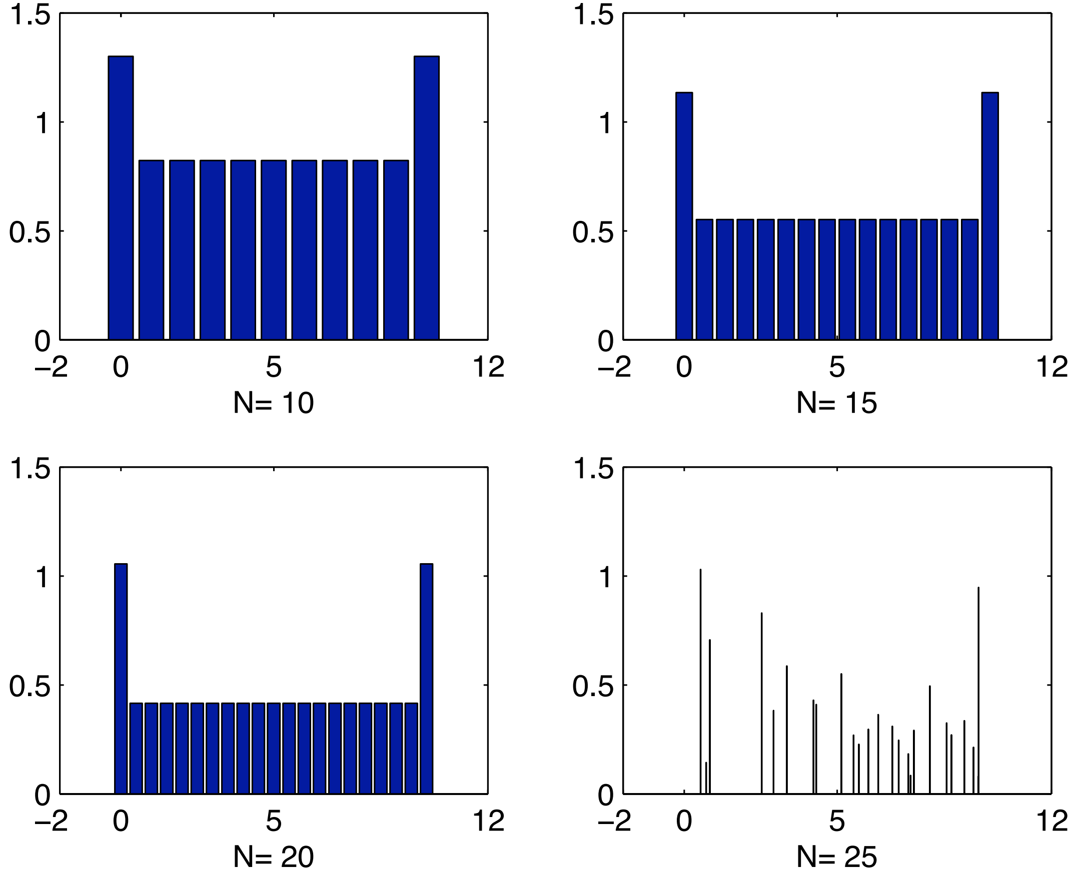
$$\psi(t) = e^{-\rho t},$$

$\psi(|\cdot|)$ is the Fourier transform of the positive measure

$$\mu(dt) = \frac{1}{\pi}\frac{\rho}{\rho^2 + t^2}\,dt$$

Hence, ψ is strictly positive definite.

Optimal strategies for exponential resilience $\psi(t) = e^{-\rho t}$



The optimal strategy can in fact be computed explicitly for any time grid [**Alfonsi, Fruth, A.S. (2008)**]:
Letting

$$\lambda_0 = \frac{X_0}{\mathbf{1}^\top M^{-1}\mathbf{1}} = \frac{X_0}{\frac{2}{1+a_1} + \sum_{n=2}^N \frac{1-a_n}{1+a_n}},$$

the initial market order of the optimal strategy is

$$x_0^* = \frac{\lambda_0}{1+a_1},$$

the intermediate market orders are given by

$$x_n^* = \lambda_0\Big(\frac{1}{1+a_n} - \frac{a_{n+1}}{1+a_{n+1}}\Big), \qquad n = 1, \dots, N-1,$$

and the final market order is

$$x_N^* = \frac{\lambda_0}{1+a_N}.$$

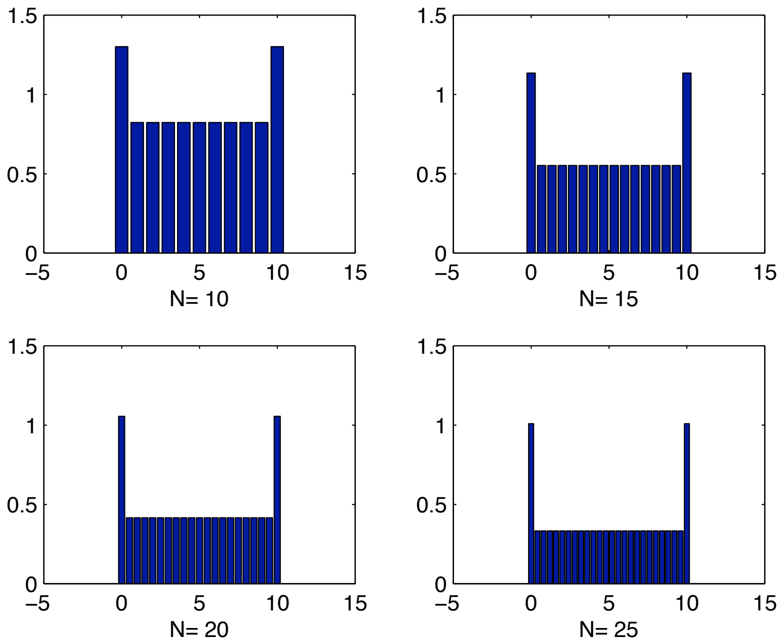
all components of $\boldsymbol{x}^*$ are strictly positive

For the equidistant time grid $t_n = nT/N$ the solution simplifies:

$$x_0^* = x_N^* = \frac{X_0}{(N-1)(1-a)+2}$$

and

$$x_1^* = \cdots = x_{N-1}^* = \xi_0^*(1-a).$$



The symmetry of the optimal strategy is a general fact:

Proposition 3. *Suppose that ψ is strictly positive definite and that the time grid is symmetric, i.e.,*

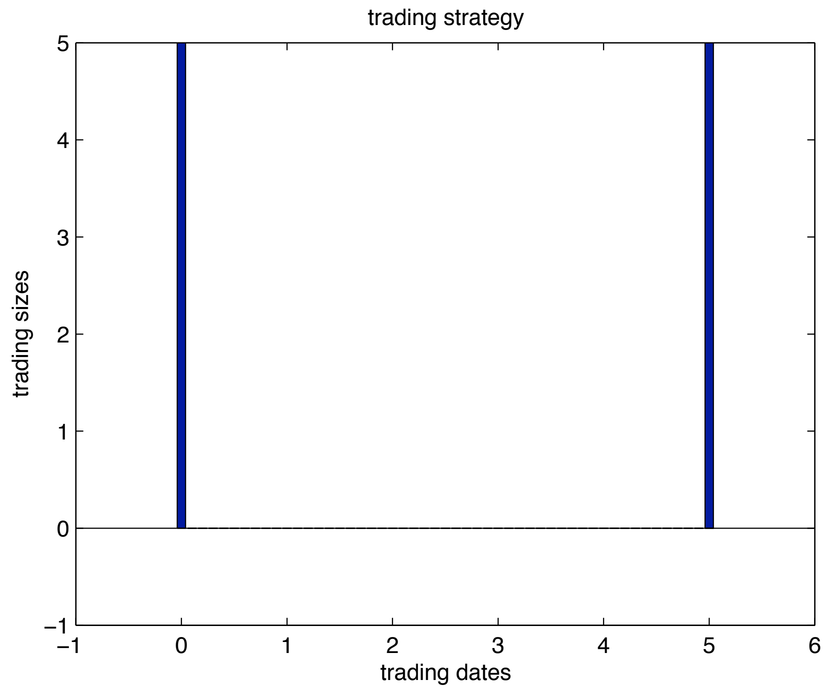
$$t_i = t_N - t_{N-i} \qquad \textit{for all } i,$$

then the optimal strategy is reversible, i.e.,

$$x^*_{t_i} = x^*_{t_{N-i}} \qquad \textit{for all } i.$$

Example 2: Linear resilience $\psi(t) = 1 - \rho t$ for some $\rho \leq 1/T$

The optimal strategy is always of this form:



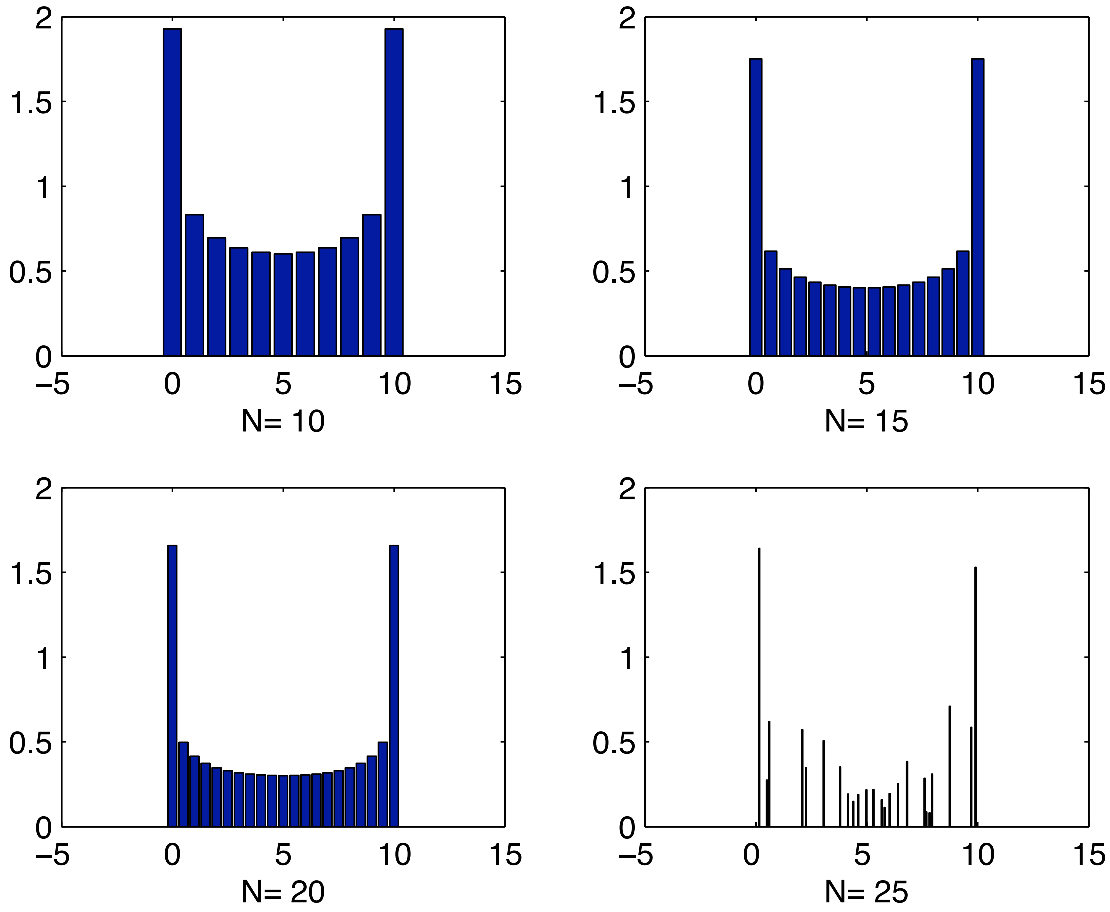
It is independent of the underlying time grid and consists of two symmetric trades of size $X_0/2$ at $t = 0$ and $t = T$, all other trades are zero.

More generally: Convex resilience

Theorem 4.
[Carathéodory (1907), Toeplitz (1911), Young (1912)]

ψ is convex, decreasing, nonnegative, and nonconstant $\Longrightarrow$
$\psi(|\cdot|)$ is strictly positive definite.

Example 3: Power law resilience $\psi(t) = (1 + \beta t)^{-\alpha}$



Example 4: Trigonometric resilience

The function

$$\cos \rho x$$

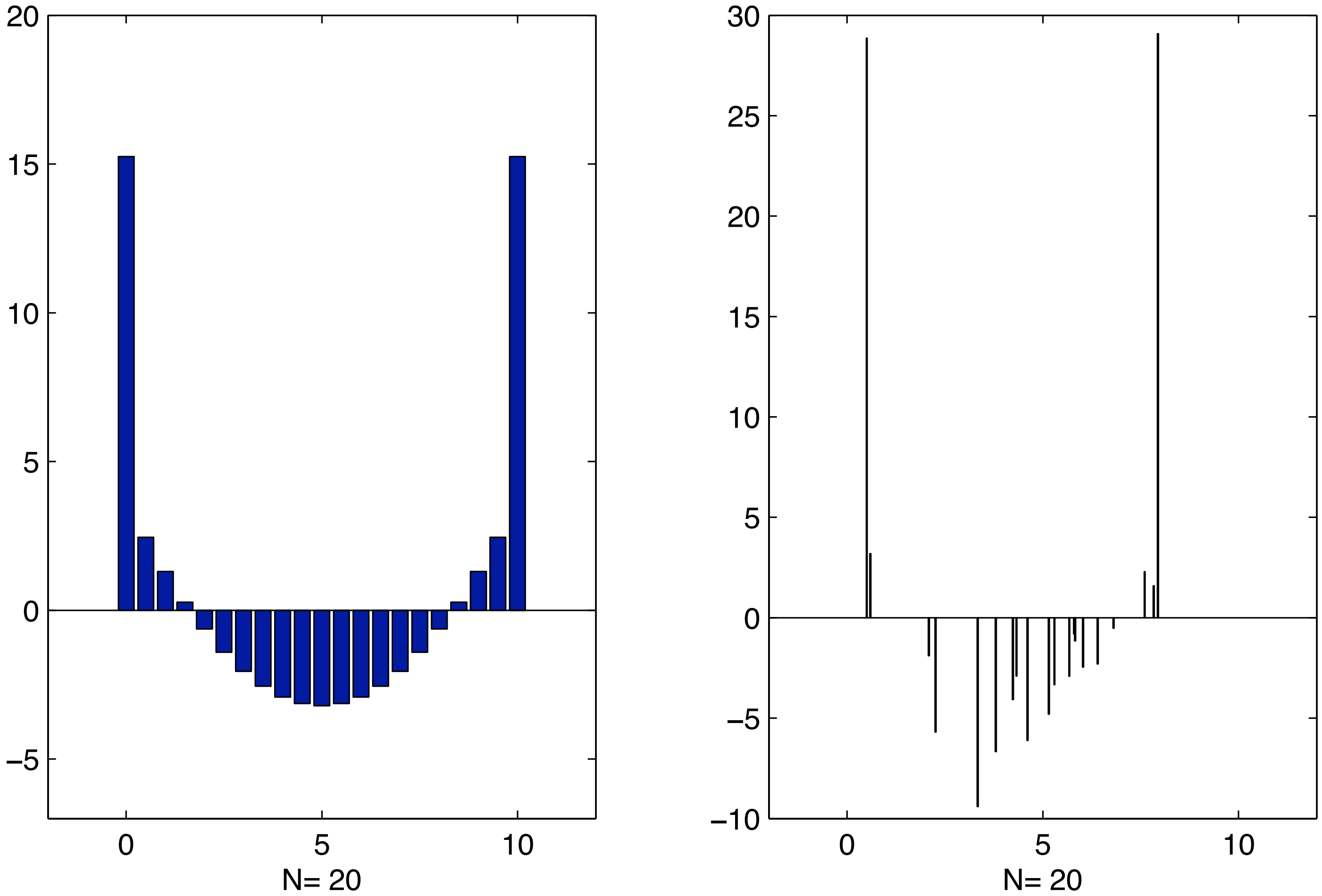
is the Fourier transform of the positive finite measure

$$\mu = \frac{1}{2}(\delta_{-\rho} + \delta_{\rho})$$

Since it is not strictly positive definite, we take

$$\psi(t) = (1-\varepsilon)\cos \rho t + \varepsilon e^{-t} \qquad \text{for some } \rho \leq \frac{\pi}{2T}.$$

Trigonometric resilience $\psi(t) = 0.999\cos(t\pi/2T) + 0.001e^{-t}$



Example 5: Gaussian resilience

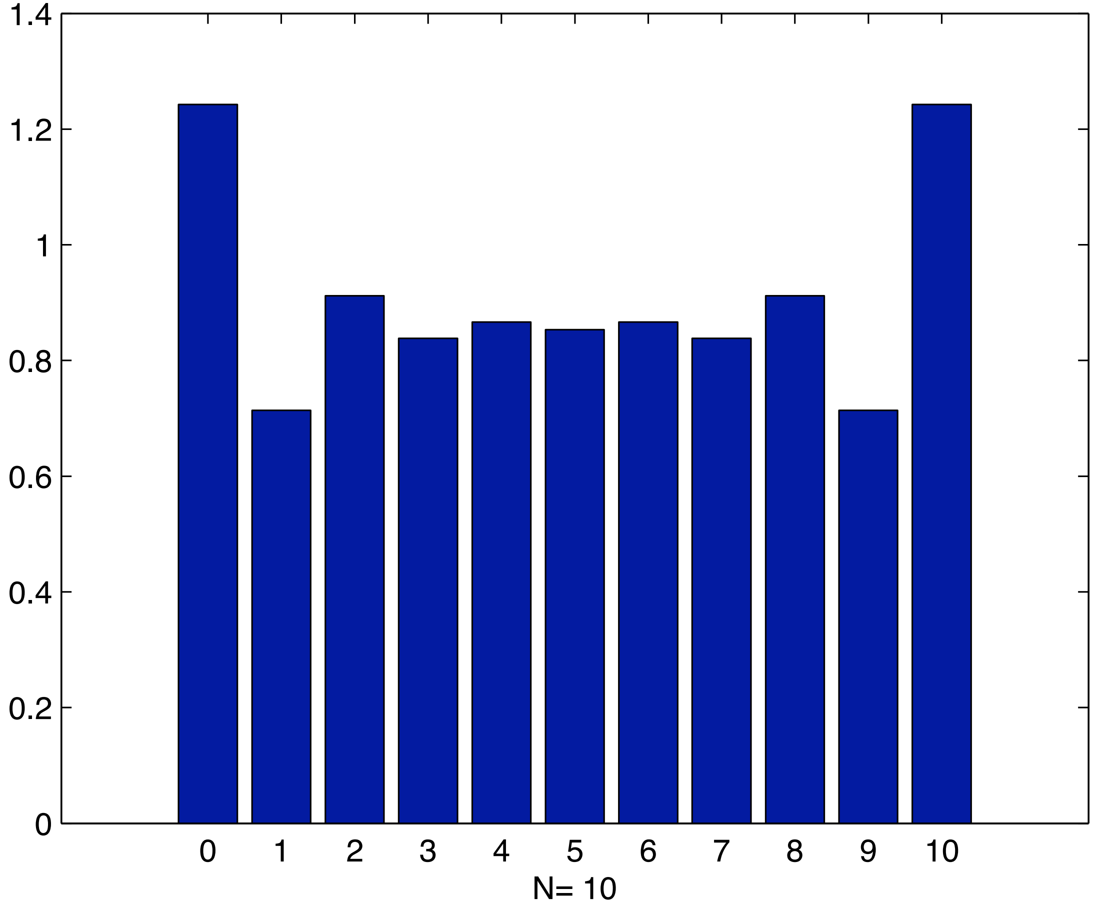
The Gaussian resilience function

$$\psi(t) = e^{-t^2}$$

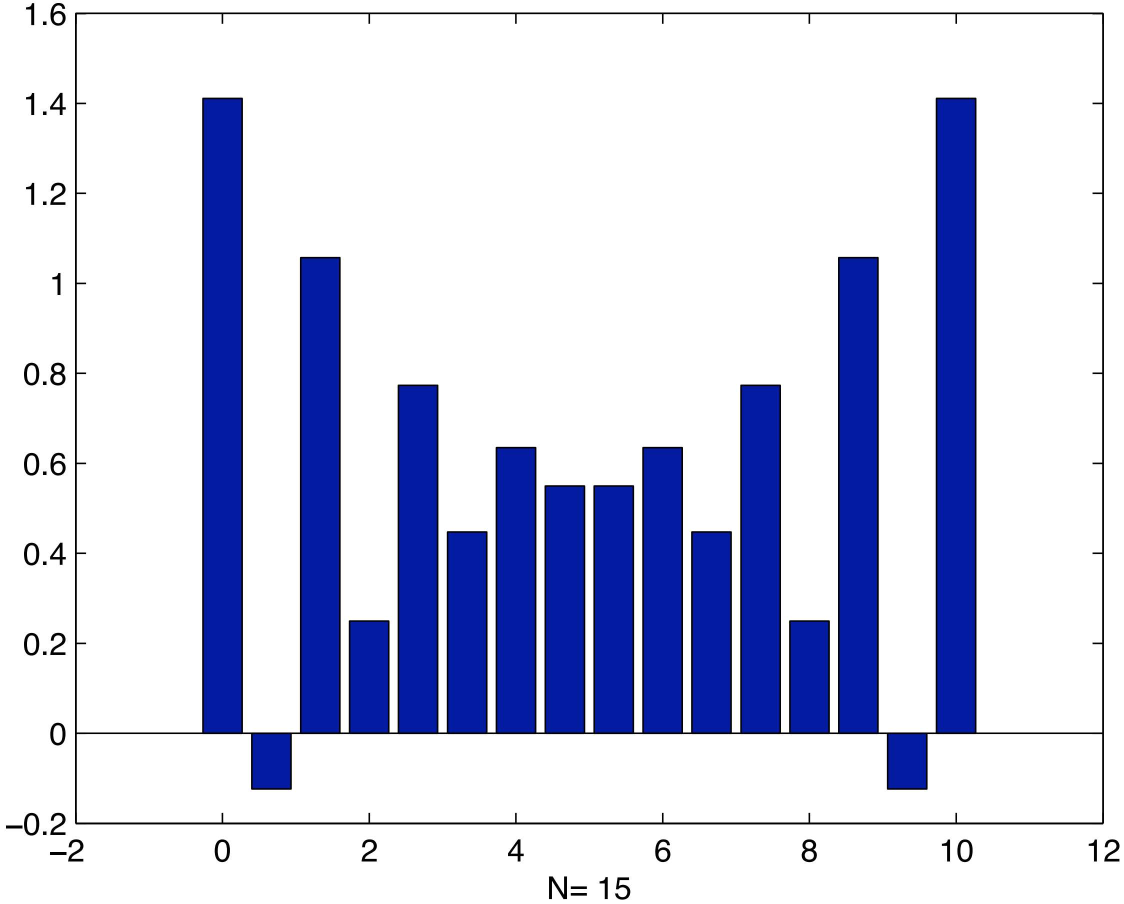
is its own Fourier transform (modulo constants). The corresponding quadratic form is hence positive definite.

Nevertheless.....

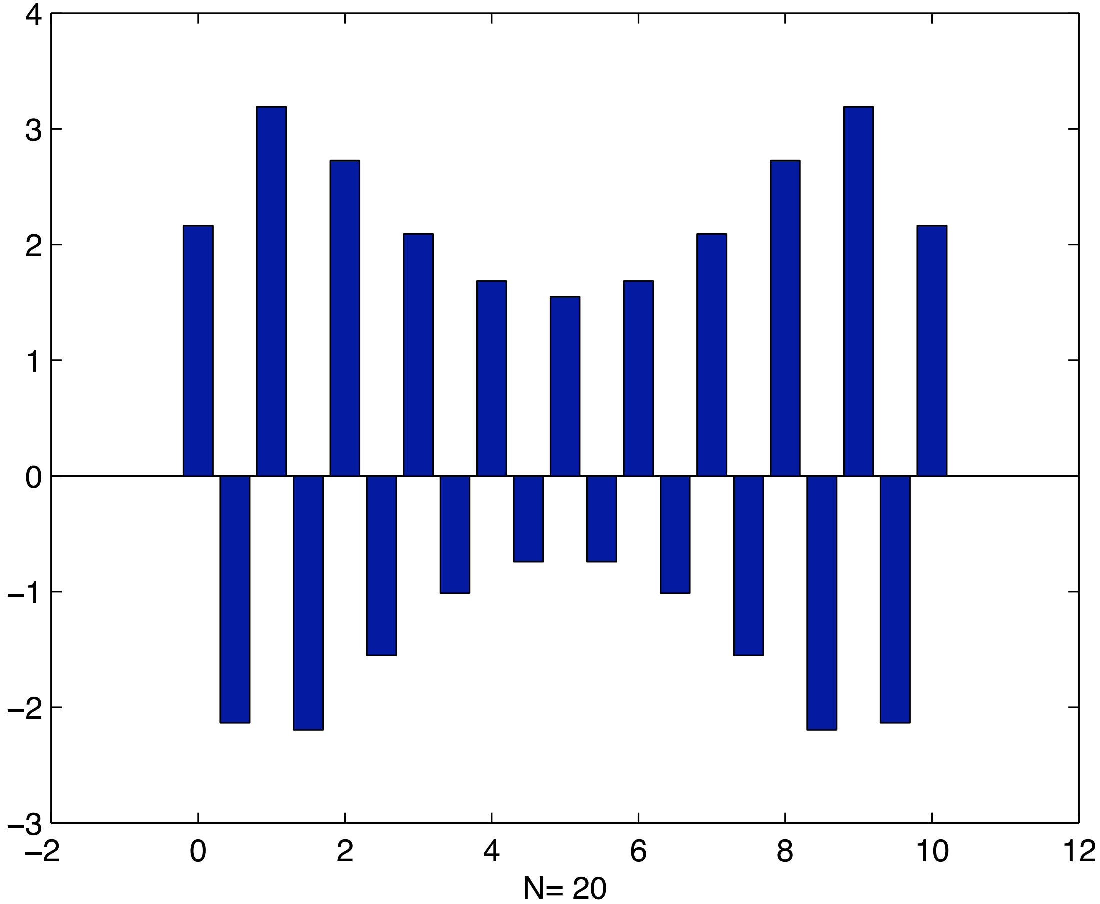
Gaussian resilience $\psi(t) = e^{-t^2}$, $N = 10$



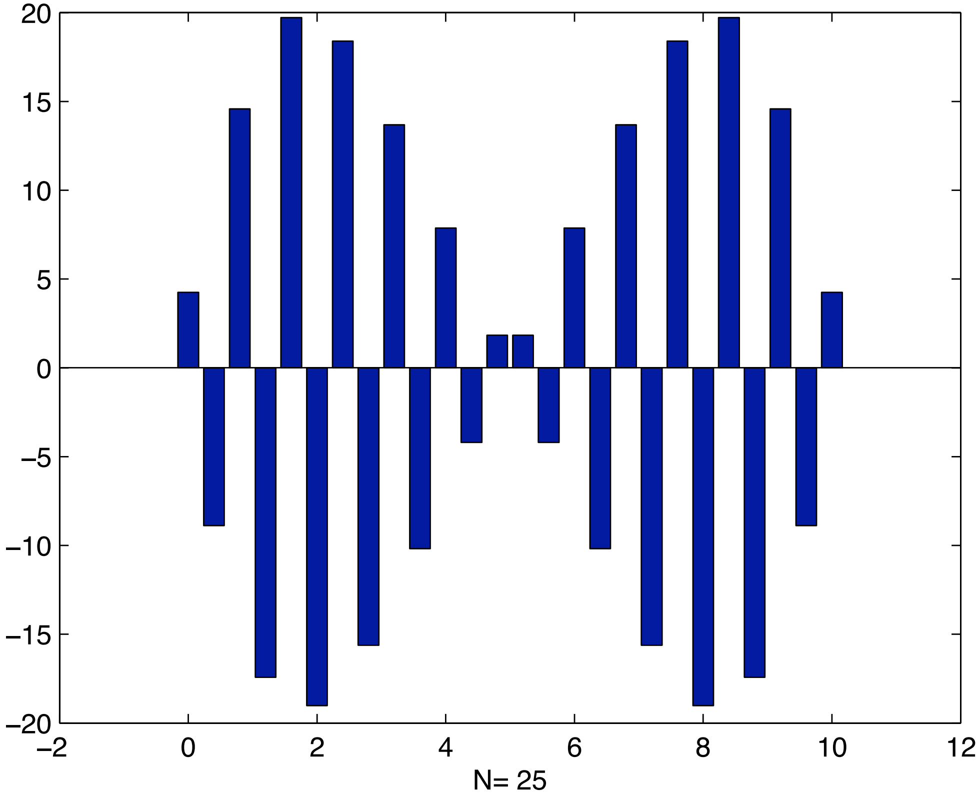
Gaussian resilience $\psi(t) = e^{-t^2}$, $N = 15$



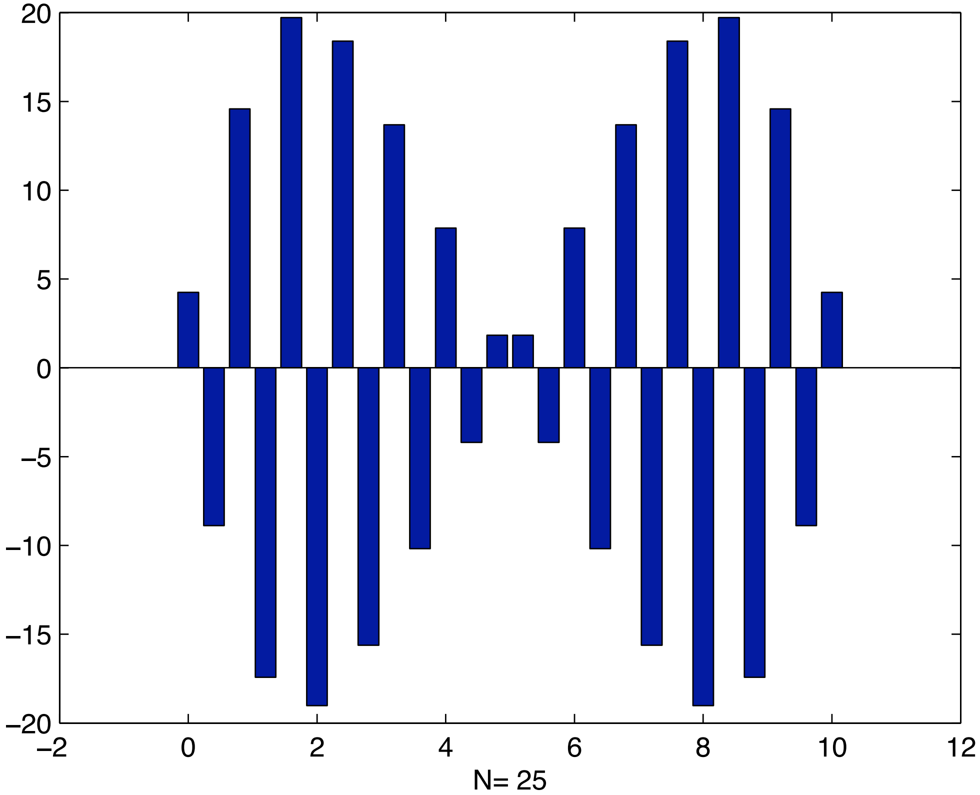
Gaussian resilience $\psi(t) = e^{-t^2}$, $N = 20$



Gaussian resilience $\psi(t) = e^{-t^2}$, $N = 25$



Gaussian resilience $\psi(t) = e^{-t^2}$**,** $N = 25$



$\Rightarrow$ **absence of price manipulation strategies is not enough**

Definition [Hubermann & Stanzl (2004)]
A market impact model admits

price manipulation strategies in the strong sense

if there is a round trip with negative expected liquidation costs.

Definition:
A market impact model admits

price manipulation strategies in the weak sense

if the expected liquidation costs of a sell (buy) program can be decreased by intermediate buy (sell) trades.

Question: When does the minimizer $\boldsymbol{x}^*$ of

$$\sum_{i,j} x_i x_j \psi(|t_i - t_j|) \qquad \text{with} \quad \sum_i x_i = X_0$$

have only nonnegative components?

Related to the **positive portfolio problem** in finance:

When are there no short sales in a Markowitz portfolio?

I.e. when is the solution of the following problem nonnegative

$$\boldsymbol{x}^\top M \boldsymbol{x} - \boldsymbol{m}^\top \boldsymbol{x} \to \min \qquad \text{for } \boldsymbol{x}^\top \mathbf{1} = X_0,$$

where M is a covariance matrix of assets and $\boldsymbol{m}$ is the vector of returns?

Partial results, e.g., by Gale (1960), Green (1986), Nielsen (1987)

Proposition 5. **[Alfonsi, A.S., Slynko (2009)]**
When ψ is strictly positive definite and trading times are equidistant, then

$$x_0^* > 0 \qquad \textit{and} \qquad x_N^* > 0.$$

Proof relies on Trench algorithm for inverting the Toeplitz matrix

$$M_{ij} = \psi(|i-j|/N), \qquad i,j = 0,\dots,N$$

$\square$

Theorem 6. [Alfonsi, A.S., Slynko (2009)]

- *If* ψ *is convex then all components of* $\boldsymbol{x}^*$ *are nonnegative.*

- *If* ψ *is strictly convex, then all components are strictly positive.*

- *Conversely,* $\boldsymbol{x}^*$ *has negative components as soon as, e.g.,* ψ *is strictly concave in a neighborhood of* 0.

Qualitative properties of optimal strategies?

Qualitative properties of optimal strategies?

Proposition 8. [Alfonsi, A.S., Slynko (2009)]
When ψ is convex and nonconstant, the optimal $\boldsymbol{x}^*$ satisfies

$$x_0^* \geq x_1^* \qquad \textit{and} \qquad x_{N-1}^* \leq x_N^*$$

Proof: Equating the first and second equations in $M\boldsymbol{x}^* = \lambda_0 \mathbf{1}$ gives

$$\sum_{j=0}^{N} x_j^* \psi(t_j) = \sum_{j=0}^{N} x_j^* \psi(|t_j - t_1|).$$

Thus,

$$\begin{aligned} x_0^* - x_1^* &= \sum_{j=0,\ j\neq 1}^{N} x_j^* \psi(|t_j - t_1|) - \sum_{j=1}^{N} x_j^* \psi(t_j) \\ &= x_0^* \psi(t_1) - x_1^* \psi(t_1) + \sum_{j=2}^{N} x_j^* \left[\psi(t_j - t_1) - \psi(t_j)\right] \\ &\geq (x_0^* - x_1^*)\psi(t_1), \end{aligned}$$

by convexity of ψ. Therefore

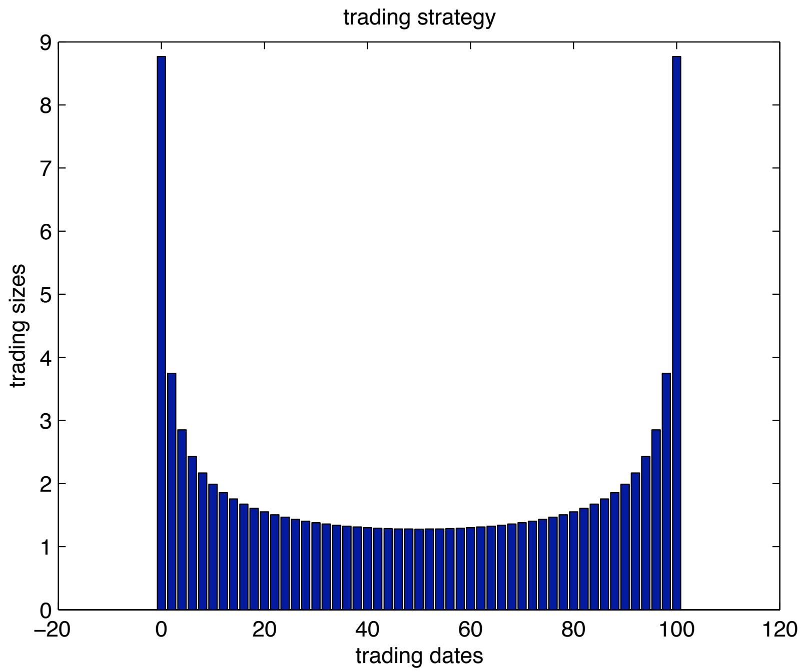
$$(x_0 - x_1)(1 - \psi(t_1)) \geq 0$$

$\square$

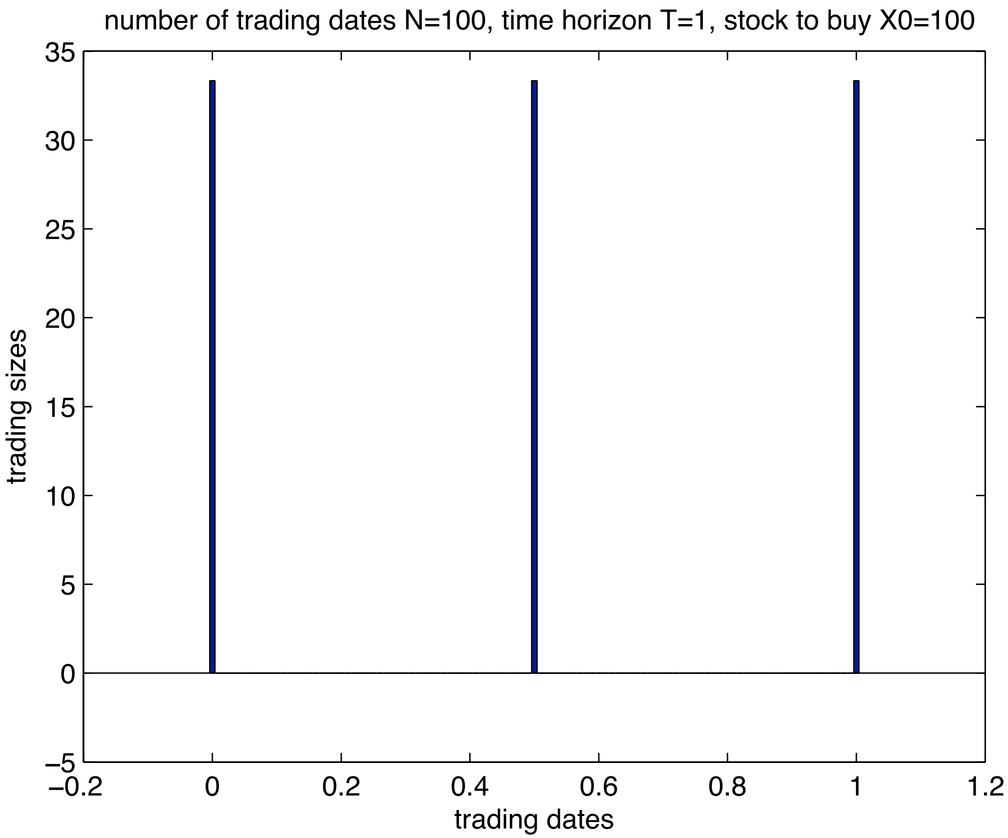
Proposition 8. [Alfonsi, A.S., Slynko (2009)] *When ψ is convex and nonconstant, the optimal $\boldsymbol{x}^*$ satisfies*

$$x_0^* \geq x_1^* \qquad \textit{and} \qquad x_{N-1}^* \leq x_N^*$$

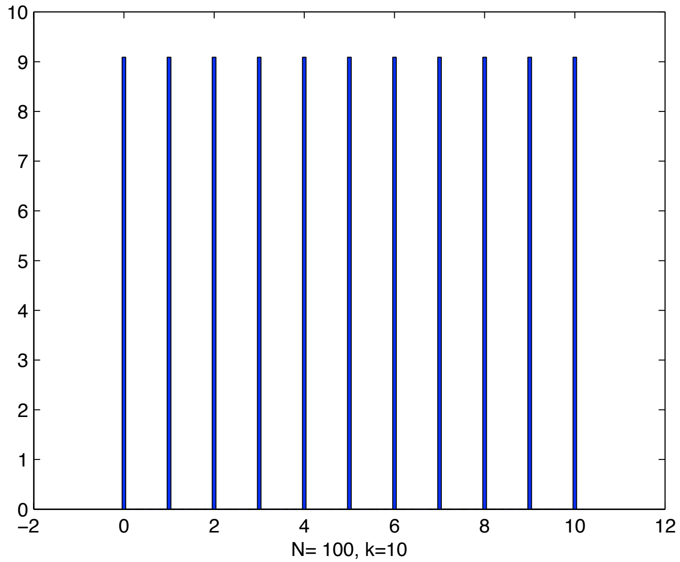
What about other trades? General pattern?



No! Capped linear resilience $\psi(t) = (1 - \rho t)^+, \rho = 2/T$

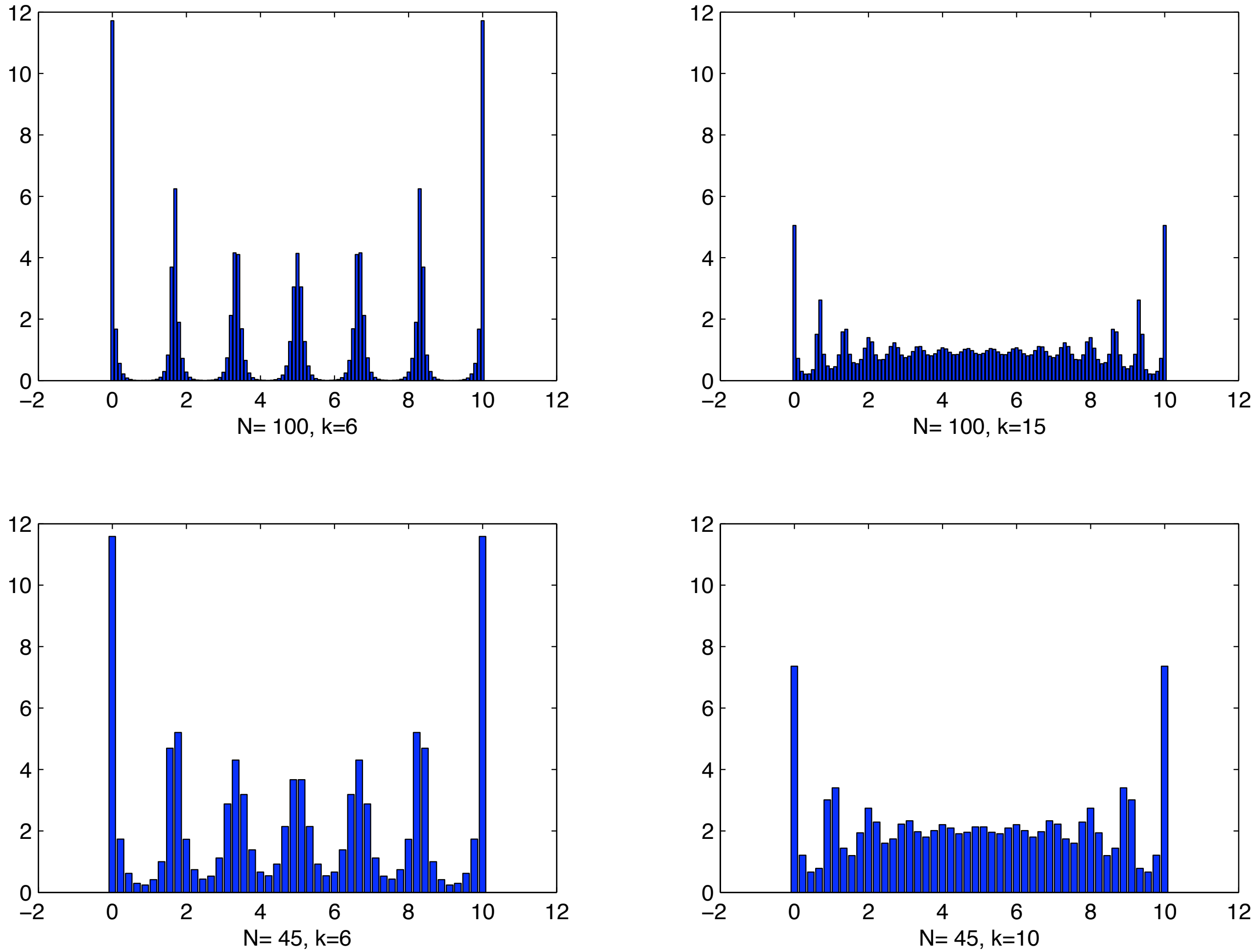


Proposition 9. [Alfonsi, A.S., Slynko (2009)]
Suppose that $\psi(t) = (1 - kt/T)^+$ *and that* k *divides* N. *Then the optimal strategy consists of* $k + 1$ *equal equidistant trades.*



Proof relies on Trench algorithm $\square$

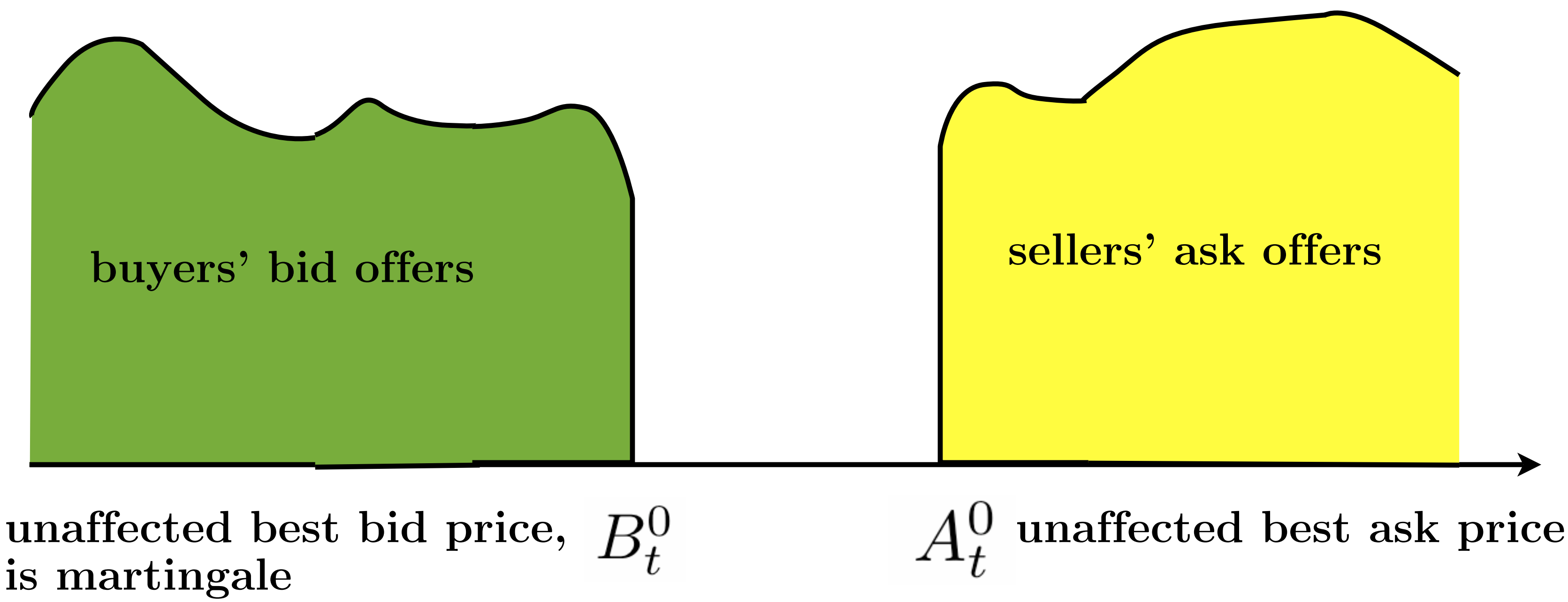
When k does not divide N, the situation becomes more complicated:



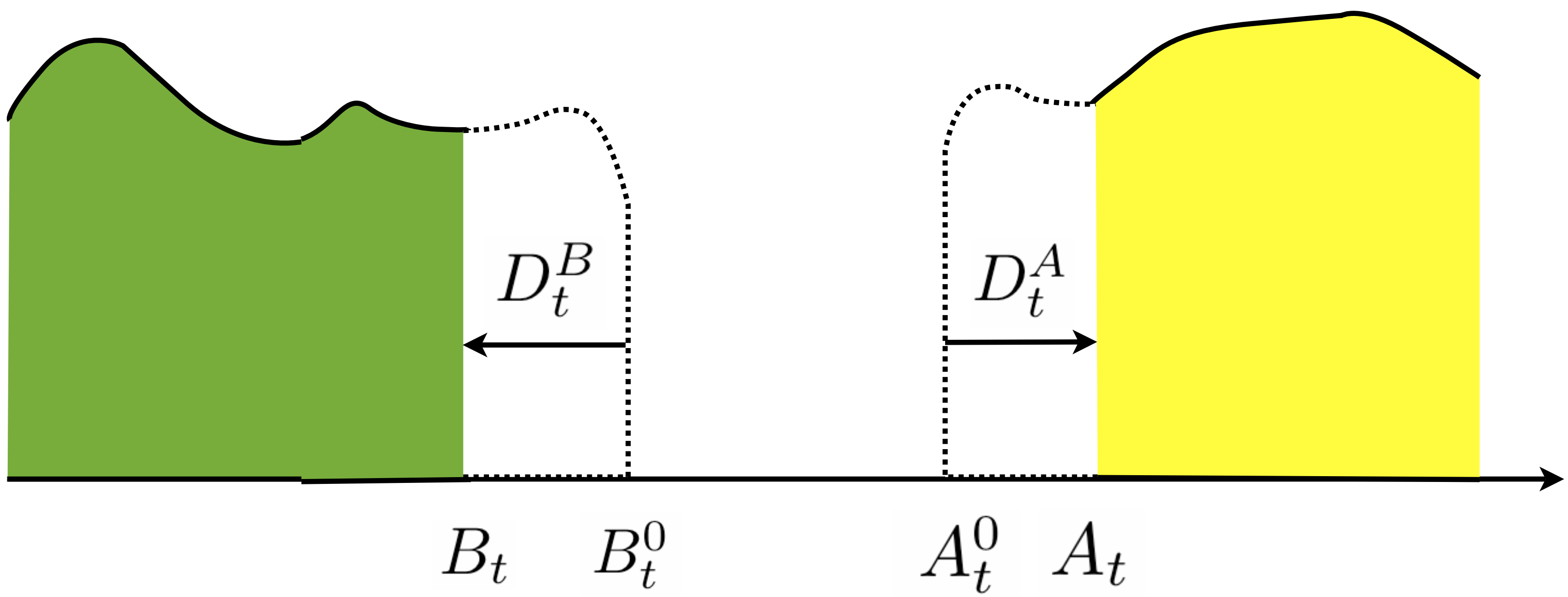
1. **Linear impact, general resilience**

2. **Nonlinear impact, exponential resilience**

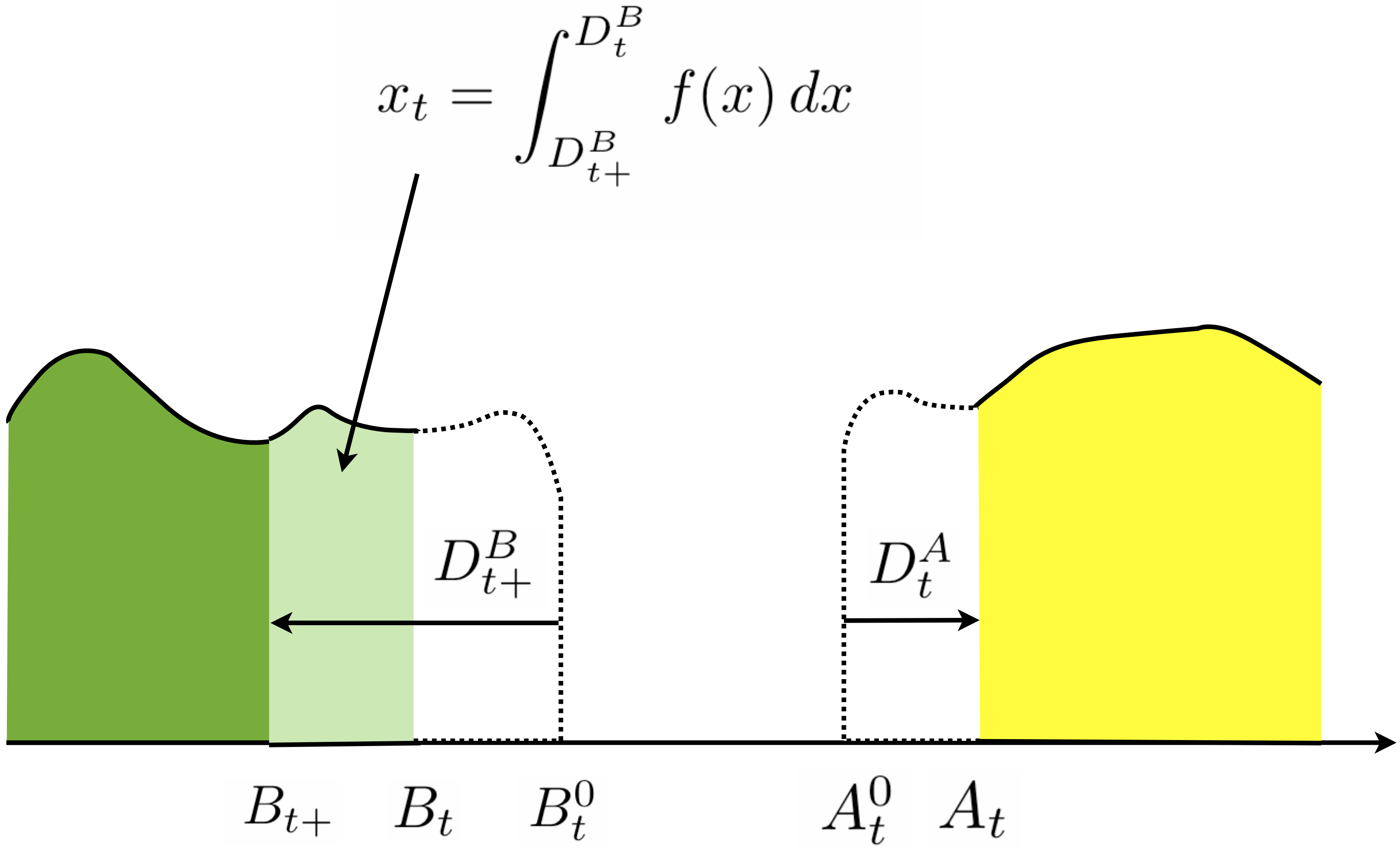
Limit order book model without large trader



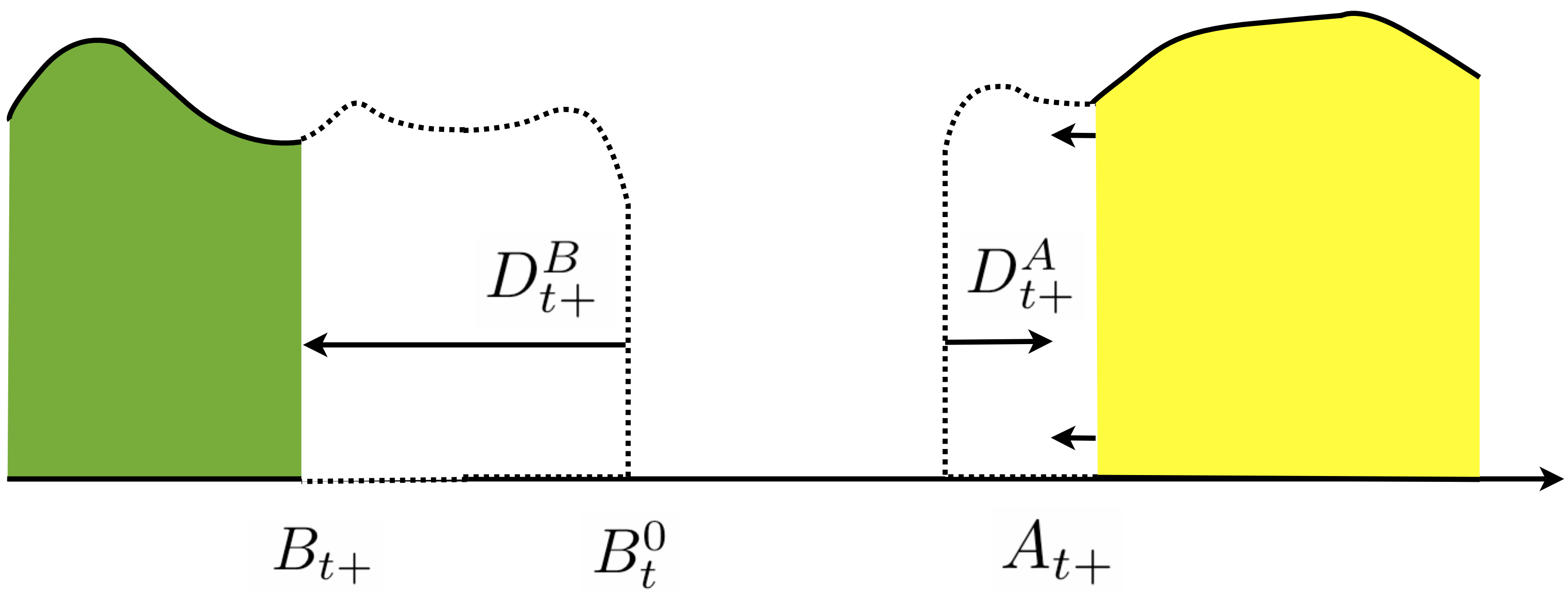
Limit order book model after large trades



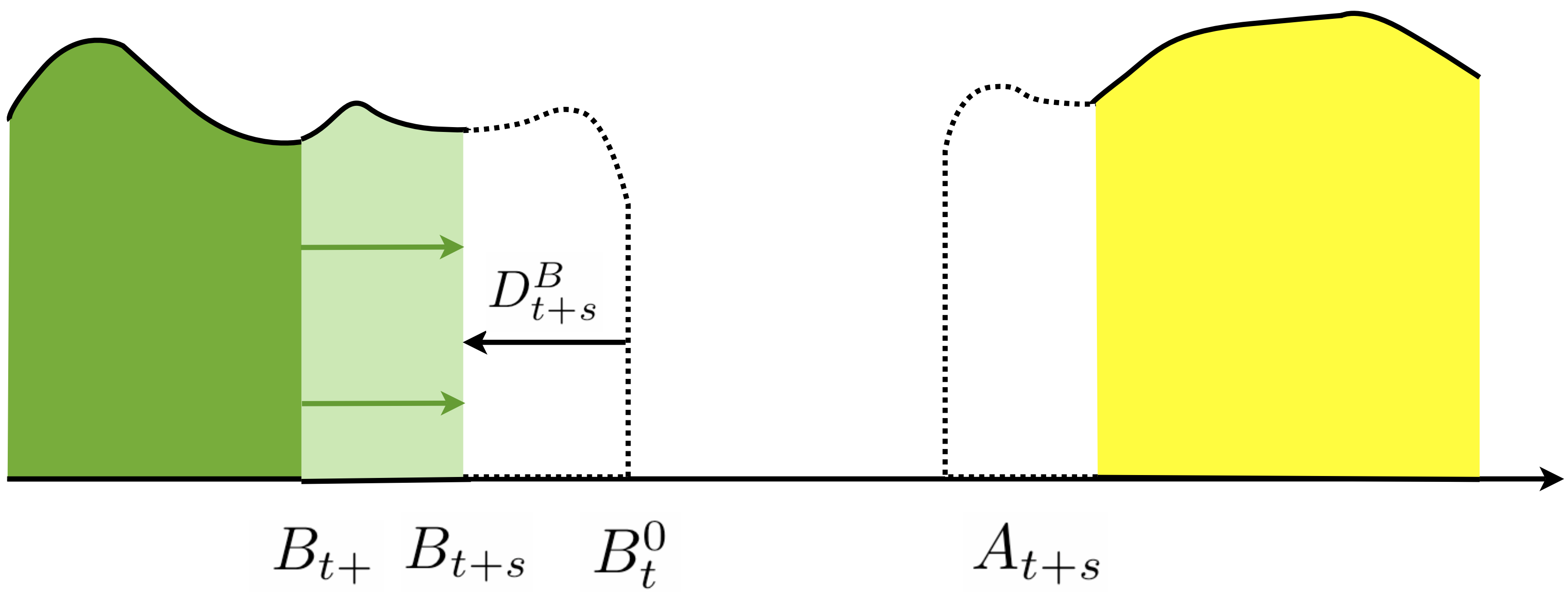
Limit order book model at large trade



Limit order book model immediately after large trade



Limit order book model with resilience



$f(x)$ = shape function = densities of bids for $x < 0$, asks for $x > 0$

B_t^0 = 'unaffected' bid price at time t, is martingale

B_t = bid price after market orders before time t

$D_t^B = B_t - B_t^0$

If sell order of $\xi_t \leq 0$ shares is placed at time t:

D_t^B changes to D_{t+}^B, where

$$\int_{D_t^B}^{D_{t+}^B} f(x)dx = \xi_t$$

and

$$B_{t+} := B_t + D_{t+}^B - D_t^B = B_t^0 + D_{t+}^B,$$

$\Longrightarrow$ nonlinear price impact

A_t^0 = 'unaffected' ask price at time t, satisfies $B_t^0 \leq A_t^0$

A_t = bid price after market orders before time t

$D_t^A = A_t - A_t^0$

If buy order of $\xi_t \geq 0$ shares is placed at time t:

D_t^A changes to D_{t+}^A, where

$$\int_{D_t^A}^{D_{t+}^A} f(x)dx = \xi_t$$

and

$$A_{t+} := A_t + D_{t+}^A - D_t^A = A_t^0 + D_{t+}^A,$$

For simplicity, we assume that the LOB has infinite depth, i.e., $|F(x)| \to \infty$ as $|x| \to \infty$, where

$$F(x) := \int_0^x f(y)\, dy.$$

If the large investor is inactive during the time interval $[t, t+s[$, there are *two* possibilities:

- **Exponential recovery of the extra spread**

$$D_t^B = e^{-\int_s^t \rho_r \, dr} D_s^B \qquad \text{for } s < t.$$

- **Exponential recovery of the order book volume**

$$E_t^B = e^{-\int_s^t \rho_r \, dr} E_s^B \qquad \text{for } s < t,$$

where

$$E_t^B = \int_{D_t^B}^0 f(x) \, dx =: F(D_t^B).$$

In both cases: analogous dynamics for D^A or E^A

Strategy:

$N+1$ market orders: ξ_n shares placed at time τ_n s.th.

a) the (τ_n) are stopping times s.th. $0=\tau_0 \le \tau_1 \le \cdots \le \tau_N = T$

b) ξ_n is $\mathcal{F}_{\tau_n}$-measurable and bounded from below,

c) we have $\displaystyle\sum_{n=0}^{N} \xi_n = X_0$

Will write

$$(\boldsymbol{\tau}, \boldsymbol{\xi})$$

and optimize jointly over $\boldsymbol{\tau}$ and $\boldsymbol{\xi}$.

- When selling $\xi_n < 0$ shares, we sell $f(x)\,dx$ shares at price $B^0_{\tau_n} + x$ with x ranging from $D^B_{\tau_n}$ to $D^B_{\tau_n+} < D^B_{\tau_n}$, i.e., the costs are negative:

$$c_n(\boldsymbol{\tau}, \boldsymbol{\xi}) := \int_{D^B_{\tau_n}}^{D^B_{\tau_n+}} (B^0_{\tau_n} + x) f(x)\,dx = \xi_n B^0_{\tau_n} + \int_{D^B_{\tau_n}}^{D^B_{\tau_n+}} x f(x)\,dx$$

- When buying shares ($\xi_n > 0$), the costs are positive:

$$c_n(\boldsymbol{\tau}, \boldsymbol{\xi}) := \xi_n A^0_{\tau_n} + \int_{D^A_{\tau_n}}^{D^A_{\tau_n+}} x f(x)\,dx$$

- The expected costs for the strategy $(\boldsymbol{\tau}, \boldsymbol{\xi})$ are

$$\mathcal{C}(\boldsymbol{\tau}, \boldsymbol{\xi}) = \mathbb{E}\Big[\sum_{n=0}^{N} c_n(\boldsymbol{\tau}, \boldsymbol{\xi})\Big]$$

Instead of the τ_k, we will use

$$\alpha_k := \int_{\tau_{k-1}}^{\tau_k} \rho_s ds, \qquad k = 1, \ldots, N. \tag{1}$$

The condition $0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_N = T$ is equivalent to $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_N)$ belonging to

$$\mathcal{A} := \left\{ \boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}_+^N \,\middle|\, \sum_{k=1}^N \alpha_k = \int_0^T \rho_s \, ds \right\}.$$

A simplified model without bid-ask spread

S_t^0 = unaffected price, is (continuous) martingale.

$$S_{t_n} = S_{t_n}^0 + D_n$$

where D and E are defined as follows:

$$E_0 = D_0 = 0, \qquad E_n = F(D_n) \qquad \text{and} \qquad D_n = F^{-1}(E_n).$$

For $n = 0, \ldots, N$, regardless of the sign of ξ_n,

$$E_{n+} = E_n - \xi_n \qquad \text{and} \qquad D_{n+} = F^{-1}(E_{n+}) = F^{-1}\left(F\left(D_n\right) - \xi_n\right).$$

For $k = 0, \ldots, N-1$,

$$E_{k+1} = e^{-\alpha_{k+1}} E_{k+} = e^{-\alpha_{k+1}} (E_k - \xi_k)$$

The costs are

$$\overline{c}_n(\boldsymbol{\tau}, \boldsymbol{\xi}) = \xi_n S_{\tau_n}^0 + \int_{D_{\tau_n}}^{D_{\tau_n+}} x f(x)\, dx$$

Lemma 10. *Suppose that $S^0 = B^0$. Then, for any strategy $\boldsymbol{\xi}$,*

$$\overline{c}_n(\boldsymbol{\xi}) \leq c_n(\boldsymbol{\xi}) \qquad \textit{with equality if } \xi_k \geq 0 \textit{ for all } k.$$

Moreover,

$$\overline{\mathcal{C}}(\boldsymbol{\tau}, \boldsymbol{\xi}) := \mathbb{E}\Big[\sum_{n=0}^{N} \overline{c}_n(\boldsymbol{\tau}, \boldsymbol{\xi})\Big] = \mathbb{E}\Big[C(\boldsymbol{\alpha}, \boldsymbol{\xi})\Big] - X_0 S_0^0$$

where

$$C(\boldsymbol{\alpha}, \boldsymbol{\xi}) := \sum_{n=0}^{N} \int_{D_n}^{D_{n+}} x f(x)\, dx$$

is a deterministic function of $\boldsymbol{\alpha} \in \mathcal{A}$ and $\boldsymbol{\xi} \in \mathbb{R}^{N+1}$.

Implies that is is enough to minimize $C(\boldsymbol{\alpha}, \boldsymbol{\xi})$ over $\boldsymbol{\alpha} \in \mathcal{A}$ and

$$\boldsymbol{\xi} \in \Big\{\boldsymbol{x} = (x_0, \ldots, x_N) \in \mathbb{R}^{N+1} \,\Big|\, \sum_{n=0}^{N} x_n = X_0\Big\}.$$

Theorem 11. *Suppose f is increasing on $\mathbb{R}_-$ and decreasing on $\mathbb{R}_+$. Then there is a unique optimal strategy $(\boldsymbol{\xi}^*, \boldsymbol{\tau}^*)$ consisting of homogeneously spaced trading times,*

$$\int_{\tau_i^*}^{\tau_{i+1}^*} \rho_r \, dr = \frac{1}{N} \int_0^T \rho_r \, dr =: -\log a,$$

and trades defined via

$$F^{-1}\left(X_0 - N\xi_0^*\left(1-a\right)\right) = \frac{F^{-1}(\xi_0^*) - aF^{-1}(a\xi_0^*)}{1-a},$$

and

$$\xi_1^* = \cdots = \xi_{N-1}^* = \xi_0^*\left(1-a\right),$$

as well as

$$\xi_N^* = X_0 - \xi_0^* - (N-1)\xi_0^*\left(1-a\right).$$

Moreover, $\xi_i^* > 0$ for all i.

Taking $X_0 \downarrow 0$ yields:

Corollary 12. *Both the original and simplified models admit neither strong nor weak price manipulation strategies.*

Robustness of the optimal strategy

[Plots by C. Lorenz (2009)]

First figure:

$$f(x) = \frac{1}{1+|x|}$$

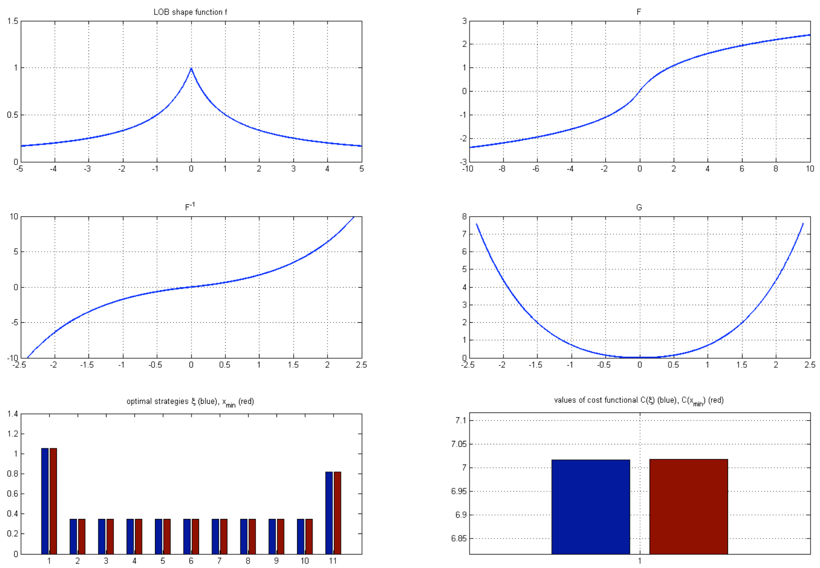


Figure 1: f, F, F^{-1}, G and optimal strategy

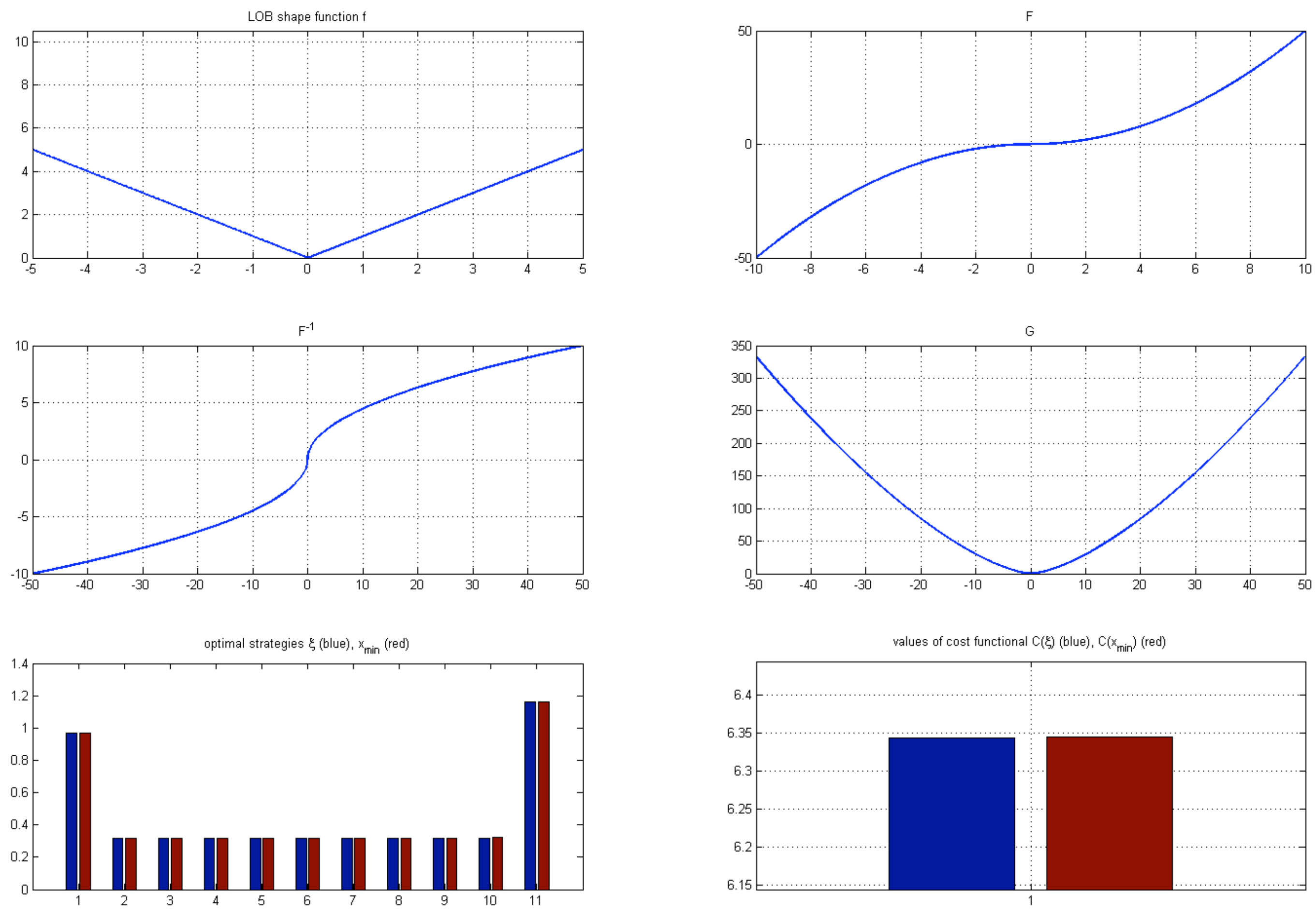

LOB shape function f
F
F^-1
G
optimal strategies ξ (blue), x_min (red)
values of cost functional C(ξ) (blue), C(x_min) (red)

Figure 2: $f(x) = |x|$

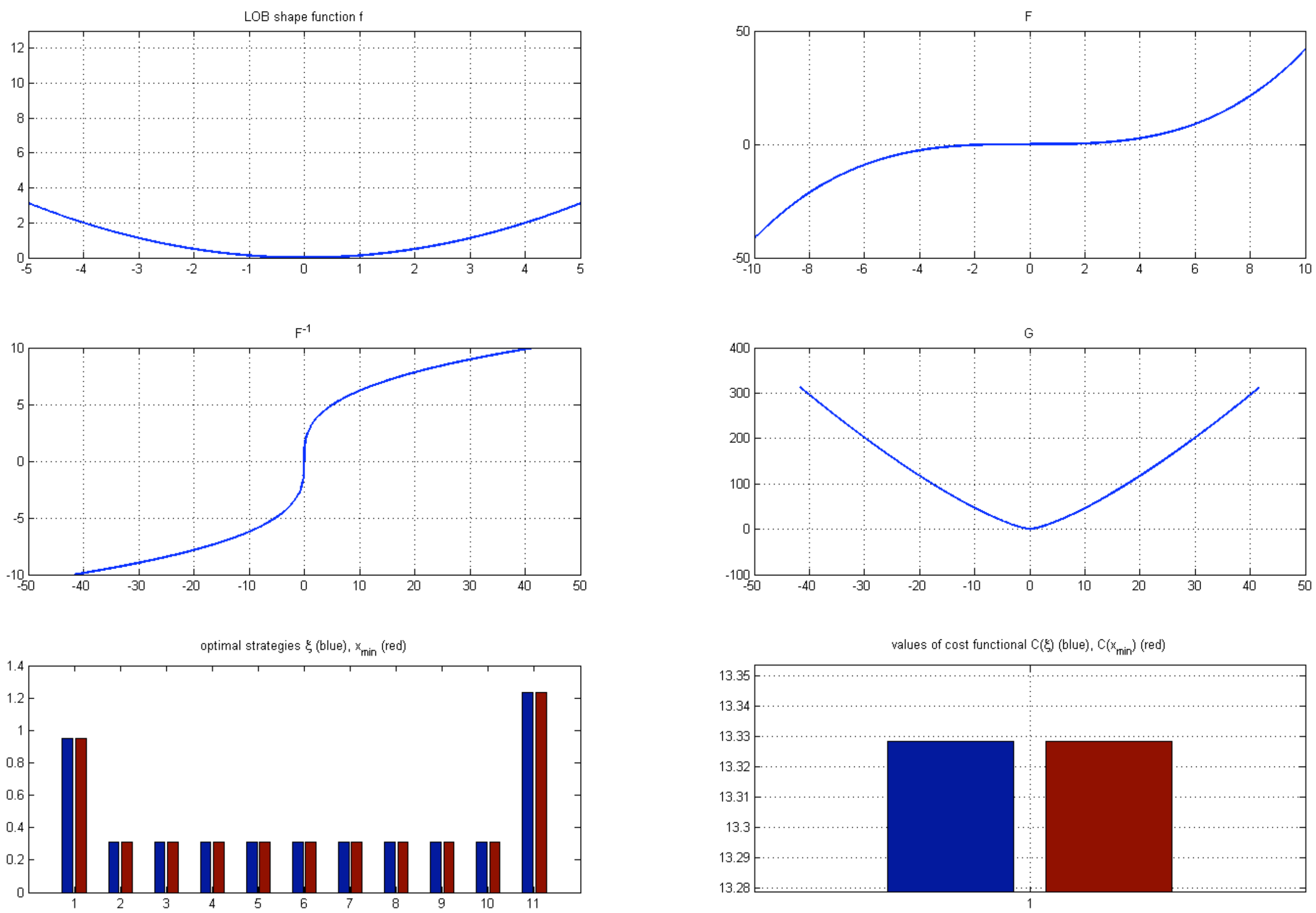


Figure 3: $f(x) = \frac{1}{8}x^2$

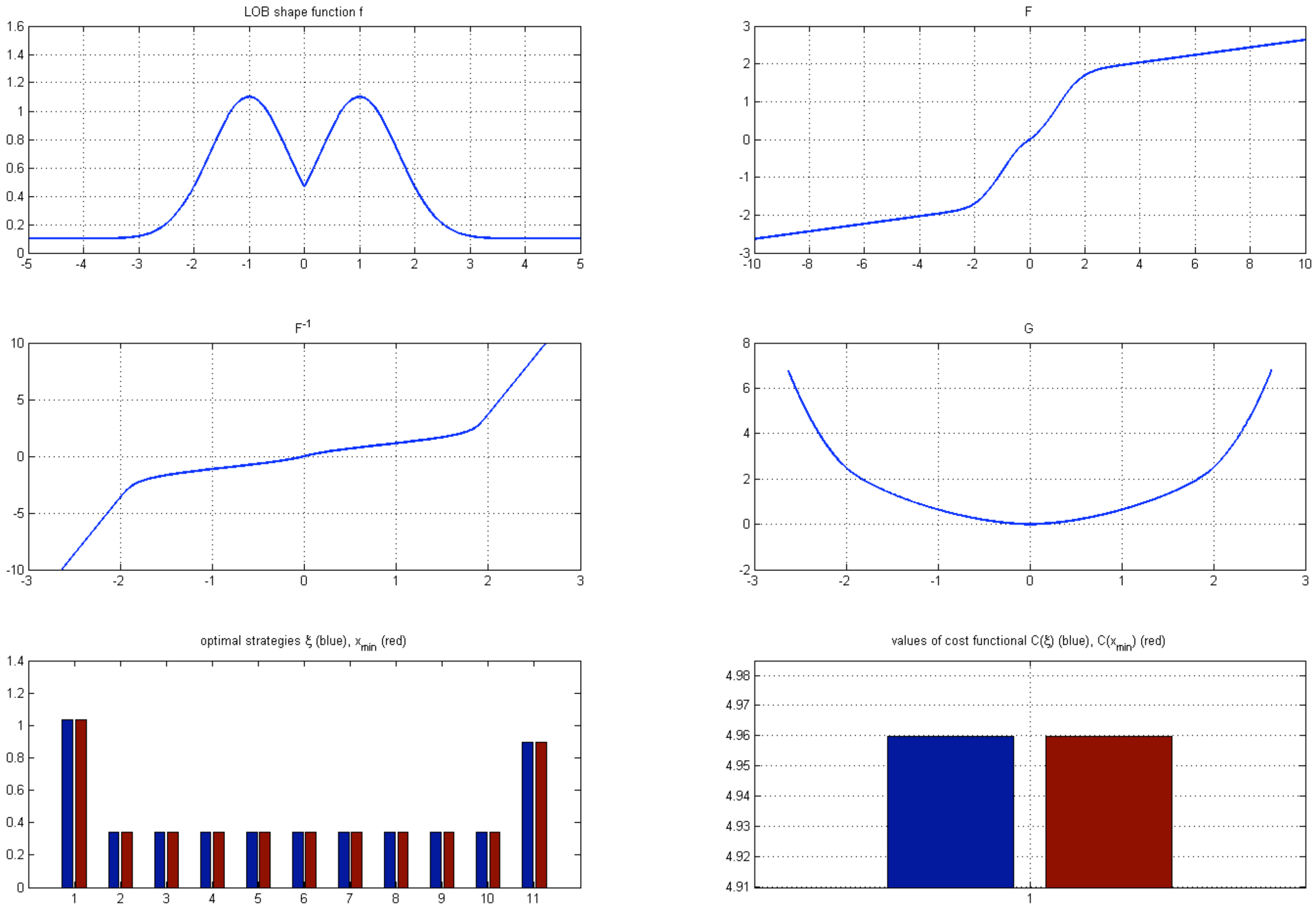


Figure 4: $f(x) = \exp(-(|x| - 1)^2) + 0.1$

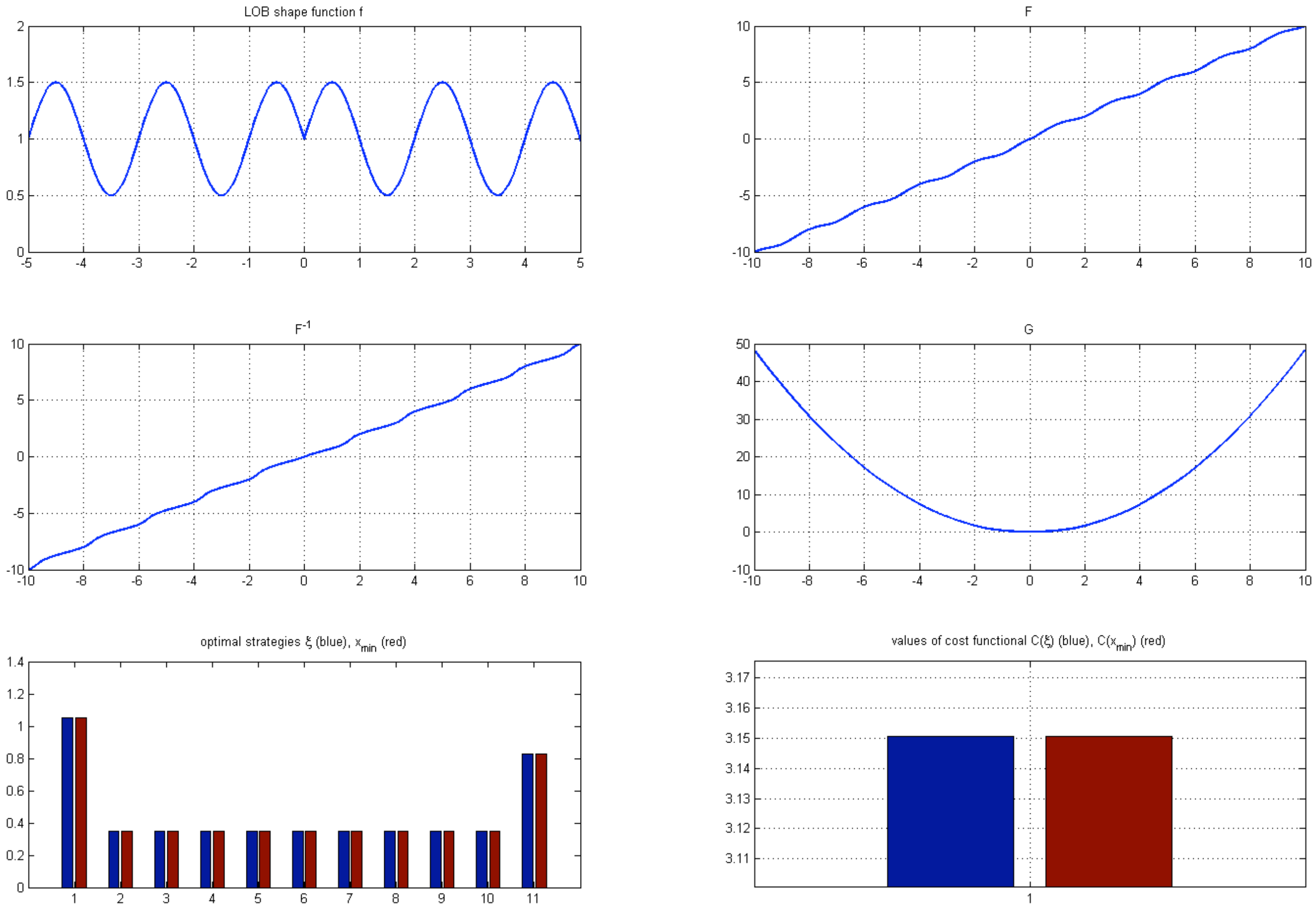


Figure 5: $f(x) = \frac{1}{2}\sin(\pi|x|) + 1$

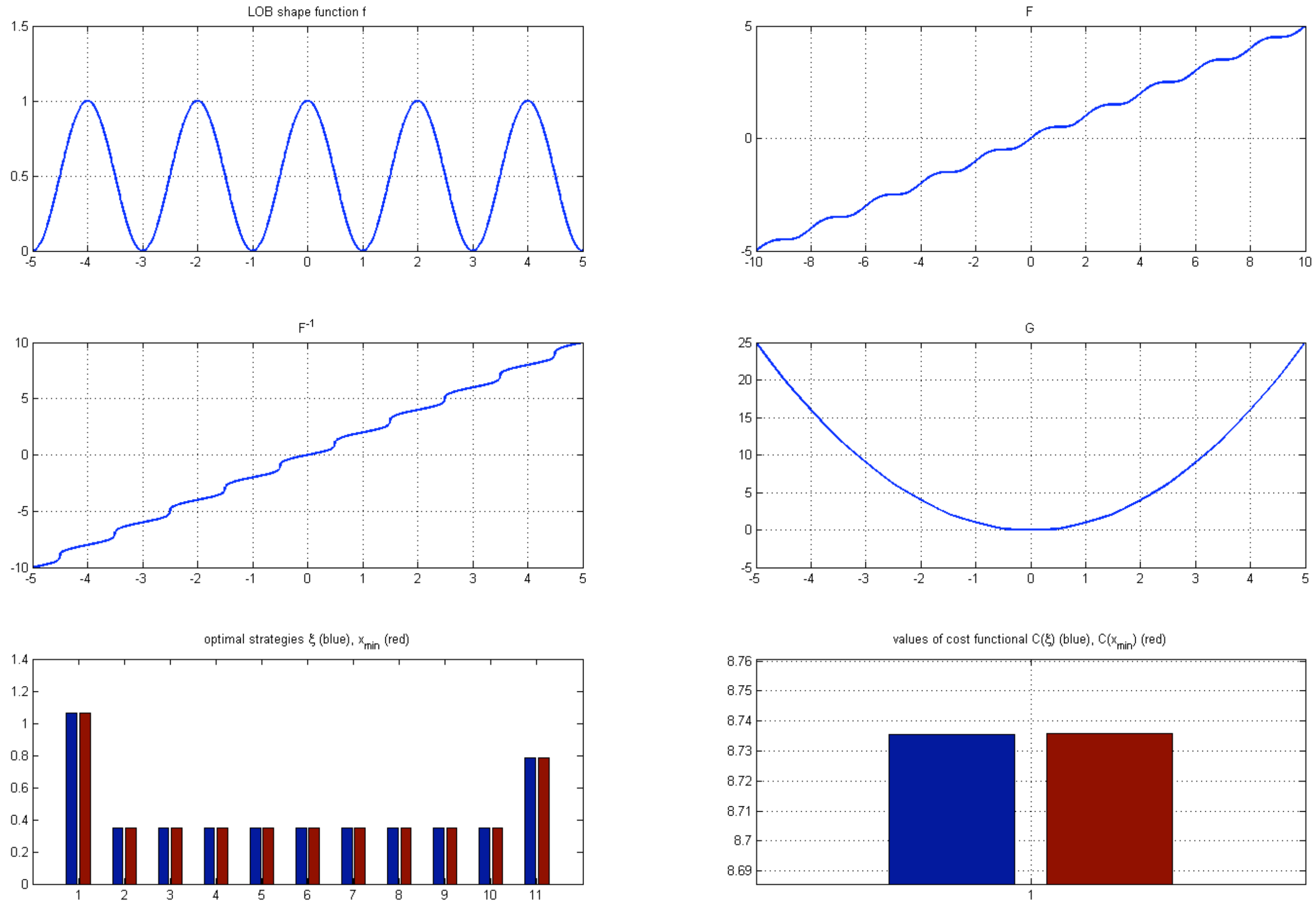


Figure 6: $f(x) = \frac{1}{2}\cos(\pi|x| + \frac{1}{2})$

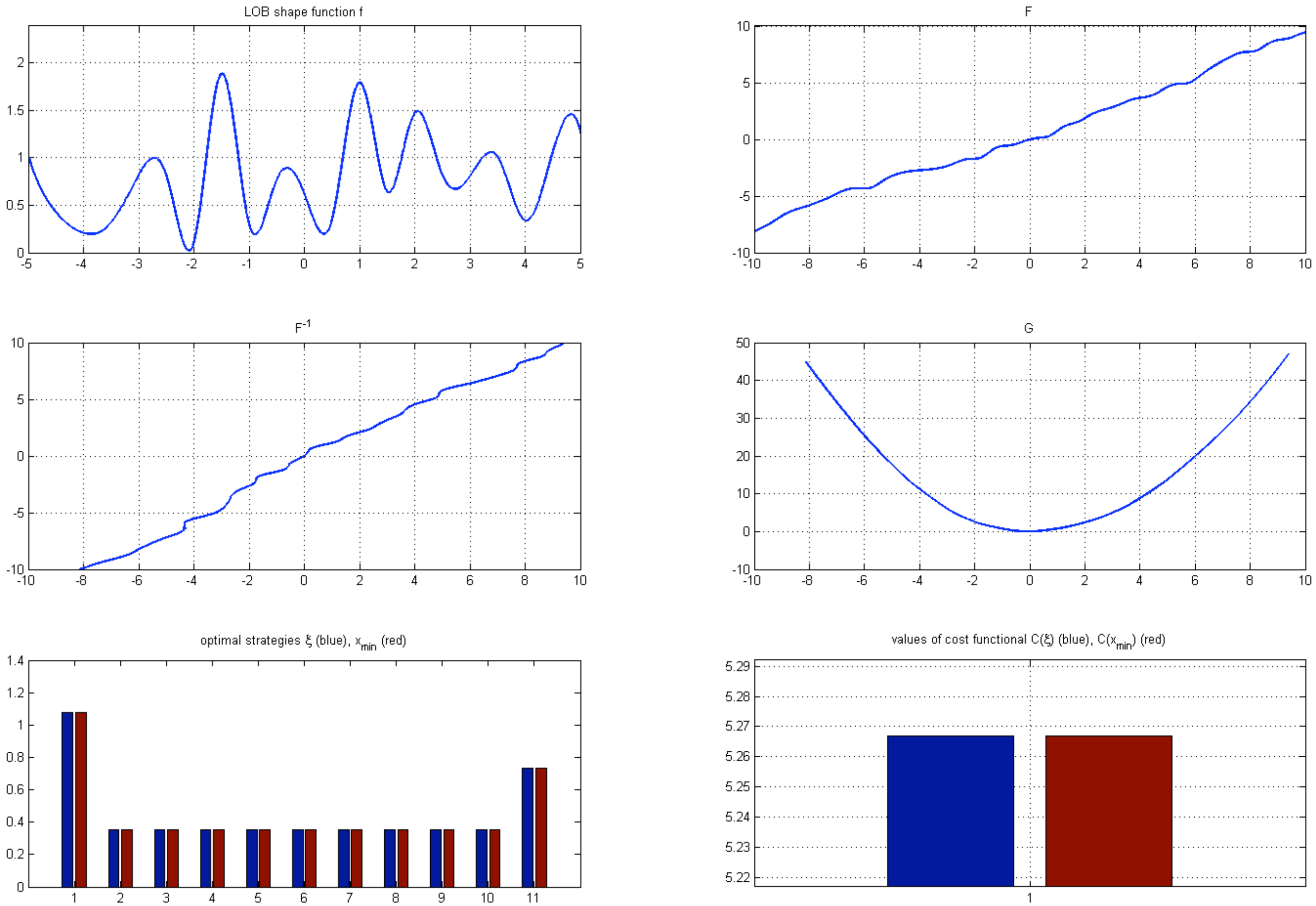


Figure 7: f random

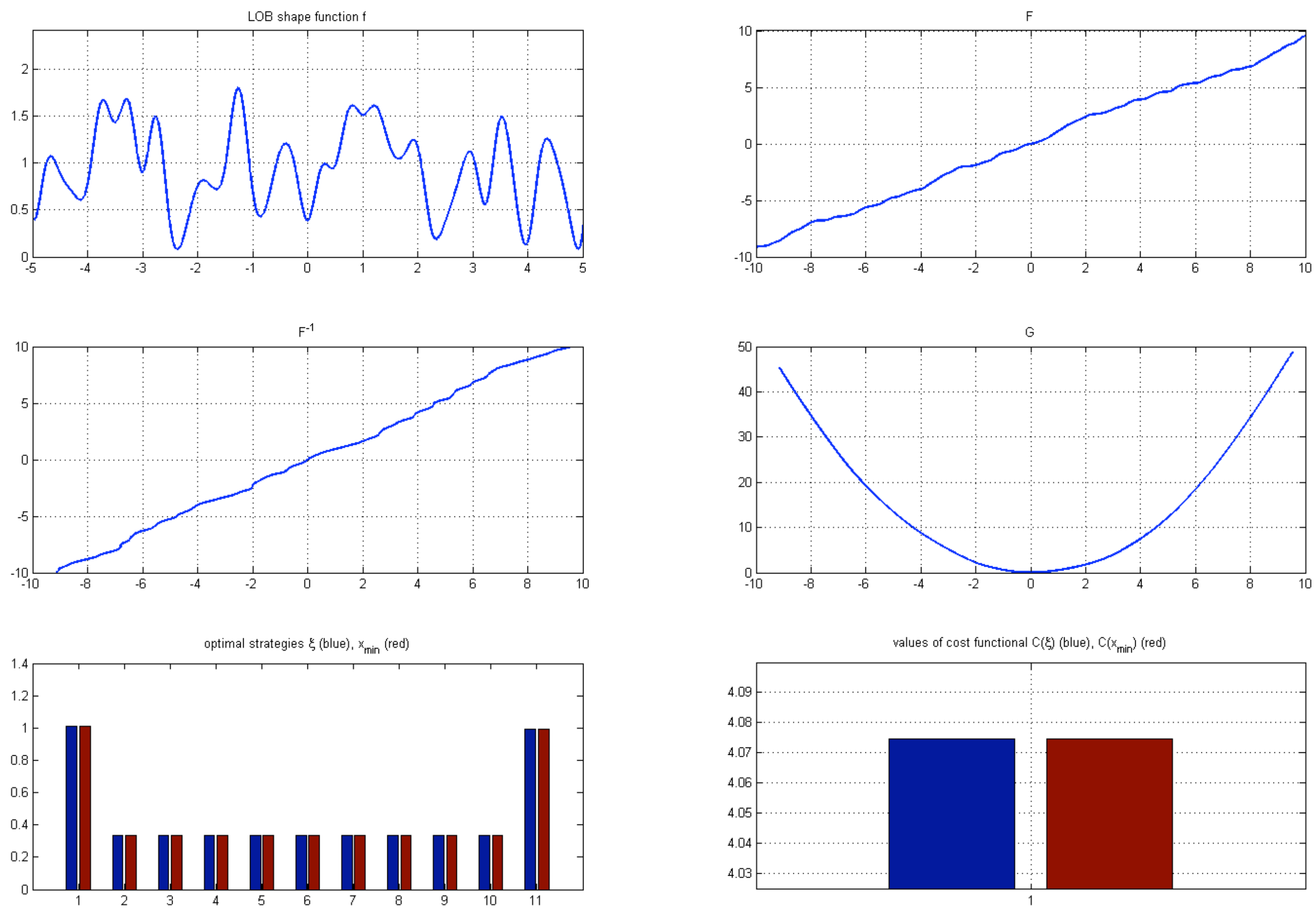


Figure 8: f random

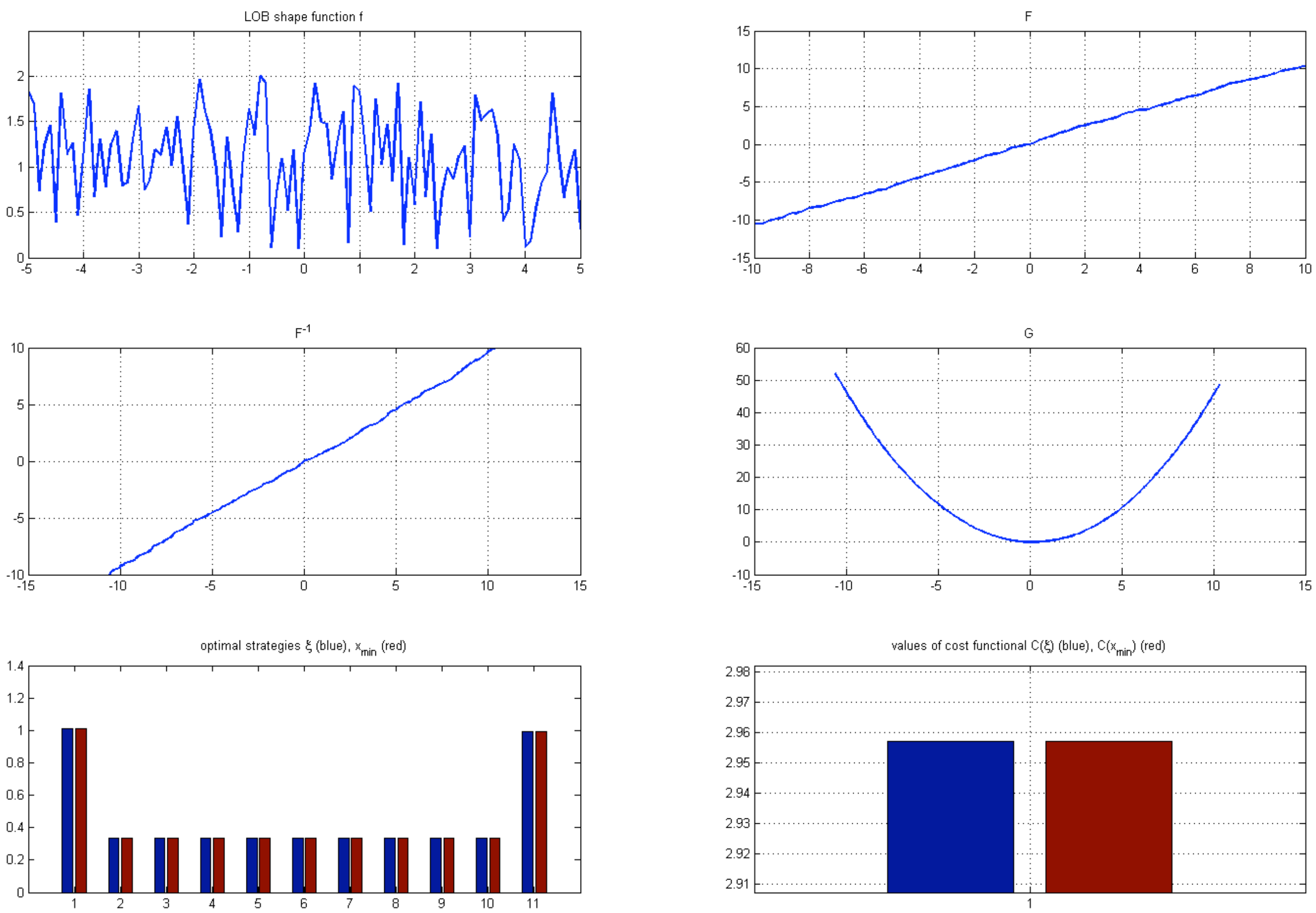


Figure 9: f random

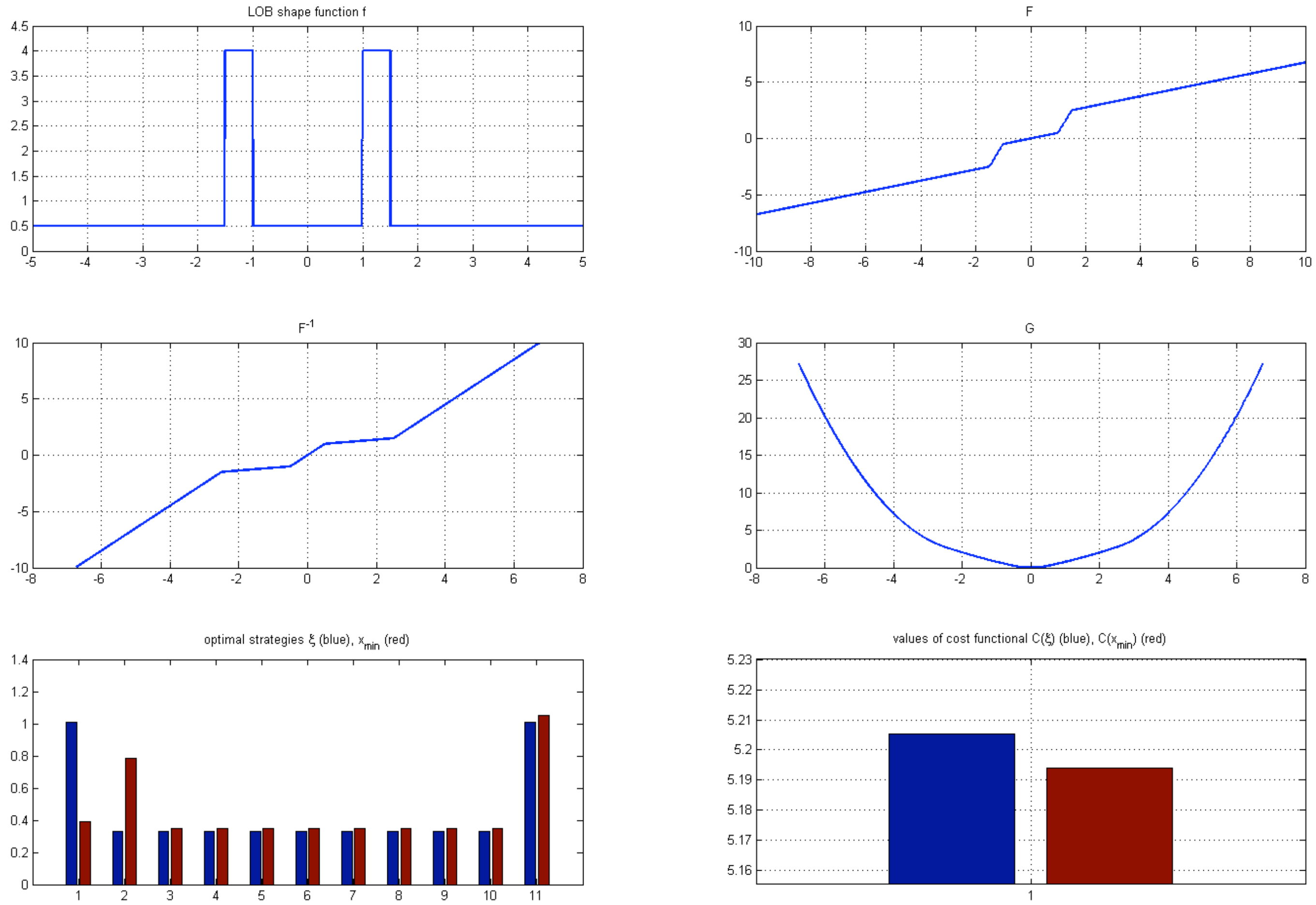


Figure 10: f piecewise constant

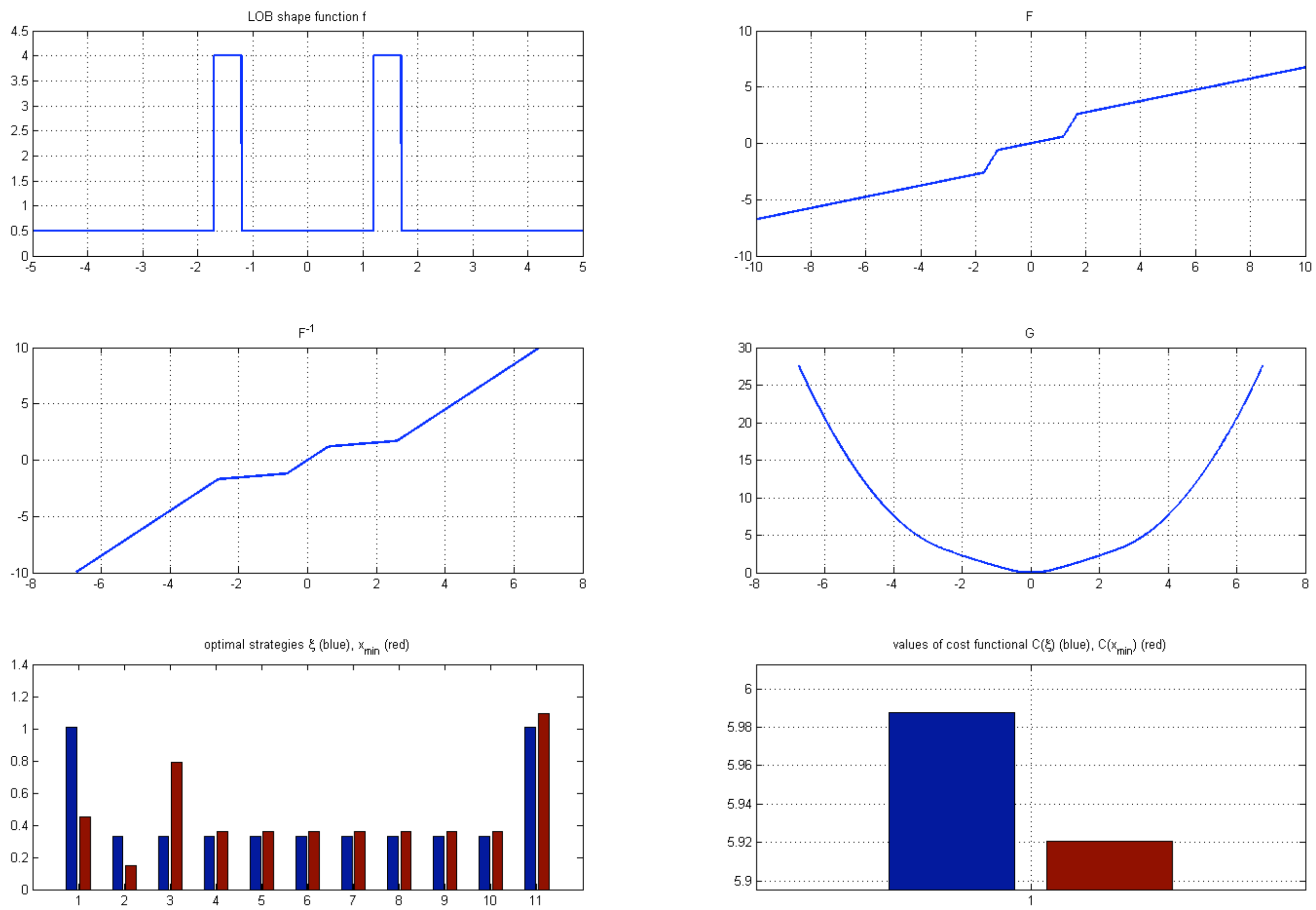


Figure 11: f piecewise constant

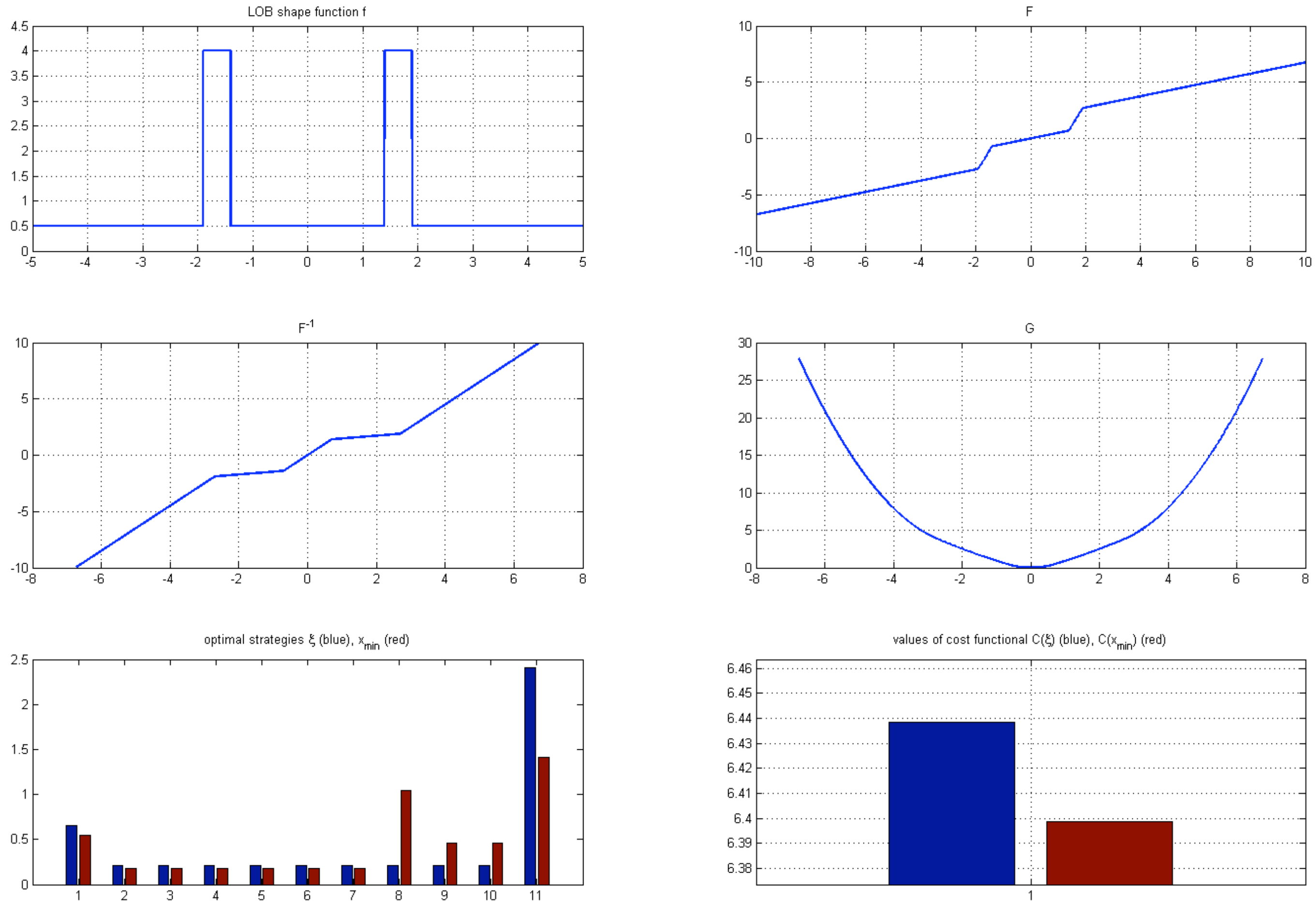


Figure 12: f piecewise constant

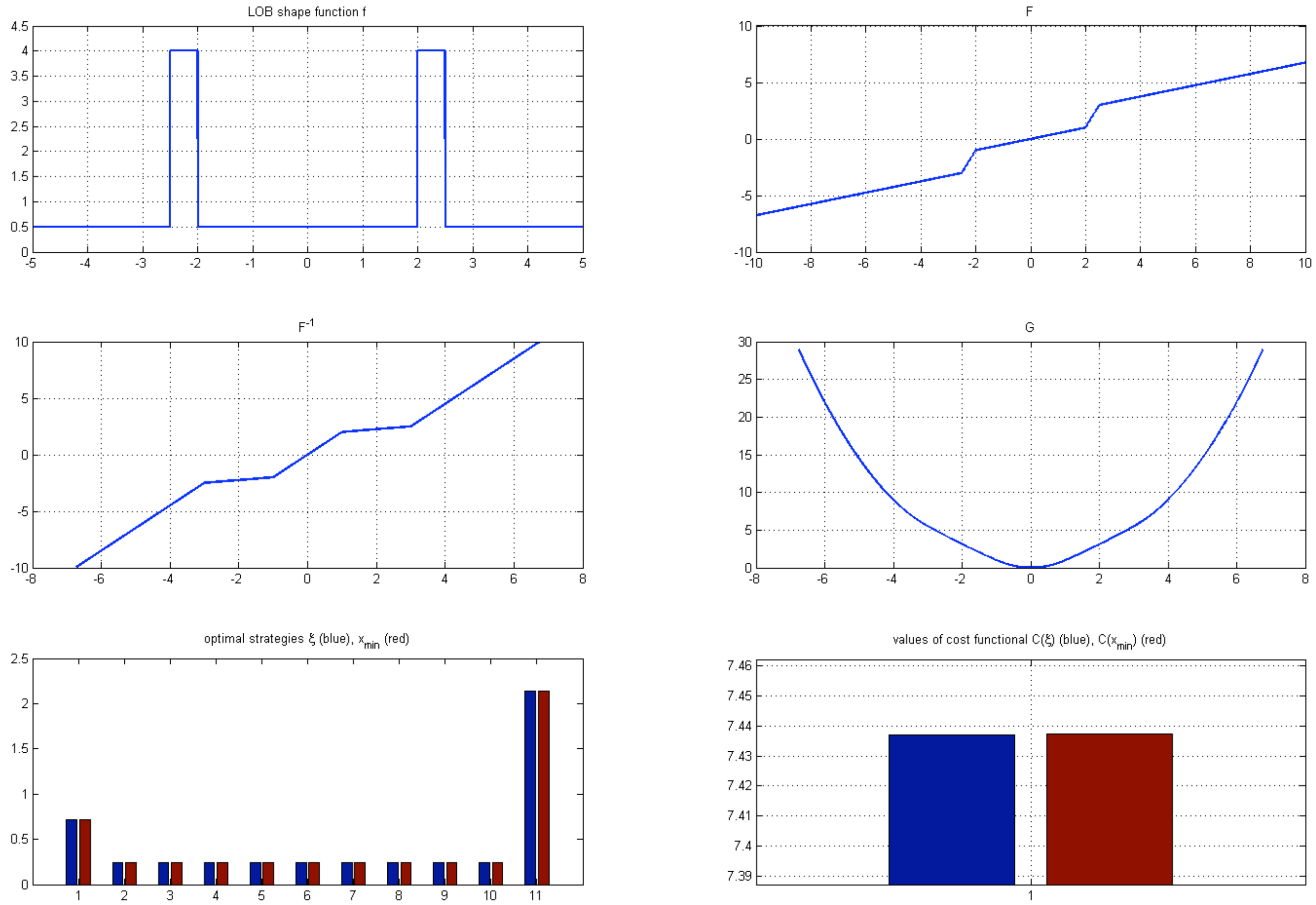


Figure 13: f piecewise constant

Continuous-time limit of the optimal strategy

- Initial block trade of size ξ_0^*, where

$$F^{-1}\Big(X_0 - \xi_0^* \int_0^T \rho_s \, ds\Big) = F^{-1}(\xi_0^*) + \frac{\xi_0^*}{f(F^{-1}(\xi_0^*))}$$

- Continuous trading in $]0, T[$ at rate

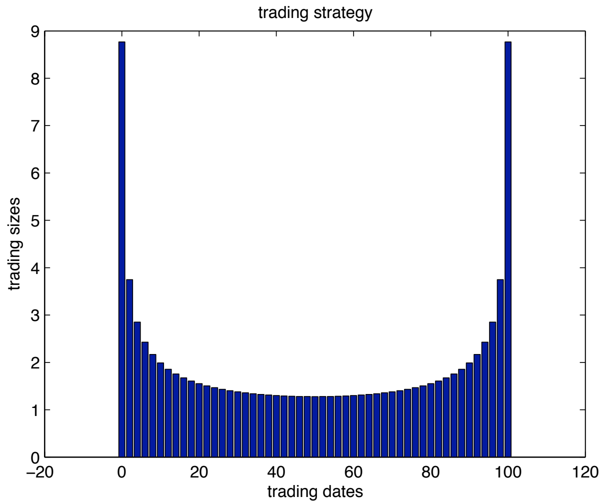
$$\xi_t^* = \rho_t \xi_0^*$$

- Terminal block trade of size

$$\xi_T^* = X_0 - \xi_0^* - \xi_0^* \int_0^T \rho_t \, dt$$

Conclusion

- Market impact should decay as a convex function of time
- Exponential or power law resilience leads to "bathtub solutions"



  which are extremely robust

- Many open problems